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THE STABILITY OF WEAK SOLUTIONS TO AN ANISOTROPIC POLYTROPIC INFILTRATION EQUATION

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Abstract. This paper considers an anisotropic polytropic infiltration equation with a source term

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u|^{\alpha_i}|u_{x_i}|^{p_i-2}u_{x_i}\right) + f(x,t,u),$$

where $p_i > 1$, $\alpha_i > 0$, $a_i(x) \geq 0$. The existence of weak solution is proved by parabolically regularized method. Based on local integrability $u_{x_i} \in W^{1,p_i}_{loc}(\Omega)$, the stability of weak solutions is proved without boundary value condition by the weak characteristic function method. One of the essential characteristics of an anisotropic equation different from an isotropic equation is found originally.

1. Introduction

In this paper, we consider the following anisotropic polytropic infiltration equation

$$u_t=\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u|^{\alpha_i}|u_{x_i}|^{p_i-2}u_{x_i}\right)+f(x,t,u),\ (x,t)\in Q_T = \Omega\times(0,T). \tag{1.1}$$

Anisotropic operators modelize directionally dependent phenomena [8, 9, 23], and have attracted many people's attention [2, 4, 5, 10, 11, 13, 16, 17]. Letting $m_i = 1 + \frac{\alpha_i}{p_i-1}$, $b_i(x) = \left(1 + \frac{\alpha_i}{p_i-1}\right)^{1-p_i} a_i(x)$, the equation (1.1) becomes

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(b_i(x)\left|u^{m_i}_{x_i}\right|^{p_i-2}u_{x_i}\right) + f(x,t,u),\ (x,t) \in Q_T, \tag{1.2}$$

where $\Omega \subset \mathbb{R}^N$ is a bounded domain with smooth boundary $\partial\Omega$, $\alpha_i > 0$, $p_i > 1$, $a_i(x) \geq 0$ is a continuous function, $f(x,t,u)$ is a Lipschitz function

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when $|u| \leq c$. A simpler version of the equation (1.2) is

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(b_i(x)|u_{x_i}^m|^{p_i-2}u_{x_i}\right) + f(x,t,u). \tag{1.3}$$

Moreover, if $p_i = p$ for all $i = 1, 2, \ldots, N$, then the equation (1.3) has the form

$$(u^\beta)_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(b_i(x)\,|u_{x_i}|^{p-2}\,u_{x_i}\right) + f(x,t,u), \tag{1.4}$$

where $\beta = \frac{1}{m}$. If $a_i(x) = 1$ and $f(x,t,u) = -\lambda u^\gamma$, Tsutsumi had studied the existence, uniqueness, regularity, and the behavior of solutions to the equation (1.4) in [22]. Here, λ and γ are positive constants. Meanwhile, many scholars have mainly paid attention on the polytropic infiltration equations with the isotropic type, i.e.,

$$u_t = \operatorname{div}(|u|^r|\nabla u|^{p-2}\nabla u) + f(x,t,u,\nabla u), \tag{1.5}$$

or its equivalent form,

$$u_t = \operatorname{div}(|\nabla u^m|^{p-2}\nabla u^m) + f(x,t,u,\nabla u). \tag{1.6}$$

There are a great deal of papers to study various subjects on polytropic infiltration equations (1.5) and (1.6). Let us give a simple review. If $f(x,t,u,\nabla u) = \nabla A(u)$ and $u_0(x) \in L^q(\Omega)$ with $q \geq 1$, by imposing some restrictions on the convection function $A(s)$, the local L^∞-estimates were made and $u_t \in L^2(\mathbb{R}^N \times (\tau,T))$ was proved in [6]. If the initial value $u_0(x) \in L^1(\mathbb{R}^N)$, the well-posedness problem was studied in [34] and $u_t \in L^1(\mathbb{R}^N \times (\tau,T))$ is true for any $\tau > 0$. If the initial value $u_0(x)$ is a measure, the Cauchy problem was considered in [12] and [18]. The large time behavior of solutions to the equations (1.5) and (1.6) had been studied in [1]. The extinction, positivity and the blow-up of solutions had been studied in [21, 25]. Also, there are a lot of papers to other subjects such the regularity, the Harnack inequality and the free boundary problem in [7, 14, 15, 19, 20, 36, 37] and the references therein.

In recent years, using some techniques of [34], the existence and the uniqueness of weak solution to the equation

$$u_t = \operatorname{div}(a(x)|\nabla u^m|^{p-2}\nabla u^m) \tag{1.7}$$

had been studied by the author in [26, 28], where $a(x)$ satisfies

$$a(x) > 0,\ x \in \Omega;\ a(x) = 0,\ x \in \partial\Omega. \tag{1.8}$$

By assuming $a_i(x) \geq 0$

$$a_i(x) > 0,\ x \in \Omega,\ a_i(x) = 0,\ x \in \partial\Omega,\ i = 1, 2, \ldots, N, \tag{1.9}$$

the anisotropic equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\right) + \sum_{i=1}^{N} \frac{\partial b^i(u)}{\partial x_i} \tag{1.10}$$

was studied by the author very recently [29]. Since $a_i(x)|_{x\in\partial\Omega}=0$, the weak solution u may not be in $L^1(0,T;W_0^{1,\vec{p}}(\Omega))$, where $\vec{p}=(p_1,p_2,\dots,p_N)$ and $W_0^{1,\vec{p}}(\Omega)$ is the vector-form Sobolev space. However, since $a_i(x)|_{x\in\Omega}>0$, $u\in L^1(0,T;W_{loc}^{1,\vec{p}}(\Omega))$ is true. Basing on this observation, the stability of weak solutions is proved in some cases [29].

In this paper, we consider the well-posedness problem of the equation (1.1) with that $f(x,t,z)$ is a Lipchitz function on $\bar{\Omega}\times[0,T]\times[-c,c]$ for any given $c>0$. For the isotropic case equation (i.e., the equation (1.7)), where $|\nabla u^m|\in L^p_{loc}(\Omega)$ can be directly deduced from the property $a(x)|\nabla u^m|^p\in L^1(\Omega)$ [26,28]. Also, for the anisotropic equation (1.10), $|u_{x_i}|\in L^{p_i}_{loc}(\Omega)$ is a direct corollary of the conclusion $a_i(x)|u_{x_i}|^{p_i}\in L^1(\Omega)$ [29]. Unlike these equations, for the equation (1.1) considered in this paper, though we can show $|u|^{\alpha_i}|u_{x_i}|^{p_i}\in L^1_{loc}(\Omega)$ easily, we find it hard to extrapolate $|u_{x_i}|\in L^{p_i}_{loc}(\Omega)$ from $|u|^{\alpha_i}|u_{x_i}|^{p_i}\in L^1_{loc}(\Omega)$. For the first time, the existence of weak solution to anisotropic equation (1.1) with $|u_{x_i}|\in L^{p_i}_{loc}(\Omega)$ is shown in this paper, a similar result has been obtained in our unpublished paper [32] in which a non-Newtonian fluid and electrorheological fluid mixed type equation is considered.

Afterwards, for a nonlinear degenerate parabolic equation, in order to study the stability of weak solutions, it is general knowledge that the initial value

$$u(x,t)=u_0(x),\ x\in\Omega, \tag{1.11}$$

is always necessary. While the boundary value condition

$$u(x,t)=0,\ (x,t)\in\partial\Omega\times[0,T), \tag{1.12}$$

maybe overdetermined generally. Actually, there are two related questions here. One is that, only if

$$\int_\Omega|\nabla u|dx<\infty, \tag{1.13}$$

then (1.12) is true in the trace sense. But unfortunately, for a nonlinear parabolic equation, especially when the coefficient is not smooth enough or the equation itself has a strongly degeneracy or singularity, how to obtain the BV estimate (1.13) becomes difficult. Another one is that even if (1.13) is true, the boundary value condition (1.12) may be still overdetermined. Although one can conjecture that only a partial boundary value condition

$$u(x,t)=0,\ (x,t)\in\Sigma\subset\partial\Omega\times[0,T), \tag{1.14}$$

is required, it is very difficult to find the geometric expression of the submanifold Σ.

Such a difficulty makes us to find the other conditions to replace the boundary value condition (1.12). In our previous works [26, 28, 29], to study the stability of weak solutions, conditions (1.8) and (1.9) act as such a role in some special senses. In this paper, we will continue to probe this problem by a new method, which was called as the weak (or general) characteristic function method, introduced in [31]. The corresponding definitions are quoted below.

Definition 1.1. For a nonnegative continuous function $\phi(x)$ in $\mathbb{R}^N$, when x is near to the boundary $\partial\Omega$, $\phi(x)$ is a C^1 function, and satisfies

$$\partial\Omega = \{x \in \mathbb{R}^N : \phi(x) = 0\},\ \Omega = \{x \in \mathbb{R}^N : \phi(x) > 0\},$$

then $\phi(x)$ is called as a weak characteristic function of Ω.

Definition 1.2. By the weak characteristic function method it means that, by choosing a suitable test function related to a weak characteristic function of Ω, one can find the explicit geometric expression of Σ in the partial boundary value condition (1.15), or one can prove the stability of weak solutions by searching for some other conditions to replace the boundary value condition.

One of the main results of this paper is given here.

Theorem 1.3. *Let* $0 \leq a_i(x) \in C(\overline{\Omega})$, $u(x,t)$ *and* $v(x,t)$ *be two solutions of the equation* (1.1) *with the initial values* $u_0(x)$ *and* $v_0(x)$ *respectively. If* $\alpha^- \geq 1$, $p^+ < 2$, *and*

$$a_i(x) \leq cd(x),\ i = 1, 2, \dots, N, \tag{1.15}$$

then

$$\int_\Omega |u(x,t) - v(x,t)|dx \leq c \int_\Omega |u_0(x) - v_0(x)|dx. \tag{1.16}$$

Moreover, if $u(x,t) > 0$ *and* $v(x,t) > 0$ *when* $x \in \Omega$, *instead of* $\alpha^- \geq 1$, *only* $\alpha_i > 0$, *then the stability* (1.16) *is still true.*

Roughly speaking, the condition (1.15) is a substitute of the boundary value condition. Here, there is a remained problem that whether Theorem 1.3 is still true only if $p^- > 1$. In this paper, solution to this question is presented in part in the last section in which the stability of weak solutions to the equation (1.3) is established. We would like to give some details at first. One can see that the condition (1.15) implies that $a_i(x)$ satisfies (1.9). But for the equation (1.3), instead of the condition (1.9), only if

$$\prod_{i=1}^{N} b_i(x) = 0, \tag{1.17}$$

the stability of weak solutions may be obtained, such a fact has been found by the authors in [33], where the equation

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(b_i(x) \left| u_{x_i} \right|^{p_i(x)-2} u_{x_i} \right) + \sum_{i=1}^{N} \frac{\partial f_i(u,x,t)}{\partial x_i} - f(x,t)|u|^{\sigma(x)-2}u$$

was considered. Since [33] is unpublished, we should give a more explanation here. We give an example to show that the condition (1.17) reflects the essential characteristic of an anisotropic equations different from the isotropic equations. Let $\Omega \subset \mathbb{R}^N$ be a ring domain

$$\Omega = \{x \in \mathbb{R}^2 : r < x_1^2 + x_2^2 + \cdots + x_N^2 < 2r\}.$$

Here $r>0$ is a given constant. If

$$b_1(x)=(x_1^2+x_2^2+\cdots+x_N^2)-r,$$
$$b_2(x)=b_3(x)=\cdots=b_N(x)=[2r-(x_1^2+x_2^2+\cdots+x_N^2)]^{\frac{1}{N-1}},$$

then the condition (1.17) is true. The conditions

$$b_1(x)>0,\ x\in\Sigma_2=\{x:x_1^2+x_2^2+\cdots+x_N^2=2r\}, \tag{1.18}$$

and

$$b_i(x)>0,\ x\in\Sigma_1=\{x:x_1^2+x_2^2+\cdots+x_N^2=r\},\ i=2,3,\dots,N, \tag{1.19}$$

seem very different from the condition (1.9). In order to ensure the well-posedness of weak solutions, my consciousness of such things is pretty much limited to that, since (1.18), maybe one should impose the boundary value on

$$\Sigma_2\times(0,T). \tag{1.20}$$

In the meantime, from (1.19), maybe one should impose the boundary value on

$$\Sigma_1\times(0,T). \tag{1.21}$$

Since $\Sigma_1\bigcup\Sigma_2=\partial\Omega$, it seems that the boundary value condition (1.12) should be imposed as usual. But this is not the truth. Only if the condition (1.17) is true, the stability of weak solutions can be proved without the boundary value condition (1.12). In other words, the condition (1.17) can be regarded as a substitute of the boundary value condition for the equation (1.3).

Throughout this paper, we denote by

$$p^+=\max\{p_1,p_2,\dots,p_N\},\ p^-=\min\{p_1,p_2,\dots,p_N\},$$
$$\alpha_-=\min\{\alpha_1,\alpha_2,\dots,\alpha_N\},$$

and

$$d(x)=\mathrm{dist}(x,\partial\Omega).$$

The paper is arranged as follows. In Section 1, we give the introduction. In Section 2, the various definitions of weak solutions are introduced, the existence of the various weak solutions are proved. In Section 3, Theorem 1.3 is proved. In Section 4, the stability of weak solutions to the equation (1.3) is proved without boundary value condition.

2. The existence of the solutions

First, we consider the equation (1.3) with the initial value condition

$$u(x,t)=u_0(x),\ x\in\Omega, \tag{2.1}$$

where $u_0(x)$ satisfies

$$\int_\Omega u_0^{m+1}(x)dx<\infty,\ \int_\Omega b_i(x)|u^m_{0x_i}|^{p_i}dx<\infty,\ i=1,2,\dots,N. \tag{2.2}$$

Definition 2.1. By a weak solution of the equation (1.3) with the initial value (2.1) we mean a nonnegative function

$$u^m \in L^\infty([\tau,T];W^{1,p_i}_{loc}(\Omega)\cap L^1(\Omega))$$

with

$$u\in C([0,T);L^1(\Omega)),\ \frac{\partial}{\partial t}u^{\frac{m+1}{2}}\in L^2(\Omega\times(\tau,T))$$

for any $\tau>0$, satisfying

$$\lim_{t\to 0}\int_\Omega |u(x,t)-u_0(x)|\,dx=0, \tag{2.3}$$

and

$$\int_0^T\int_\Omega\left(u\frac{\partial\varphi}{\partial t}-\sum_{i=1}^N b_i(x)\left|u^m_{x_i}\right|^{p_i-2}u^m_{x_i}\cdot\varphi_{x_i}\right)dxdt = \int_0^T\int_\Omega f(x,t,u)\varphi(x,t)dxdt \tag{2.4}$$

for any $\varphi\in C^1_0(\Omega\times(0,T))$.

Theorem 2.2. *Suppose that $u_0(x)\ge 0$ satisfies (2.2), $b_i(x)$ is suitable smooth. If $f(x,t,z)\ge 0$ when $z<0$, $|f(x,t,z)|\le c|z|^{\frac{m-1}{2}}$, $|f(x,t,z)|\le c\phi(x,t)$, $\phi(x,t)\in L^{q^+}(Q_T)$, then there exists a weak solution of the equation (1.3) with the initial value (2.1). Here $q^+=\frac{p^+}{p^+-1}$.*

Proof. Consider the following equation

$$u_t=\sum_{i=1}^N\frac{\partial}{\partial x_i}\left((b_i(x)+\varepsilon)|u^m_{x_i}|^{p_i-2}u^m_{x_i}\right)+f(x,t,u),\ (x,t)\in Q_T, \tag{2.5}$$

with the initial value condition (2.1) and with the homogeneous boundary value condition

$$u(x,t)=0,\ x\in\partial\Omega\times(0,T). \tag{2.6}$$

According to [22], since $f(x,t,z)\ge 0$ when $z<0$, there is a nonnegative weak solution u_ε of the equation (2.5) satisfying

$$u^m_\varepsilon\in L^\infty([\tau,T);W^{1,p_i}_0(\Omega))$$

and

$$u_\varepsilon\in C([0,T);L^1(\Omega)),\ \frac{\partial}{\partial t}u_\varepsilon^{\frac{m+1}{2}}\in L^2(\Omega\times(\tau,T))$$

for any $\tau>0$.

Multiplying (2.5) by u^m_ε and integrating it over $Q_{\tau T}=\Omega\times(\tau,T)$, we have

$$\iint_{Q_{\tau T}}\sum_{i=1}^N(b_i(x)+\varepsilon)\,|\nabla u^m_\varepsilon|^{p_i}\,dxdt \tag{2.7}$$

$$
\begin{aligned}
&+\frac{1}{m+1}\left[\int_\Omega u_\varepsilon^{m+1}(x,t)dx-\int_\Omega u_0^{m+1}(x)dx\right]\\
&=\iint_{Q_{\tau T}} f(x,t,u_\varepsilon)u_\varepsilon dxdt\\
&\le c\left(\iint_{Q_{\tau T}}\phi(x,t)^{q_i}dxdt\right)^{\frac{1}{q_i}}\left(\iint_{Q_{\tau T}}|u_\varepsilon|^{p_i}dxdt\right)^{\frac{1}{p_i}}\\
&\le c,
\end{aligned}
$$

where $q_i=\frac{p_i}{p_i-1}$, and the assumption of $|f(x,t,z)|\le c\phi(x,t)$, $\phi(x,t)\in L^{q^+}(Q_T)$, is used. Accordingly,

$$
\sum_{i=1}^N\iint_{Q_{\tau T}}(b_i(x)+\varepsilon)|\nabla u_\varepsilon^m|^{p_i}dxdt\le c. \tag{2.8}
$$

Multiplying (2.5) by $\frac{\partial u_\varepsilon^m}{\partial t}$, integrating it over $Q_{\tau T}$, yields

$$
\begin{aligned}
&\iint_{Q_{\tau T}}u_{\varepsilon t}\frac{\partial u_\varepsilon^m}{\partial t}dxdt \qquad (2.9)\\
&=\sum_{i=1}^N\iint_{Q_{\tau T}}(a_i(x)+\varepsilon)|u^m_{\varepsilon x_i}|^{p_i-2}u^m_{\varepsilon x_i}u_{\varepsilon t x_i}dxdt+\iint_{Q_{\tau T}}f(x,t,u_\varepsilon)u_{\varepsilon t}dxdt.
\end{aligned}
$$

Noticing that

$$
|u^m_{\varepsilon x_i}|^{p_i-2}u^m_{\varepsilon x_i}\frac{\partial}{\partial x_i}\frac{\partial u_\varepsilon^m}{\partial t}=\frac12\frac{d}{dt}\int_0^{\left|u^m_{\varepsilon x_i}(x,t)\right|^2}s^{\frac{p_i-2}{2}}ds,
$$

we can extrapolate that

$$
\begin{aligned}
&\iint_{\tau T}\sum_{i=1}^N\frac{\partial}{\partial x_i}\left((b_i(x)+\varepsilon)|u^m_{\varepsilon x_i}|^{p_i-2}u^m_{\varepsilon x_i}\right)\frac{\partial u_\varepsilon^m}{\partial t}dxdt \qquad (2.10)\\
&=-\sum_{i=1}^N\iint_{\tau T}(b_i(x)+\varepsilon)|u^m_{\varepsilon x_i}|^{p_i-2}u^m_{\varepsilon x_i}\frac{\partial}{\partial x_i}\frac{\partial u_\varepsilon^m}{\partial t}dxdt\\
&=-\frac12\sum_{i=1}^N\iint_{\tau T}(b_i(x)+\varepsilon)\frac{d}{dt}\int_0^{|u^m_{\varepsilon x_i}|^2}s^{\frac{p_i-2}{2}}dsdxdt\\
&=-\frac12\sum_{i=1}^N\int_\tau^T\int_\Omega(b_i(x)+\varepsilon)\int_0^{|u^m_{\varepsilon x_i}|^2}s^{\frac{p_i-2}{2}}dsdxdt\\
&\quad+\frac12\sum_{i=1}^N\int_\tau^T\int_\Omega(b_i(x)+\varepsilon)\int_0^{|u^m_{\varepsilon x_i}(x,0)|^2}s^{\frac{p_i-2}{2}}dsdxdt.
\end{aligned}
$$

By the assumption $|f(x,t,z)| \le c|z|^{\frac{m-1}{2}}$, we have

$$\left|\iint_{Q_{\tau T}} f(x,t,u_\varepsilon)u_{\varepsilon t}dxdt\right| \le \frac{1}{2m}\iint_{Q_{\tau T}} \left|f(x,t,u_\varepsilon)u_\varepsilon^{-\frac{m-1}{2}}\right|^2 dxdt + \frac{1}{2}\iint_{Q_{\tau T}} m\left|u_\varepsilon^{\frac{m-1}{2}}u_{\varepsilon t}\right|^2 dxdt. \tag{2.11}$$

Combing (2.9)-(2.11), and

$$\int_\Omega a_i(x)|\nabla u_0^m|^{p_i}dx < \infty,$$

we deduce that

$$\iint_{Q_{\tau T}} \left|u_{\varepsilon t}\frac{\partial u_\varepsilon^m}{\partial t}\right| dxdt \le c,$$

which implies that

$$\left\|\frac{\partial}{\partial t}u_\varepsilon^{\frac{m+1}{2}}\right\|_{L^2(\Omega\times(\tau,T))} \le c. \tag{2.12}$$

By (2.8), (2.12), using the usual weak convergence method, we can prove Theorem 2.2. □

Secondly, we consider the equation (1.1) with the initial value (2.1),

$$u_0 \in L^\infty(\Omega), \quad \sum_{i=1}^N b_i(x)|u_0|^{\alpha_i}|u_{0x_i}|^{p_i} \in L^1(\Omega). \tag{2.13}$$

Definition 2.3. A function $u(x,t)$ is said to be a weak solution of the equation (1.1) with the initial value (2.1), if

$$u \in L^\infty(Q_T), \ \frac{\partial u}{\partial t} \in L^2(Q_T), \ \sum_{i=1}^N a_i(x)|u|^{\alpha_i}|u_{x_i}|^{p_i} \in L^1(Q_T),$$

and for any function $\varphi \in C_0^1(Q_T)$,

$$\iint_{Q_T}\left(\frac{\partial u}{\partial t}\varphi + \sum_{i=1}^N a_i(x)|u|^{\alpha_i}|u_{x_i}|^{p_i-2}u_{x_i}\varphi_{x_i}\right)dxdt = \iint_{Q_T} f(x,t,u)\varphi(x,t)dxdt.$$

The initial value (2.1) is satisfied in the sense of (2.3).

Theorem 2.4. *Suppose that $u_0(x)$ satisfies (2.13). If $f(x,t,z)$ is a C^1 function on $\overline{\Omega}\times[0,T]\times[-c,c]$ for any bounded interval $[-c,c]\subset\mathbb{R}$, then there is a weak solution of the equation (1.1) with the initial value (2.1).*

Proof. For small $\varepsilon > 0$, $u_{0\varepsilon}(x) = J_\varepsilon * u_0(x)$ is the mollified function of $u_0(x)$ as above, then it is easily to show that $\|u_{0\varepsilon}\|_{L^\infty(\Omega)}$ and $\|a_i(x)|u_{0\varepsilon}|^{\alpha_i}|u_{0\varepsilon x_i}|^{p_i}\|_{L^1(\Omega)}$ are uniformly bounded, and $|u_{0\varepsilon}|^{\alpha_i}|u_{0\varepsilon x_i}|^{p_i}$ converges to $u_{0x_i}|u_{0x_i}|^{p_i}$ in $L^1_{\text{loc}}(\Omega)$.

Consider the approximate equation

$$u_{\varepsilon t} = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(A_{i\varepsilon}(u_\varepsilon, x, t)(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}} u_{\varepsilon x_i}\right) + f(x,t,u_\varepsilon),\ (x,t)\in Q_T, \tag{2.14}$$

with the initial value

$$u_\varepsilon(x,0) = u_{0\varepsilon}(x),\ x\in\Omega, \tag{2.15}$$

and with the homogeneous boundary value (2.6). Here,

$$A_{i\varepsilon}(u_\varepsilon, x, t) = (a_i(x)+\varepsilon)(\varepsilon + |u_\varepsilon|)^{\gamma_i(p_i-1)},\ \gamma_i = \frac{\alpha_i}{p_i-1}.$$

By [24], there is a weak solution u_ε, $a_i(x)|u_\varepsilon|^{\alpha_i}|u_{\varepsilon x_i}|^{p_i} \in L^1(Q_T)$. Application of maximum value principle yields

$$\|u_\varepsilon\|_{L^\infty(Q_T)} \le c. \tag{2.16}$$

By multiplying $\int_0^{u_\varepsilon}(\varepsilon+|s|)^{\gamma_i} ds$ into (2.14), we have

$$\begin{aligned}
&\int_\Omega \int_0^{u_\varepsilon(x,t)} (\varepsilon+|s|)^{\gamma_i} ds dx \\
&+ \int_0^t \int_\Omega (a_i(x)+\varepsilon)(\varepsilon+|u_\varepsilon|)^{p_i\gamma_i}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}|u_{\varepsilon x_i}|^2 dxdt \\
=& \int_\Omega \int_0^{u_\varepsilon(x,0)} (\varepsilon+|s|)^{\gamma_i} ds dx + \int_0^t \int_\Omega f(x,t,u_\varepsilon)\int_0^{u_\varepsilon}(\varepsilon+|s|)^{\gamma_i} dsdxdt.
\end{aligned}$$

Then

$$\begin{aligned}
&\int_0^T \int_\Omega (a_i(x)+\varepsilon)(\varepsilon+|u_\varepsilon|)^{p_i\gamma_i}|u_{\varepsilon x_i}|^{p_i} dxdt \\
&\le c\int_0^T \int_\Omega (a_i(x)+\varepsilon)(\varepsilon+|u_\varepsilon|)^{p_i\gamma_i}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}|u_{\varepsilon x_i}|^2 dxdt \\
&\le c,
\end{aligned} \tag{2.17}$$

and in particular, for any $\Omega_1 \subset\subset \Omega$, we have

$$\int_0^T \int_{\Omega_1} (|u_\varepsilon|^{\gamma_i}|u_{\varepsilon x_i}|)^{p_i} dxdt \le c(\Omega_1). \tag{2.18}$$

At the same time, we can rewrite (2.14) as

$$u_{\varepsilon t} = \sum_{i=1}^{N}\left[\frac{\partial A_{i\varepsilon}}{\partial x_i}\left(|u_{\varepsilon x_i}|^2+\varepsilon\right)^{\frac{p_i-2}{2}} u_{\varepsilon x_i} + A_{i\varepsilon}\alpha_{ii}u_{\varepsilon x_i x_i} + f(x,t,u_\varepsilon)\right], \tag{2.19}$$

where

$$\alpha_{ii} = \left(|u_{\varepsilon x_i}|^2+\varepsilon\right)^{\frac{p_i-2}{2}}[(p_i-2)\left(|u_\varepsilon|^2+\varepsilon\right)^{-1}u_{\varepsilon x_i}u_{\varepsilon x_i}+1].$$

Making differential with t, then

(2.20)

$$\begin{aligned}\frac{\partial w}{\partial t} = &\sum_{i=1}^{N}\Big[A_{i\varepsilon}\alpha_{ii}\frac{\partial^2 w}{\partial x_i\partial x_i}+\frac{\partial(A_{i\varepsilon}\alpha_{ii})}{\partial x_i}\frac{\partial w}{\partial x_i}\\ &+\left(|u_{\varepsilon x_i}|^2+\varepsilon\right)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)-1}\gamma_i(p_i-1)(a_i+\varepsilon)\mathrm{sign}(u_\varepsilon)w\frac{\partial w}{\partial x_i}\\ &+\frac{\partial A_{i\varepsilon}}{\partial x_i}\left(|u_{\varepsilon x_i}|^2+\varepsilon\right)^{\frac{p_i-2}{2}-4}((p_i-1)|u_{\varepsilon x_i}|^2+\varepsilon)\frac{\partial w}{\partial x_i}\Big]\\ &+F(x,t,u_\varepsilon,u_{\varepsilon x_i},w),\end{aligned}$$

where $w=\frac{\partial u_\varepsilon}{\partial t}$, and

$$\begin{aligned}&F(x,t,u_\varepsilon,u_{\varepsilon x_i},w)\\ =&\sum_{i=1}^{N}\left(|u_{\varepsilon x_i}|^2+\varepsilon\right)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}\Big[(\gamma_i(p_i-1)-1)(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)-2}\mathrm{sign}(u_\varepsilon)w\\ &\cdot\left(\frac{\partial a_i}{\partial x_i}(|u_\varepsilon|+\varepsilon)+\gamma_i(p_i-1)(a_i+\varepsilon)\mathrm{sign}(u_\varepsilon)u_{\varepsilon x_i}\right)\\ &+(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)-1}\frac{\partial a_i}{\partial x_i}\mathrm{sign}(u_\varepsilon)w\Big]\\ &+\frac{\partial f}{\partial t}+\frac{\partial f}{\partial z}w.\end{aligned}$$

Clearly, $w(x,t)$ satisfies

$$\begin{aligned}&w(x,t)=0,(x,t)\in\partial\Omega\times(0,T),\\ &w(x,0)=\sum_{i=1}^{N}\frac{\partial}{\partial x_i}(A_{i\varepsilon}(u_{0\varepsilon},x,0)(|u_{0\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{0\varepsilon x_i})+f(x,0,u_{0\varepsilon}),\ x\in\Omega.\end{aligned}$$

Denoting that

$$\alpha_{i0}=\left(|u_{0\varepsilon x_i}|^2+\varepsilon\right)^{\frac{p_i-2}{2}},\ \alpha_0=\min_{i=1}^{N}\{\alpha_{i0}\},\ \alpha^0=\max_{i=1}^{N}\{\alpha_{i0}\}$$

for any $\xi\in\mathbb{R}^N$, we have

$$\min\{p_i-1,1\}\alpha_0|\xi|^2\le\alpha_{ii}\xi^i\xi^i\le\max\{p_i-1,1\}\alpha^0|\xi|^2,\ \forall\xi\in\mathbb{R}^N.$$

Since

$$\varepsilon\le(A_{i\varepsilon}(u_{0\varepsilon},x,0)\le c(\varepsilon)$$

and

$$\varepsilon\min\{p_i-1,1\}\alpha_0|\xi|^2\sum_{i=1}^{N}A_{i\varepsilon}\alpha_{ii}\xi^i\xi^i\le c(\varepsilon)\max\{p_i-1,1\}\alpha^0|\xi|^2, \tag{2.21}$$

we can employ the maximal value principle to show that

(2.22)

$$\max_{Q_T}|u_{\varepsilon t}| \leq \max\sum_{i=1}^{N}\left|\frac{\partial}{\partial x_i}(A_{i\varepsilon}(u_{0\varepsilon},x,0)(|u_{0\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{0\varepsilon x_i})+f(x,0,u_{0\varepsilon})\right| \leq c,$$

which implies

$$\iint_{Q_T}|u_{\varepsilon t}|^s dxdt \leqslant c,\ \forall s\geq 1. \tag{2.23}$$

Thus there are a function $u\in L^\infty(Q)$ and a subsequence of $\{u_\varepsilon\}$ (we conserve for this subsequence the same notation u_ε) such that

$$\begin{aligned}&u_{\varepsilon t}\rightharpoonup u_t,\ \text{weakly star in } L^\infty(Q_T),\\ &u_\varepsilon\to u,\ \text{in } L^s_{loc}(Q_T),\\ &u_\varepsilon\to u,\ \text{a.e. in } Q_T,\\ &u_{\varepsilon t}\rightharpoonup u_t,\ \text{weakly in } L^2(Q_T),\end{aligned}$$

$$(a_i+\varepsilon)(|u_\varepsilon|+\varepsilon)^{p_i\gamma_i}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}\rightharpoonup *\ \xi_i,\ \text{weakly star in } L^\infty\left(0,\infty;L^{\frac{p_i}{p_i-1}}(\Omega)\right),$$

where $\xi=\{\xi_i:1\leq i\leq N\}$ and every ξ_i is a function in $L^\infty\left(0,\infty;L^{\frac{p_i}{p_i-1}}(\Omega)\right)$, $s=2$ when $p_-\geq 2$, $1<s<\frac{Np_-}{N-p^+}$ when $1<p^+<2$. In order to prove the theorem, we only need to prove that

$$\xi_i=a_i(x)|u|^{p_i\gamma_i}|u_{x_i}|^{p_i-2}u_{x_i}\ \text{in } L^\infty\left(0,\infty;L^{\frac{p_i}{p_i-1}}(\Omega)\right). \tag{2.24}$$

For any $\phi(x)\in C_0^1(\Omega)$, we choose $(u_\varepsilon-u)\phi$ as the test function.

$$\begin{aligned}&\int_\Omega(u_\varepsilon-u)\phi\frac{\partial u_\varepsilon}{\partial t}dx\\ &+\sum_{i=1}^N\int_\Omega\phi(x)(a_i(x)+\varepsilon)(|u_{\varepsilon x_i}|+\varepsilon)^{\gamma_i(p_i-1)}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}(u_\varepsilon-u)_{x_i}dx\\ &+\sum_{i=1}^N\int_\Omega(a_i(x)+\varepsilon)(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}(\phi(x))_{x_i}(u_\varepsilon-u)dx\\ =&\int_\Omega f(x,t,u_\varepsilon)\phi(u_\varepsilon-u)dx.\end{aligned} \tag{2.25}$$

Since $|\frac{\partial u_\varepsilon}{\partial t}|\leq c$ holds, letting $\varepsilon\to 0$ in this equality, we have

(2.26)

$$\lim_{\varepsilon\to 0}\int_\Omega\phi(x)(a_i(x)+\varepsilon)(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}(u_\varepsilon-u)_{x_i}dx=0,$$

which yields that

$$\begin{aligned}
&\lim_{\varepsilon\to 0}\int_\Omega \phi(x)(a_i(x)+\varepsilon)(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}u_{x_i}dx\\
=&\lim_{\varepsilon\to 0}\int_\Omega \phi(x)(a_i(x)+\varepsilon)(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}u_{\varepsilon x_i}dx\\
\leq& c+\lim_{\varepsilon\to 0}\int_\Omega \phi(x)(a_i(x)+\varepsilon)(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)}|u_{\varepsilon x_i}|^{p_i}dx\\
\leq& c.
\end{aligned}$$

Then

$$\int_\Omega \phi(x)(a_i(x)+\varepsilon)(|u_\varepsilon|+\varepsilon)^{\gamma_i(p_i-1)}(|u_{\varepsilon x_i}|^2+\varepsilon)^{\frac{p_i-2}{2}}u_{\varepsilon x_i}u_{x_i}dx\leq c.$$

Since $|u_\varepsilon|^{\gamma_i(p_i-1)}|u_{\varepsilon x_i}|^{p_i-2}u_{\varepsilon x_i}\in L^r\left(0,T;L_{loc}^{\frac{p_i}{p_i-1}}(\Omega)\right)$ for any $r\geq 1$, by the arbitrary of ϕ, we know that

$$u_{x_i}\in L^{r'}\left(0,T;L^{p_i}_{loc}(\Omega)\right). \tag{2.27}$$

By this property, we are able to deduce (2.24), we omit the details here. □

3. The proof of Theorem 1.3

In this section, we will use the general characteristic method to prove Theorem 1.3. For small $\eta>0$, set

$$S_\eta(s)=\int_0^s h_\eta(\tau)d\tau,\ h_\eta(s)=\frac{2}{\eta}\left(1-\frac{|\,s\,|}{\eta}\right)_+.$$

Then $h_\eta(s)\in C(\mathbb{R})$ and satisfies

$$h_\eta(s)\geq 0,\ |\,sh_\eta(s)\,|\leq 1,\ |\,S_\eta(s)\,|\leq 1;\ \lim_{\eta\to 0}S_\eta(s)=\text{sgn}s,\ \lim_{\eta\to 0}sh_\eta(s)=0. \tag{3.1}$$

Let $\phi(x)\in C^1(\overline{\Omega})$ be a weak characteristic function of Ω, i.e.,

$$\phi(x)=0,\ x\in\partial\Omega,\ \phi(x)>0,\ x\in\Omega, \tag{3.2}$$

and define

$$\phi_\lambda(x)=\begin{cases}0, & \phi(x)\leq\lambda,\\ \frac{\phi(x)-\lambda}{\lambda}, & \lambda<\phi(x)\leq 2\lambda,\\ 1, & \phi(x)\geq 2\lambda,\end{cases} \tag{3.3}$$

for small $\lambda>0$.

Lemma 3.1. *Let $a_i(x)\geq 0$, and $u(x,t)$ and $v(x,t)$ be two solutions of the equation* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively. If $\alpha^-\geq 1$, and there exists a weak characteristic function of Ω, $\phi(x)$, satisfying*

$$\sum_{i=1}^N\int_{\Omega_\lambda\setminus\Omega_{2\lambda}}a_i(x)\left|\frac{\phi_{x_i}}{\phi(x)-\lambda}\right|^{p_i}dx\leq c, \tag{3.4}$$

then

$$\int_\Omega |u(x,t)-v(x,t)|dx \le c\int_\Omega |u_0(x)-v_0(x)|dx, \tag{3.5}$$

where $\Omega_\lambda = \{x\in\Omega : \phi(x)>\lambda\}$.

Proof. Choosing $S_\eta(\phi_\lambda(u-v))$ as the test function, then we have

$$\begin{aligned}
&\int_\Omega S_\eta(\phi_\lambda(u-v))\frac{\partial(u-v)}{\partial t}dx \\
&+\sum_{i=1}^N\int_\Omega a_i(x)\phi_\lambda(x)|u|^{\alpha_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)(u-v)_{x_i}S'_\eta(\phi_\lambda(u-v))dx \\
&+\sum_{i=1}^N\int_\Omega a_i(x)\phi_\lambda(x)\left(|u|^{\alpha_i}-|v|^{\alpha_i}\right)|v_{x_i}|^{p_i-2}v_{x_i}(u-v)_{x_i}S'_\eta(\phi_\lambda(u-v))dx \\
&+\sum_{i=1}^N\int_\Omega a_i(x)|u|^{\alpha_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)\phi_{\lambda x_i}(u-v)S'_\eta(\phi_\lambda(u-v))dx \\
&+\sum_{i=1}^N\int_\Omega a_i(x)\left(|u|^{\alpha_i}-|v|^{\alpha_i}\right)|v_{x_i}|^{p_i-2}v_{x_i}\phi_{\lambda x_i}(u-v)S'_\eta(\phi_\lambda(u-v))dx \\
=&\int_\Omega[f(x,t,u)-f(x,t,v)]S_\eta(\phi_\lambda(u-v))dx.
\end{aligned} \tag{3.6}$$

Since the weak characteristic function of Ω, $\phi(x)$ satisfies (3.2), we have

$$\lim_{\lambda\to0}\lim_{\eta\to0}\int_\Omega S_\eta(\phi_\lambda(u-v))\frac{\partial(u-v)}{\partial t}dx = \frac{d}{dt}\int_\Omega |u(x,t)-v(x,t)|dx. \tag{3.7}$$

The monotone characteristic of operator $|u_{x_i}|^{p_i-2}u_{x_i}$ yields

$$\int_\Omega \phi_\lambda(x)a_i(x)|u|^{\alpha_i}\left(|u_{x_i}|^{p_i-2}u_{x_i}-|v_{x_i}|^{p_i-2}v_{x_i}\right)(u-v)_{x_i}S'_\eta(\phi_\lambda(u-v))dx \geqslant 0. \tag{3.8}$$

By $\left|\phi_\lambda(u-v)S'_\eta(\phi_\lambda(u-v))\right| \le c$, $|v_{x_i}|^{p_i}\in L^1_{loc}(Q_T)$, $\alpha^-\ge 1$, using the fact $\lim_{\eta\to0}S'_\eta(s)s=0$, by the Lebesgue dominated convergence theorem, we have

$$\begin{aligned}
&\lim_{\eta\to0}\int_\Omega \phi_\lambda(x)a_i(x)\left||u|^{\alpha_i}-|v|^{\alpha_i}\right||v_{x_i}|^{p_i}S'_\eta(\phi_\lambda(u-v))dx \\
=&\lim_{\eta\to0}\int_{\Omega\setminus\Omega_\lambda} a_i(x)\phi_\lambda(x)\alpha_i|\xi|^{\alpha_i-1}|u-v|\,|v_{x_i}|^{p_i}S'_\eta(\phi_\lambda(u-v))dx \\
=&\,0.
\end{aligned} \tag{3.9}$$

Similarly, we have

$$\lim_{\eta\to0}\int_\Omega \phi_\lambda(x)a_i(x)\left||u|^{\alpha_i}-|v|^{\alpha_i}\right||u_{x_i}|^{p_i}S'_\eta(\phi_\lambda(u-v))dx = 0. \tag{3.10}$$

By (3.9)-(3.10), we have

$$\lim_{\eta\to0}\left|\int_\Omega \phi_\lambda(x)a_i(x)(|u|^{\alpha_i}-|v|^{\alpha_i})|v_{x_i}|^{p_i-2}v_{x_i}(u-v)_{x_i}S'_\eta(\phi_\lambda(u-v))dx\right| \tag{3.11}$$

$$
\begin{aligned}
&\leq \lim_{\eta\to 0}\int_\Omega \phi_\lambda(x)a_i(x)||u|^{\alpha_i}-|v|^{\alpha_i}||\,|v_{x_i}|^{p_i-1}\,(|u_{x_i}|+|v_{x_i}|)S'_\eta(\phi_\lambda(u-v))dx\\
&\leq c\lim_{\eta\to 0}\int_\Omega \phi_\lambda(x)a_i(x)||u|^{\alpha_i}-|v|^{\alpha_i}|\,|v_{x_i}|^{p_i-1}\,|u_{x_i}|S'_\eta(\phi_\lambda(u-v))dx\\
&\quad+\lim_{\eta\to 0}\int_\Omega \phi_\lambda(x)a_i(x)||u|^{\alpha_i}-|v|^{\alpha_i}||\,|v_{x_i}|^{p_i}\,S'_\eta(\phi_\lambda(u-v))dx\\
&\leq c\lim_{\eta\to 0}\left(\int_\Omega \phi_\lambda(x)a_i(x)||u|^{\alpha_i}-|v|^{\alpha_i}||\,|v_{x_i}|^{p_i}\,S'_\eta(\phi_\lambda(u-v))dx\right)^{\frac{p_i-1}{p_i}}\\
&\quad\cdot\left(\int_\Omega \phi_\lambda(x)a_i(x)||u|^{\alpha_i}-|v|^{\alpha_i}|\,|u_{x_i}|^{p_i}\,S'_\eta(\phi_\lambda(u-v))dx\right)^{\frac{1}{p_i}}\\
&\quad+\lim_{\eta\to 0}\int_\Omega \phi_\lambda(x)a_i(x)|\,|v_{x_i}|^{p_i}\,||u|^{\alpha_i}-|v|^{\alpha_i}||S'_\eta(\phi_\lambda(u-v))dx\\
&=0.
\end{aligned}
$$

Moreover, by the assumption (3.4), we have,

$$
\begin{aligned}
(3.12)\quad &\lim_{\eta\to 0}\left|\int_\Omega a_i(x)|u|^{\alpha_i}(|u_{x_i}|^{p_i-2}\,u_{x_i}-|v_{x_i}|^{p_i-2}\,v_{x_i})\phi_{\lambda x_i}(u-v)S'_\eta(\phi_\lambda(u-v))dx\right|\\
&=\lim_{\eta\to 0}\left|\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)|u|^{\alpha_i}(|u_{x_i}|^{p_i-2}\,u_{x_i}-|v_{x_i}|^{p_i-2}\,v_{x_i})\phi_{\lambda x_i}(u-v)S'_\eta(\phi_\lambda(u-v))dx\right|\\
&\leq c\lim_{\eta\to 0}\left(\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)|u|^{\alpha_i}\,(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i})\,\phi_\lambda(u-v)S'_\eta(\phi_\lambda(u-v))dx\right)^{\frac{p_i-1}{p_i}}\\
&\quad\cdot\left(\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)|u|^{\alpha_i}\left|\frac{\phi_{\lambda x_i}}{\phi_\lambda}\right|^{p_i}\phi_\lambda(u-v)S'_\eta(\phi_\lambda(u-v))dx\right)^{\frac{1}{p_i}}\\
&\leq c\lim_{\eta\to 0}\left(\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)|u|^{\alpha_i}\,(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i})\,\phi_\lambda(u-v)S'_\eta(\phi_\lambda(u-v))dx\right)^{\frac{p_i-1}{p_i}}\\
&\quad\cdot\left(\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)|u|^{\alpha_i}\left|\frac{\phi_{\lambda x_i}}{\phi_\lambda}\right|^{p_i}\phi_\lambda(u-v)S'_\eta(\phi_\lambda(u-v))\right)^{\frac{1}{p_i}}\\
&\leq c\lim_{\eta\to 0}\left(\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)|u|^{\alpha_i}\,(|u_{x_i}|^{p_i}+|v_{x_i}|^{p_i})\,\phi_\lambda(u-v)S'_\eta(\phi_\lambda(u-v))dx\right)^{\frac{p_i-1}{p_i}}\\
&\quad\cdot\left(\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)|u|^{\alpha_i}\left|\frac{\phi_{\lambda x_i}}{\phi_\lambda}\right|^{p_i}dx\right)^{\frac{1}{p_i}}\\
&=0.
\end{aligned}
$$

Similarly, we can show that

$$
(3.13)\quad \lim_{\eta\to 0}\left|\int_\Omega a_i(x)\,(|u|^{\alpha_i}-|v|^{\alpha_i})\,|v_{x_i}|^{p_i-2}\,v_{x_i}\phi_{\lambda x_i}(u-v)S'_\eta(\phi_\lambda(u-v))dx\right|
$$

$$= \lim_{\eta\to 0}\left|\int_{\Omega_\lambda\setminus\Omega_{2\lambda}} a_i(x)\left(|u|^{\alpha_i}-|v|^{\alpha_i}\right)|v_{x_i}|^{p_i-2}v_{x_i}\frac{\phi_{\lambda x_i}}{\phi_\lambda}\phi_\lambda(u-v)S'_\eta(\phi_\lambda(u-v))dx\right|$$
$$= 0.$$

Once more, since f is a Lipschitz function, $u \in L^\infty(Q_T)$,

$$\begin{aligned}(3.14)\qquad &\lim_{\lambda\to 0}\lim_{\eta\to 0}\left|\int_\Omega [f(x,t,u)-f(x,t,v)]S_\eta(\phi_\lambda(u-v))dx\right|\\ &\le c\int_\Omega |u(x,t)-v(x,t)|dx\\ &= \|u(x,t)-v(x,t)\|_1.\end{aligned}$$

After letting $\eta \to 0$ in (3.6), let $\lambda \to 0$. Then

$$\frac{d}{dt}\|u(x,t)-v(x,t)\|_1 \le c\|u(x,t)-v(x,t)\|_1.$$

By the Gronwall's inequality, we have

$$\int_\Omega |u(x,t)-v(x,t)|\,dx \leqslant \int_\Omega |u_0(x)-v_0(x)|\,dx. \qquad \square$$

Since condition $\alpha^- \ge 1$ is only used in the proof of (3.12), if

$$u(x,t)>0,\ v(x,t)>0,\quad x\in\Omega, \tag{3.15}$$

then, we find that only if $\alpha_i > 0$, (3.12) is still true. Thus we have the following conclusion.

Lemma 3.2. *Let $a_i(x) \in C(\overline{\Omega})$, and $u(x,t)$ and $v(x,t)$ be two solutions of the equation* (1.1) *with the initial values $u_0(x)$ and $v_0(x)$ respectively, $u(x,t) > 0$ and $v(x,t) > 0$ when $x \in \Omega$. If there exists a weak characteristic function of Ω, $\phi(x)$, satisfying* (3.4)*, then the stability conclusion* (3.5) *is still true.*

Since the weak characteristic function $\phi(x)$ satisfies (3.4), only if we choose $\phi(x) = d(x) = \mathrm{dist}(x, \partial\Omega)$, since $p^+ < 2$ and $a_i(x) \le cd(x)$, then the condition (3.4) is naturally true. Thus, Theorem 1.3 is obviously deduced from Lemma 3.1 and Lemma 3.2.

4. The stability of the equation (1.3)

In this section, we will prove the following theorem.

Theorem 4.1. *Let $b_i(x) \ge 0$, $u(x,t)$ and $v(x,t)$ be two solutions of the equation* (1.3) *with the initial values $u_0(x)$ and $v_0(x)$ respectively. If $p_i > 1, m > 0$, $f(s,t,v)$ is a Lipschitz function when $|v| < c$, and*

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)^{1-p_i}\left|\prod_{k=1}^N b_k(x)b_{ix_i}\right|^{p_i}dx\right)^{\frac{1}{p_i}} \le c, \tag{4.1}$$

then

$$\int_\Omega |u(x,t)-v(x,t)|dx \le c\int_\Omega |u_0(x)-v_0(x)|dx.$$

For a small positive constant $\lambda>0$, $\phi(x)$ is a general characteristic function, let

$$\phi_\lambda(x)=\begin{cases} 1, & \text{if } x\in\Omega_\lambda,\\ \frac{1}{\lambda}\phi(x), & \text{if } x\in\Omega\setminus\Omega_\lambda.\end{cases}$$

Here, $\Omega_\lambda=\{x\in\Omega:\phi(x)>\lambda\}$ as before.

Lemma 4.2. *Let $b_i(x)\in C(\overline{\Omega})$, and $u(x,t)$ and $v(x,t)$ be two solutions of the equation (1.3) with the initial values $u_0(x)$ and $v_0(x)$ respectively. If $p_i>1, m>0$, $f(x,t,s)$ is a Lipschitz function when $|s|<c$, and there exists a weak characteristic function of Ω, $\phi(x)$, satisfying*

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)|\phi_{x_i}|^{p_i}dx\right)^{\frac{1}{p_i}} \le c,\ \ i=1,2,\dots,N, \tag{4.2}$$

then

$$\int_\Omega |u(x,t)-v(x,t)|dx \le c\int_\Omega |u_0(x)-v_0(x)|dx.$$

Proof. We choose $\chi_{[\tau,x]}\phi_\lambda(x)S_\eta(u^m-v^m)$ as the test function, where $\chi_{[\tau,x]}$ is the characteristic function of $[\tau,s]\subset(0,T)$. Then

$$\begin{aligned}
&\int_\tau^s\int_\Omega \phi_\lambda(x)S_\eta(u^m-v^m)\frac{\partial(u-v)}{\partial t}dxdt\\
&+\sum_{i=1}^N\int_\tau^s\int_\Omega \phi_\lambda(x)b_i(x)\left(\left|u^m_{x_i}\right|^{p_i-2}u^m_{x_i}-\left|v^m_{x_i}\right|^{p_i-2}v^m_{x_i}\right)(u^m-v^m)_{x_i}S'_\eta(u^m-v^m)dxdt\\
&+\sum_{i=1}^N\int_\tau^s\int_\Omega b_i(x)\left(\left|u^m_{x_i}\right|^{p_i-2}u^m_{x_i}-\left|v^m_{x_i}\right|^{p_i-2}v^m_{x_i}\right)\phi_{\lambda x_i}(x)S_\eta(u^m-v^m)dxdt\\
&-\int_\tau^s\int_\Omega [f(x,t,u)-f(x,t,v)]\phi_\lambda(x)S_\eta(u^m-v^m)dxdt\\
&=0.
\end{aligned}\tag{4.3}$$

At first, we still have

$$\int_\tau^s\int_\Omega \phi_\lambda(x)b_i(x)\left(\left|u^m_{x_i}\right|^{p_i-2}u^m_{x_i}-\left|v^m_{x_i}\right|^{p_i-2}v^m_{x_i}\right)(u^m-v^m)_{x_i}S'_\eta(u^m-v^m)dxdt\ge 0. \tag{4.4}$$

Moreover, we have

$$\begin{aligned}
&\left|\int_\Omega b_i(x)\left(\left|u^m_{x_i}\right|^{p_i-2}u^m_{x_i}-\left|v^m_{x_i}\right|^{p_i-2}v^m_{x_i}\right)\phi_{\lambda x_i}(x)S_\eta(u^m-v^m)dx\right|\\
\le&\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|\left(\left|u^m_{x_i}\right|^{p_i-2}u^m_{x_i}-\left|v^m_{x_i}\right|^{p_i-2}v^m_{x_i}\right)\phi_{\lambda x_i}(x)S_\eta(u^m-v^m)\right|dx\\
\le&\, c\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|\left(\left|u^m_{x_i}\right|^{p_i-2}u^m_{x_i}-\left|v^m_{x_i}\right|^{p_i-2}v^m_{x_i}\right)\phi_{\lambda x_i}(x)\right|dx
\end{aligned}\tag{4.5}$$

$$\leq \frac{c}{\lambda}\left[\int_{\Omega\setminus\Omega_\lambda} b_i(x)|u_{x_i}^m|^{p_i-1}|\phi_{x_i}|dx + \int_{\Omega\setminus\Omega_\lambda} b_i(x)|v_{x_i}^m|^{p_i-1}|\phi_{x_i}|dx\right].$$

Using the Hölder inequality, by the condition (4.2), we have

$$\left|\int_\Omega b_i(x)\left(\left|u_{x_i}^m\right|^{p_i-2}u_{x_i}^m - \left|v_{x_i}^m\right|^{p_i-2}v_{x_i}^m\right)\phi_{\lambda x_i}S_\eta(u^m-v^m)dx\right| \tag{4.6}$$

$$\leq \frac{c}{\lambda}\left[\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|u_{x_i}^m\right|^{p_i-1}|\phi_{x_i}|dx + \int_\tau^s\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|v_{x_i}^m\right|^{p_i-1}|\phi_{x_i}|dx\right]$$

$$\leq \frac{c}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)|\phi_{x_i}|^{p_i}dx\right)^{\frac{1}{p_i}}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|u_{x_i}^m\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}$$

$$+\frac{c}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)|\phi_{x_i}|^{p_i}dx\right)^{\frac{1}{p_i}}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|v_{x_i}^m\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}$$

$$\leq c\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|u_{x_i}^m\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}} + c\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)\left|v_{x_i}^m\right|^{p_i}dx\right)^{\frac{p_i-1}{p_i}}.$$

Since

$$\int_\Omega b_i(x)\left|u_{x_i}^m\right|^{p_i}dx \leq c, \int_\Omega b_i(x)\left|v_{x_i}^m\right|^{p_i}dx \leq c,$$

(4.6) yields

$$\lim_{\lambda\to 0}\int_\tau^s\left|\int_\Omega b_i(x)\left(\left|u_{x_i}^m\right|^{p_i-2}u_{x_i}^m - \left|v_{x_i}^m\right|^{p_i-2}v_{x_i}^m\right)\phi_{\lambda x_i}S_\eta(u^m-v^m)dxdt\right| = 0. \tag{4.7}$$

Once more, since $m > 0$,

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\left|\int_\tau^s\int_\Omega [f(x,t,u)-f(x,t,v]\phi_\lambda S_\eta(u^m-v^m)dxdt\right| \leq c\int_\tau^s \|u-v\|_{L^1(\Omega)}dt. \tag{4.8}$$

At last,

$$\lim_{\eta\to 0}\lim_{\lambda\to 0}\int_\tau^s\int_\Omega \phi_\lambda(x)S_\eta(u^m-v^m)\frac{\partial(u-v)}{\partial t}dxdt \tag{4.9}$$

$$= \lim_{\eta\to 0}\int_\tau^s\int_\Omega S_\eta(u^m-v^m)\frac{\partial(u-v)}{\partial t}dxdt$$

$$= \int_\tau^s\int_\Omega \text{sign}(u-v)\frac{\partial(u-v)}{\partial t}dxdt$$

$$= \int_\tau^s \frac{d}{dt}\|u-v\|_{L^1(\Omega)}dt.$$

Now, after letting $\lambda \to 0$ in (4.3), integrating over $[\tau, s]$ and letting $\eta \to 0$, using the Gronwall inequality, we have,

$$\int_\Omega |u(x,s)-v(x,s)|\,dx \leqslant c\int_\Omega |u(x,\tau)-v(x,\tau)|\,dx,$$

which implies the conclusion of the lemma. □

At last, we give the proof of Theorem 4.1.

Proof. If we choose $\phi(x) = \prod_{k=1}^{N} b_k(x)$, then (4.2) becomes

$$\begin{aligned} &\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)|\phi_{x_i}|^{p_i}dx\right)^{\frac{1}{p_i}} \\ &= \frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)|\prod_{k=1,k\neq i}^{N} b_k(x)b_{ix_i}|^{p_i}dx\right)^{\frac{1}{p_i}} \\ &= \frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_\lambda} b_i(x)^{1-p_i}|\prod_{k=1}^{N} b_k(x)b_{ix_i}|^{p_i}dx\right)^{\frac{1}{p_i}} \\ &\leq c. \end{aligned} \tag{4.10}$$

According to Lemma 4.2, we have the conclusion of Theorem 4.1. □

Conclusion

The equation considered in this paper is anisotropic, not only p_i is different from one to another, but also α_i is different from one to another. Such differences make the method used in proving the uniqueness of weak solution to the infiltration equation

$$u_t = \operatorname{div}(|u|^r|\nabla u|^{p-2}\nabla u) + f(x,t,u,\nabla u),$$

or the

$$u_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(a_i(x)|u_{x_i}|^{p_i-2}u_{x_i}\right) + f(x,t,u,\nabla u),$$

is invalid. In this paper, using some innovative method, by show that the gradient of weak solution is local integral, using the weak characteristic function method introduced in [31], we can prove the stability of positive weak solutions. As we have said in the introduction, it is hard to extrapolate $|u_{x_i}| \in L^{p_i}_{loc}(\Omega)$ from $|u|^{\alpha_i}|u_{x_i}|^{p_i} \in L^1_{loc}(\Omega)$. So the conclusions in this paper are interesting. Recently the well-posedness of weak solutions to parabolic equation with variable exponent

$$(|v|^{\beta-1}v)_t = \operatorname{div}\left(b(x,t)|\nabla v|^{p(x,t)-2}\nabla v\right) + \sum_{i=1}^{N} g_i(x,t)\frac{\partial \gamma_i(v)}{\partial x_i}.$$

has been studied by the author in [30]. It is well-known that the evolutionary $p(x)$-Laplacian equations is new and interesting topic in this century. Since it is with variable exponent, many mathematical difficulties arise, one can refer to [3], [27] and [35]. If variable exponent $p(x)$ satisfies the so-called logarithmic Hölder continuity condition, i.e.,

$$|p(x) - p(y) \leq \omega(|x - y|), \forall x, y \in \Omega, |x - y| < \frac{1}{2},$$

with

$$\overline{\lim_{s \to 0^+}}\, \omega(s) \ln(\frac{1}{s}) = C < \infty,$$

then we believe that the methods used in this paper can be generalized to study the well-posedness of weak solutions to the following equation

$$(|u|^{\beta-1}u)_t = \sum_{i=1}^{N} \frac{\partial}{\partial x_i} \left(a_i(x)|u|^{\alpha_i}|u_{x_i}|^{p_i(x)-2}u_{x_i} \right) + f(x, t, u, \nabla u),$$

in the future.

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