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LONG TIME BEHAVIOR OF SOLUTIONS TO SEMILINEAR HYPERBOLIC EQUATIONS INVOLVING STRONGLY DEGENERATE ELLIPTIC DIFFERENTIAL OPERATORS

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Abstract. The aim of this paper is to prove the existence of the global attractor of the Cauchy problem for a semilinear degenerate hyperbolic equation involving strongly degenerate elliptic differential operators. The attractor is characterized as the unstable manifold of the set of stationary points, due to the existence of a Lyapunov functional.

1. Introduction

The understanding of asymptotic behavior of dynamical systems is one of the most important problems of modern mathematical physics. One way to attack the problem for a dissipative dynamical system is to consider its global attractor. The existence of global attractors has been proved for various nonlinear dissipative parabolic and hyperbolic PDEs that contains elliptic operators (see, for example [2, 4, 5, 23, 24, 26] and the references therein).

One of the classes of degenerate elliptic equations that has been studied widely in recent years is the class of equations involving strongly degenerate elliptic differential operators (see [27])

$$P_{\alpha,\beta} := \Delta_x + \Delta_y + |x|^{2\alpha}|y|^{2\beta}\Delta_z, \quad \alpha, \beta \geq 0.$$

Note that $P_{0,0} = \Delta$ is the Laplacian operator and $P_{\alpha,\beta}$, when $\alpha, \beta > 0$, is not elliptic in domains intersecting the surface $x = 0$ and $y = 0$. Many aspects of the theory of degenerate elliptic differential operators are presented in monographs [30, 31] (see also some recent results in [11–21, 27–29]).

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In this paper we are interested in the global existence and the long-time behavior of solutions to the following problem

$$u_{tt} + \lambda u_t + \gamma(X)u = P_{\alpha,\beta}u + f(X,u),\ t > 0, \qquad X = (x,y,z) \in \mathbb{R}^{N_1} \times \mathbb{R}^{N_2} \times \mathbb{R}^{N_3} := \mathbb{R}^N, \tag{1}$$

$$u(X,0) = u_0(X), u_t(X,0) = u_1(X), \tag{2}$$

where λ is a positive constant, $u_0(X) \in \mathbb{S}^2_1(\mathbb{R}^N), u_1(X) \in L^2(\mathbb{R}^N)$ and

$$\Delta_x := \sum_{i=1}^{N_1} \frac{\partial^2}{\partial x_i^2},\ \Delta_y := \sum_{j=1}^{N_2} \frac{\partial^2}{\partial y_j^2},\ \Delta_z := \sum_{l=1}^{N_3} \frac{\partial^2}{\partial z_l^2},\ u_t := \frac{\partial u}{\partial t}, u_{tt} := \frac{\partial^2 u}{\partial t^2},$$

$$|x|^{2\alpha} := \left(\sum_{i=1}^{N_1} x_i^2\right)^{\alpha}, |y|^{2\beta} := \left(\sum_{j=1}^{N_2} y_j^2\right)^{\beta}, N_{\alpha,\beta} := N_1+N_2+(1+\alpha+\beta)N_3 > 2.$$

We assume that $\gamma : \mathbb{R}^N \longrightarrow \mathbb{R}$ and $f(X,\xi) : \mathbb{R}^N \times \mathbb{R} \longrightarrow \mathbb{R}$ satisfy the following conditions:

i) $\gamma : \mathbb{R}^N \longrightarrow \mathbb{R}$ is a measurable function with the following properties:

(i_1) For every $\theta \in (0,\infty)$ there is a $C_\theta \in (0,\infty)$ such that for all $u \in \mathbb{S}^2_1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |\gamma(X)||u(X)|^2 \mathrm{d}X \le \theta \left\|u\right\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + C_\theta \left\|u\right\|^2_{L^2(\mathbb{R}^N)}. \tag{3}$$

(i_2) There is a $\lambda_0 > 0$ such that for all $u \in \mathbb{S}^2_1(\mathbb{R}^N)$,

$$\left\|\nabla_{\alpha,\beta} u\right\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \gamma(X)|u(X)|^2 \mathrm{d}X \ge \lambda_0 \left\|u\right\|^2_{L^2(\mathbb{R}^N)}. \tag{4}$$

ii) $f : \mathbb{R}^N \times \mathbb{R} \to \mathbb{R},\ (X,\xi) \mapsto f(X,\xi)$ satisfies Carathéodory condition, i.e., for every $\xi \in \mathbb{R}$ the map $X \mapsto f(X,\xi)$ is Lebesgue measurable and for a.e., $X \in \mathbb{R}^N$ the map $\xi \mapsto f(X,\xi)$ is continuous. The canonical primitive of f is defined by

$$F(X,\xi) = \int_0^\xi f(X,\tau)\mathrm{d}\tau,$$

and f satisfies the following properties:

(f_1) $f(X,0) = h(X) \in L^2(\mathbb{R}^N)$;

(f_2) For all $X \in \mathbb{R}^N$ and $\xi_1, \xi_2 \in \mathbb{R}$ such that

$$\begin{aligned} &|f(X,\xi_1) - f(X,\xi_2)| \\ &\le C_1|\xi_1 - \xi_2| \left(g(X) + |\xi_1|^\rho + |\xi_2|^\rho\right) \text{ with } 0 < \rho \le \frac{2}{N_{\alpha,\beta} - 2}, \end{aligned} \tag{5}$$

and $g : \mathbb{R}^N \to \mathbb{R}$ is a measurable function such that for all $u \in \mathbb{S}^2_1(\mathbb{R}^N)$,

$$\int_{\mathbb{R}^N} |g(X)|^2|u(X)|^2 \mathrm{d}X \le C_2 \left\|u\right\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)}, \tag{6}$$

where C_1, C_2 are positive constants.

(f_3) There are measurable functions $g_1, g_2 : \mathbb{R}^N \to \mathbb{R}, g_1, g_2 \in L^1(\mathbb{R}^N)$ such that

$$f(X,\xi)\xi \le g_1(X) \text{ for a.e., } X \in \mathbb{R}^N, \xi \in \mathbb{R}, \tag{7}$$

$$F(X,\xi) \le g_2(X) \text{ for a.e., } X \in \mathbb{R}^N, \xi \in \mathbb{R}. \tag{8}$$

The major techniques that seem to be valid to get a global attractor in the natural energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ are: working with a weighted Sobolev space as phase space and using the method of "tail estimates" see in [3,9,10,32].

We would like to mention the results for the case $\alpha = \beta = 0, \gamma(X) = 1$. The existence of a global attractor in $H^1(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$ for the problem (1)-(2) was proved by Feireisl [8] for $\xi - f(X,\xi) = \mathcal{F}(X,\xi)$ satisfying for $N_0 := N = 3$ the growth condition

$$\mathcal{F} \in C^2(\mathbb{R}^4), \quad \mathcal{F}(\cdot,0) \in H^1(\mathbb{R}^3), \quad \left|\frac{\partial \mathcal{F}}{\partial \xi}(X,0)\right| \le C \quad \text{for all } X \in \mathbb{R}^3,$$
$$\left|\frac{\partial^2 \mathcal{F}}{\partial \xi^2}(X,\xi)\right| \le C\Big(1 + |\xi|\Big) \quad \text{for all } X \in \mathbb{R}^3, \xi \in \mathbb{R},$$
$$\liminf_{|\xi|\to\infty} \frac{\mathcal{F}(X,\xi)}{\xi} \ge 0 \quad \text{uniformly in} \quad X \in \mathbb{R}^3$$

and

$$\Big(\mathcal{F}(X,\xi) - \mathcal{F}(X,0)\Big)\xi \ge C\xi^2 \quad \text{for all } \xi \in \mathbb{R}, |X| > r_1, C > 0.$$

Fall [7] used the method of "tail estimates" and showed the existence of a global attractor in the natural energy space $H^1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ for the problem (1)-(2) (when $\alpha = \beta = 0, \gamma(X) = 1$) under strictly restrained conditions

$$\xi - f(X,\xi) = \xi + g_1(\xi) - g_2(X),\ g_2(X) \in L^2(\mathbb{R}^N),$$
$$g_1 \in C^1(\mathbb{R},\mathbb{R}),\ g_1(0) = 0,\ g_1(\xi)\xi \ge CG_1(\xi) \ge 0,\ \forall \xi \in \mathbb{R},$$
$$0 \le \limsup_{|\xi|\to+\infty} \frac{g_1(\xi)}{\xi} < \infty,$$

where C is a positive constant and $G_1(\xi) = \int_0^\xi g_1(\tau)\mathrm{d}\tau$.

Very recently, in [18] the authors studied the existence of the global attractor of the Cauchy problem for a semilinear degenerate damped hyperbolic equation involving the Grushin operator with a locally Lipschitz nonlinearity satisfying a subcritical growth condition.

In the present paper, by using the analytical techniques of [10, 33] and the method of "tail estimates", we prove that there also exist global attractors of the problem (1)-(2) in the natural energy space $\mathbb{S}_1^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$ under the conditions i) and ii).

The structure of our note is as follows: In Section 2 we give some preliminary results on the existence of global mild solutions. In Section 3 we establish the existence of the global attractor for the problem (1)-(2). In the last section, we prove the existence of a global attractor which is characterized as the unstable manifold of the set of stationary points.

2. Existence and uniqueness of a global mild solution

2.1. Function spaces and operators

We use the space $\mathbb{S}_1^2(\mathbb{R}^N)$ defined as the completion of $C_0^\infty(\mathbb{R}^N)$ in the norm

$$\|u\|_{\mathbb{S}_1^2(\mathbb{R}^N)} = \left\{ \int_{\mathbb{R}^N} \left(|u|^2 + |\nabla_{\alpha,\beta} u|^2 \right) \mathrm{d}X \right\}^{\frac{1}{2}},$$

where $\nabla_{\alpha,\beta} u = \left(\frac{\partial u}{\partial x_1}, \ldots, \frac{\partial u}{\partial x_{N_1}}, \frac{\partial u}{\partial y_1}, \ldots, \frac{\partial u}{\partial y_{N_2}}, |x|^\alpha |y|^\beta \frac{\partial u}{\partial z_1}, \ldots, |x|^\alpha |y|^\beta \frac{\partial u}{\partial z_{N_3}} \right)$, and

$$|\nabla_{\alpha,\beta} u| := \left(\sum_{i=1}^{N_1} \left| \frac{\partial u}{\partial x_i} \right|^2 + \sum_{j=1}^{N_2} \left| \frac{\partial u}{\partial y_j} \right|^2 + |x|^{2\alpha} |y|^{2\beta} \sum_{l=1}^{N_3} \left| \frac{\partial u}{\partial z_l} \right|^2 \right)^{\frac{1}{2}}.$$

Then $\mathbb{S}_1^2(\mathbb{R}^N)$ is a Hilbert space with the inner product

$$(u,v)_{\mathbb{S}_1^2(\mathbb{R}^N)} = (u,v)_{L^2(\mathbb{R}^N)} + (\nabla_{\alpha,\beta} u, \nabla_{\alpha,\beta} v)_{L^2(\mathbb{R}^N)}.$$

The following embedding inequality was proved in [1]

$$\left(\int_{\mathbb{R}^N} |u|^p \mathrm{d}X \right)^{\frac{1}{p}} \leq C(p) \, \|u\|_{\mathbb{S}_1^2(\mathbb{R}^N)},$$

where $2 \leq p \leq 2^*_{\alpha,\beta} = \frac{2N_{\alpha,\beta}}{N_{\alpha,\beta}-2}$, $C(p) > 0$.

We denote by $L^p_\nu(\mathbb{R}^N)$ the set of all measurable functions $u : \mathbb{R}^N \to \mathbb{R}$ such that

$$\|u\|_{L^p_\nu(\mathbb{R}^N)} := \sup_{Y \in \mathbb{R}^N} \left(\int_{B(Y)} |u(X)|^p \mathrm{d}X \right)^{\frac{1}{p}} < \infty,$$

where $Y \in \mathbb{R}^N$ and $B(Y) = \{X \in \mathbb{R}^N : Y < X < Y + 1_e, 1_e = (1, 1, \ldots, 1)\}$.

The following lemma contains a condition ensuring that γ satisfies the condition (i_1).

Lemma 2.1. *Let $p > 1$ and $\phi : \mathbb{R}^N \to \mathbb{R}$ be a measurable function such that $\phi \in L^p_\nu(\mathbb{R}^N)$.*

(i) *If $p \geq \frac{N_{\alpha,\beta}}{2}$, then there is a $C \in (0,\infty)$ such that*

$$\int_{\mathbb{R}^N} |\phi(X)||u(X)|^2 \mathrm{d}X \leq C \, \|u\|^2_{\mathbb{S}_1^2(\mathbb{R}^N)} \text{ for all } u \in \mathbb{S}_1^2(\mathbb{R}^N).$$

(ii) *If $p > \frac{N_{\alpha,\beta}}{2}$, then for every $\theta \in (0,\infty)$ there is a constant $C_\theta \in (0,\infty)$ such that*

$$\int_{\mathbb{R}^N} |\phi(X)||u(X)|^2 \mathrm{d}X \leq \theta \, \|u\|^2_{\mathbb{S}_1^2(\mathbb{R}^N)} + C_\theta \, \|u\|^2_{L^2(\mathbb{R}^N)} \text{ for all } u \in \mathbb{S}_1^2(\mathbb{R}^N).$$

Proof. There is a family $(Y_j)_{j\in\mathbb{N}}$ of points in $\mathbb{R}^N$ such that $\mathbb{R}^N = \bigcup_{j\in\mathbb{N}} \overline{B(Y_j)}$ and the sets $\overline{B(Y_j)}$, $j \in \mathbb{N}$, are pairwise non-overlapping. Write $B_j = B(Y_j)$, $j \in \mathbb{N}$. Let $p' = p/(p-1)$. Since $p \geq N_{\alpha,\beta}/2$ we have $2p' \leq 2^*_{\alpha,\beta}$. Let $C(2p')$ be the best constant of the imbedding $\mathbb{S}^2_1(B) \hookrightarrow L^{2p'}(B)$ where $B = B_1(0)$. Then, by translation, $C(2p')$ is the best constant of the imbedding $\mathbb{S}^2_1(B_j(Y)) \hookrightarrow L^{2p'}(B_j(Y))$ for any $Y \in \mathbb{R}^N$. Let $u \in \mathbb{S}^2_1(\mathbb{R}^N)$ be arbitrary. Then

$$\begin{aligned}
\int_{\mathbb{R}^N} |\phi(X)||u(X)|^2 \mathrm{d}X &= \sum_{j\in\mathbb{N}} \int_{B_j} |\phi(X)||u(X)|^2 \mathrm{d}X \\
&\leq \sum_{j\in\mathbb{N}} \left(\int_{B_j} |\phi(X)|^p \mathrm{d}X\right)^{\frac{1}{p}} \left(\int_{B_j} |u(X)|^{2p'} \mathrm{d}X\right)^{\frac{1}{p'}} \\
&\leq \|\phi\|_{L^p_\nu(\mathbb{R}^N)} \sum_{j\in\mathbb{N}} \left(\int_{B_j} |u(X)|^{2p'} \mathrm{d}X\right)^{\frac{1}{p'}} \\
&\leq \|\phi\|_{L^p_\nu(\mathbb{R}^N)} C^2(2p') \sum_{j\in\mathbb{N}} \|u\|^2_{\mathbb{S}^2_1(B_j)} \\
&= \|\phi\|_{L^p_\nu(\mathbb{R}^N)} C^2(2p') \|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} .
\end{aligned}$$

Hence (i) holds. If $p > N_{\alpha,\beta}/2$ we may choose q such that $2p' < q$ and $q < 2^*_{\alpha,\beta}$. We may then interpolate between 2 and q and so, for every $\theta \in (0,\infty)$ there is a constant $C_\theta \in (0,\infty)$ such that for all $j \in \mathbb{N}, u \in \mathbb{S}^2_1(\mathbb{R}^N)$

$$\begin{aligned}
\Big(\int_{B_j} |u(X)|^{2p'} \mathrm{d}X\Big)^{\frac{1}{2p'}} &\leq \theta\Big(\int_{B_j} |u(X)|^q \mathrm{d}X\Big)^{\frac{1}{q}} + C_\theta\Big(\int_{B_j} |u(X)|^2 \mathrm{d}X\Big)^{\frac{1}{2}} \\
&\leq \theta C(q) \|u\|_{\mathbb{S}^2_1(B_j)} + C_\theta \|u\|_{L^2(B_j)} .
\end{aligned}$$

Hence

$$\Big(\int_{B_j} |u(X)|^{2p'} \mathrm{d}X\Big)^{\frac{1}{p'}} \leq 2\theta^2 C^2(q) \|u\|^2_{\mathbb{S}^2_1(B_j)} + 2C^2_\theta \|u\|^2_{L^2(B_j)} .$$

Thus, by the above computation,

$$\begin{aligned}
\int_{\mathbb{R}^N} |\phi(X)||u(X)|^2 \mathrm{d}X &\leq \|\phi\|_{L^p_\nu(\mathbb{R}^N)} \sum_{j\in\mathbb{N}} \left(\int_{B_j} |u(X)|^{2p'} \mathrm{d}X\right)^{\frac{1}{p'}} \\
&\leq \|\phi\|_{L^p_\nu(\mathbb{R}^N)} \sum_{j\in\mathbb{N}} 2\theta^2 C^2(q) \|u\|^2_{\mathbb{S}^2_1(B_j)} + 2C^2_\theta \|u\|^2_{L^2(B_j)} \\
&= \|\phi\|_{L^p_\nu(\mathbb{R}^N)} \left(2\theta^2 C^2(q) \|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + 2C^2_\theta \|u\|^2_{L^2(\mathbb{R}^N)}\right).
\end{aligned}$$

Now an obvious change of notation completes the proof of the second part of the lemma. $\square$

Lemma 2.2. *Assume* i)*, let* $0<\kappa<\lambda_0, 0<\theta<1$*. Then for all* $u \in \mathbb{S}_1^2(\mathbb{R}^N)$

$$\widehat{C_1}\,\|u\|^2_{\mathbb{S}_1^2(\mathbb{R}^N)} \le \|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N}\gamma(X)|u(X)|^2\mathrm{d}X - \kappa\,\|u\|^2_{L^2(\mathbb{R}^N)}$$
$$\le \widehat{C_2}\,\|u\|^2_{\mathbb{S}_1^2(\mathbb{R}^N)},$$

where $\widehat{C_1} = \min\{(\lambda_0-\kappa)(1-\theta)/2(\lambda_0+\theta+C_\theta), (\lambda_0-\kappa)/2\}, \widehat{C_2} = \max\{1+\theta, \theta+C_\theta\}$*, and* θ, C_θ *are as in* (3)*.*

Proof. This is just a simple computation. □

From Lemma 2.2, we have:

Lemma 2.3. *Assume* i)*. For* $u, v \in \mathbb{S}_1^2(\mathbb{R}^N)$ *define*

$$((u,v))_{\mathbb{S}_1^2(\mathbb{R}^N)} = (\nabla_{\alpha,\beta}u, \nabla_{\alpha,\beta}v)_{L^2(\mathbb{R}^N)} + (\gamma u, v)_{L^2(\mathbb{R}^N)}.$$

Then $((\cdot,\cdot))_{\mathbb{S}_1^2(\mathbb{R}^N)}$ *is a scalar product on* $\mathbb{S}_1^2(\mathbb{R}^N)$ *and the norm defined by this scalar product is equivalent to the usual norm on* $\mathbb{S}_1^2(\mathbb{R}^N)$*.*

We put

$$U = \begin{pmatrix} u \\ v \end{pmatrix},\ A = \begin{pmatrix} 0 & I \\ P_{\alpha,\beta} - \gamma(X)I & 0 \end{pmatrix},$$
$$f^*(U)(X) = \begin{pmatrix} 0 \\ -\lambda v(X) + f(X,u(X)) \end{pmatrix},\ U_0 = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix},$$

where I is the unit operator on $\mathbb{S}_1^2(\mathbb{R}^N)$. Then the problem (1)-(2) can be formulated as an abstract evolutionary equation

$$\frac{\mathrm{d}U}{\mathrm{d}t} = AU + f^*(U), \tag{9}$$
$$U(0) = U_0. \tag{10}$$

We set $H = \mathbb{S}_1^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. We regard H as a Hilbert space with the inner product

$$(U,\overline{U})_H = \left(\begin{pmatrix} u \\ v \end{pmatrix}, \begin{pmatrix} \overline{u} \\ \overline{v} \end{pmatrix}\right) = ((u,\overline{u}))_{\mathbb{S}_1^2(\mathbb{R}^N)} + (v,\overline{v})_{L^2(\mathbb{R}^N)}.$$

The domains $D(A)$ of A is given by

$$D(A) = \left\{\begin{pmatrix} u \\ v \end{pmatrix} : u, v \in \mathbb{S}_1^2(\mathbb{R}^N); P_{\alpha,\beta}u - \gamma(X)u \in L^2(\mathbb{R}^N)\right\}.$$

Lemma 2.4. *The adjoint* A^* *of* A *is given by*

$$A^* = -\begin{pmatrix} 0 & I \\ P_{\alpha,\beta} - \gamma(X)I & 0 \end{pmatrix}$$

with

$$D(A^*) = \left\{\begin{pmatrix} \chi \\ \psi \end{pmatrix} : \chi, \psi \in \mathbb{S}_1^2(\mathbb{R}^N); P_{\alpha,\beta}\chi - \gamma(X)\chi \in L^2(\mathbb{R}^N)\right\}.$$

Proof. The proof is similar to the one of Lemma 1 in [17]. We therefore omit the details. □

2.2. Global solutions

Lemma 2.5. *Suppose that* $f(X,\xi)$ *satisfies the conditions* ii)(f_1) *and* ii)(f_2). *Then*

a) *The Nemytskii map*

$$\begin{aligned}\widehat{f} : \mathbb{S}_1^2(\mathbb{R}^N) &\longrightarrow L^2(\mathbb{R}^N)\\ u &\longmapsto \widehat{f}(u)(X) := f(X, u(X))\end{aligned}$$

is Lipschitzian on every bounded set of $\mathbb{S}_1^2(\mathbb{R}^N)$.

b) *The map*

$$\begin{aligned}f^* : H &\longrightarrow H\\ U &\longmapsto f^*(U) := \begin{pmatrix} 0 \\ -\lambda v(X) + f(X, u(X)) \end{pmatrix}\end{aligned}$$

is Lipschitzian on every bounded set of H.

Proof. This is just a simple computation use the Sobolev embedding and the Hölder inequality. □

Lemma 2.4 together with Theorem 10.8 (p. 41) in [22] imply that A generates a C_0-semi-group e^{At} on H.

Definition (see [25])**.** Let $T > 0$, $T \in \mathbb{R}$. A (strongly) continuous mapping $U : [0,T) \to H$ is said to be a mild solution of the problem (9)-(10) if it solves the following integral equation

$$U(X,t) = e^{At}U_0 + \int_0^t e^{A(t-s)} f^*(U(s))\mathrm{d}s, \quad t \in [0,T).$$

If U is (strongly) differentiable almost everywhere in $[0,T)$ with U_t and AU in $L^1_{loc}([0,T),H)$, and satisfies the differential equation

$$\frac{dU}{dt} \overset{a.e.}{=} AU + f^*(U) \quad \text{on } (0,T), \text{ and } U(0) = U_0,$$

then U is called a strong solution of the problem (9)-(10).

Using Lemma 2.5 and Theorem 46.1 (p. 235), Theorem 46.2 (p. 236) in [25] it is not difficult to establish the following

Proposition 2.6. *Assume* i) *and* ii). *Then for any* $R > 0$ *and* $U_0 \in H$ *such that* $\|U_0\|_H \le R$, *there exists* $T = T(R) > 0$ *such that the Cauchy problem* (9)-(10) *has a unique* (*mild*) *solution* $U(t)$ *in* H *satisfying*

$$U(X,t) = e^{At}U_0 + \int_0^t e^{A(t-s)} f^*(U(s))\mathrm{d}s, \quad t \in [0,T),$$

on the time interval $[0, T)$*. The solution can be extended to a maximal life span* $[0, \tau)$*, and we have either* $\tau = +\infty$ *or* $\lim\limits_{t \to \tau^-} \|U(t)\|_H = +\infty$*.*

From conditions ii)(f_1) and ii)(f_2) we obtain

$$|F(X, \xi)| \le C\Big(g(X)|\xi|^2 + |\xi|^{2+\rho} + |f(X, 0)||\xi|\Big).$$

Hence

$$\begin{aligned}\int_{\mathbb{R}^N} |F(X, u)| \mathrm{d}X &\le C_3 \int_{\mathbb{R}^N} \Big(g(X)|u|^2 + |u|^{2+\rho} + |h(X)||u|\Big) \mathrm{d}X \\ &\le C_4 \Big(\|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u\|^{2+\rho}_{L^{2+\rho}(\mathbb{R}^N)} + \|h\|_{L^2(\mathbb{R}^N)} \|u\|_{L^2(\mathbb{R}^N)} \Big) \\ &< +\infty \end{aligned} \tag{11}$$

for all $u \in \mathbb{S}^2_1(\mathbb{R}^N)$.

Lemma 2.7. *Assume* i) *and* ii)*. Then any solution* $u(t)$ *of the problem* (1)-(2) *satisfies*

$$\|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} \le M, \quad \forall t \ge 0, \tag{12}$$

where M *is a constant depending only on* $\gamma(X), g(X), g_1(X), g_2(X)$ *and* R *when* $\|u_0\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u_1\|^2_{L^2(\mathbb{R}^N)} \le R$*.*

Proof. Let $U(t)$ be the solution of the problem (9)-(10) with the initial condition U_0. Define

$$E(u(t), u_t(t)) = \|\nabla_{\alpha,\beta} u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \phi(X)|u(X)|^2 \mathrm{d}X + \|u_t\|^2_{L^2(\mathbb{R}^N)}.$$

Letting κ, θ in Lemma 2.2 be fixed, we have, for $t \ge 0$,

$$\begin{aligned} C_5 \Big(\|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} \Big) &\le E(u(t), u_t(t)) \\ &\le C_6 \Big(\|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} \Big), \end{aligned} \tag{13}$$

where

$$C_5 = \min\Big\{\widehat{C_1}, 1\Big\}, \qquad C_6 = \max\Big\{\widehat{C_2} + \kappa, 1\Big\}.$$

Taking an inner product of (1) with $u_t(X, t)$ in $L^2(\mathbb{R}^N)$ and integrating over $[0, t]$ with respect to time, we get

$$\begin{aligned} &\frac{1}{2} E(u(t), u_t(t)) + \lambda \int_0^t \|u_t(\tau)\|^2_{L^2(\mathbb{R}^N)} \mathrm{d}\tau \\ &= \frac{1}{2} E(u(0), u_t(0)) + \int_{\mathbb{R}^N} F(X, u(X, t)) \mathrm{d}X - \int_{\mathbb{R}^N} F(X, u(X, 0)) \mathrm{d}X \\ &\le \frac{1}{2} E(u(0), u_t(0)) + \left| \int_{\mathbb{R}^N} F(X, u(X, 0)) \mathrm{d}X \right| + \int_{\mathbb{R}^N} F(X, u(X, t)) \mathrm{d}X. \end{aligned} \tag{14}$$

From (11) and (8), we have the following basic inequalities:

$$\int_{\mathbb{R}^N} |F(X,u(X,0))| \mathrm{d}X \le C_7 \|u_0\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)},$$
$$\int_{\mathbb{R}^N} F(X,u(X,t)) \mathrm{d}X \le \int_{\mathbb{R}^N} g_2(X) \mathrm{d}X.$$

Hence

$$\frac{1}{2}E(u(t),u_t(t)) \le \frac{1}{2}E(u(0),u_t(0)) + C_7 \|u_0\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \int_{\mathbb{R}^N} g_2(X)\mathrm{d}X,$$

or

$$\|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} + \|u_t\|^2_{L^2(\mathbb{R}^N)} \le M, \quad \forall t \ge 0.$$ □

Remark 2.8. From the proof of Lemma 2.7, we have a byproduct: let B be a bounded subset in H, then there exists a constant $C_8 = C(B)$ such that for any solution $U = (u(t), u_t(t))$ with initial data $(u_0, u_1) \in B$ and for any $t \ge 0$,

$$\int_0^t \|u_t(\tau)\|^2_{L^2(\mathbb{R}^N)} \mathrm{d}\tau \le C_8. \tag{15}$$

Theorem 2.9. *Assume* i), ii) *and* $U_0 \in H$. *Then the problem* (1)-(2) *has a unique global solution* $U \in C([0,\infty); H)$. *Moreover, for each fixed* t *the map* $U_0 \to S(t)U_0 := U(t)$ *is continuous on* H.

Proof. The proof is just a simple modification of the proof of Theorem 2.6 in [18]. □

3. Existence of a global attractor in $\mathbb{S}^2_1(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$

In view of Theorem 2.9, we can define a continuous semigroup $S(t) : H \to H$ as follows

$$S(t)U_0 := U(t),$$

where $U(t)$ is the unique global mild solution of the problem (1)-(2) subject to U_0 as the initial datum.

From Lemma 2.7, the semigroup $\{S(t)\}_{t\ge 0}$ is ultimate bounded in the sense for any bounded subset B in H, there exists a bounded subset $\mathcal{B}$ in H, which depends on B such that

$$\bigcup_{t\ge 0} S(t)B \subset \mathcal{B}. \tag{16}$$

Lemma 3.1. *Assume* i), ii) *and* $\mathcal{B}$ *is a bounded subset in* H. *Then any solution* $U(t) = (u(X,t), u_t(X,t))$ *of the problem* (9)-(10) *with initial data* $U_0 \in \mathcal{B}$ *satisfies*

(17)
$$\lim_{T,R\to+\infty} \frac{1}{T}\int_0^T \int_{|X|_{\alpha,\beta}\ge R} \Big(|u(X,t)|^2 + |u_t(X,t)|^2 + |\nabla_{\alpha,\beta} u(X,t)|^2\Big) \mathrm{d}X\mathrm{d}t = 0,$$

where

$$|X|_{\alpha,\beta} = \Big[|x|^{2(1+\alpha+\beta)} + |y|^{2(1+\alpha+\beta)} + (1+\alpha+\beta)^2|z|^2\Big]^{\frac{1}{2(1+\alpha+\beta)}}.$$

Proof. Choose a smooth function ϑ such that $0 \leq \vartheta(s) \leq 1$ for $s \in \mathbb{R}^+$ and

$$\vartheta(s) = 0 \quad \text{for } 0 \leq s \leq 1; \qquad \vartheta(s) = 1 \quad \text{for } s \geq 2.$$

Then

$$\nabla_{\alpha,\beta}\vartheta\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right) = \frac{1}{R^{2(1+\alpha+\beta)}}\vartheta'\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\nabla_{\alpha,\beta}|X|_{\alpha,\beta}^{2(1+\alpha+\beta)},$$

where

$$\begin{aligned}\nabla_{\alpha,\beta}|X|_{\alpha,\beta}^{2(1+\alpha+\beta)} = {} & 2(1+\alpha+\beta)\Big(x_1|x|^{2(\alpha+\beta)}, \ldots, x_{N_1}|x|^{2(\alpha+\beta)}, \\ & y_1|y|^{2(\alpha+\beta)}, \ldots, y_{N_2}|y|^{2(\alpha+\beta)}, (1+\alpha+\beta)|x|^{\alpha}|y|^{\beta}z_1, \\ & \ldots, (1+\alpha+\beta)|x|^{\alpha}|y|^{\beta}z_{N_3}\Big),\end{aligned}$$

hence

$$\begin{aligned}&\Big|\nabla_{\alpha,\beta}|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}\Big|^2 \\ &= 4(1+\alpha+\beta)^2\Big[|x|^{4(\alpha+\beta)+2} + |y|^{4(\alpha+\beta)+2} + (1+\alpha+\beta)^2|z|^2|x|^{2\alpha}|y|^{2\beta}\Big].\end{aligned}$$

Notice that there exists a constant $C_\vartheta > 0$ such that $|\vartheta'(s)| \leq C_\vartheta$ for $s \in \mathbb{R}^+$ and if $|X|_{\alpha,\beta}^{2(1+\alpha+\beta)} \leq R^{2(1+\alpha+\beta)}$, then

$$\frac{\Big|\nabla_{\alpha,\beta}|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}\Big|}{R^{2(1+\alpha+\beta)}} \leq \frac{\sqrt{12}}{R}(1+\alpha+\beta) = \frac{C_9}{R}. \tag{18}$$

Taking an inner product of (1) with $\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)u$ in $L^2(\mathbb{R}^N)$, and integrating over $[0,T]$ with respect to time t, we get

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^N} u_{tt}u\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\mathrm{d}X\mathrm{d}t \\ &+ \lambda\int_0^T\int_{\mathbb{R}^N} u_t u\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\mathrm{d}X\mathrm{d}t \\ &+ \int_0^T\int_{\mathbb{R}^N} \gamma(X)\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)u^2\mathrm{d}X\mathrm{d}t \\ = &\int_0^T\int_{\mathbb{R}^N} P_{\alpha,\beta}u\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)u\mathrm{d}X\mathrm{d}t\end{aligned} \tag{19}$$

$$+\int_0^T\int_{\mathbb{R}^N} f(X,u)u\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\mathrm{d}X\mathrm{d}t.$$

But

$$\begin{aligned}
&-\int_0^T\int_{\mathbb{R}^N} P_{\alpha,\beta}u\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)u\mathrm{d}X\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)|\nabla_{\alpha,\beta}u|^2\mathrm{d}X\mathrm{d}t\\
&\quad+\frac{2}{R^{2(1+\alpha+\beta)}}\int_0^T\int_{\mathbb{R}^N}\vartheta\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\vartheta'\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\\
&\qquad\qquad u\nabla_{\alpha,\beta}|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}\cdot\nabla_{\alpha,\beta}u\mathrm{d}X\mathrm{d}t\\
&=\int_0^T\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)|\nabla_{\alpha,\beta}u|^2\,\mathrm{d}X\mathrm{d}t=I_1,
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)u_tu\mathrm{d}X\mathrm{d}t\right|\\
&=\left|\frac{1}{2}\int_0^T\left(\frac{d}{dt}\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)|u(X,t)|^2\mathrm{d}X\right)\mathrm{d}t\right|\\
&\le\frac{1}{2}\left|\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)|u(X,T)|^2\mathrm{d}X\right|\\
&\quad+\frac{1}{2}\left|\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)|u(X,0)|^2\mathrm{d}X\right|=I_2,
\end{aligned}$$

$$\begin{aligned}
&\left|\int_0^T\int_{\mathbb{R}^N}u_{tt}(X,t)u(X,t)\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\mathrm{d}X\mathrm{d}t\right|\\
&=\left|\int_{\mathbb{R}^N}u_t(X,T)u(X,T)\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\mathrm{d}X\right.\\
&\quad-\int_{\mathbb{R}^N}u_t(X,0)u(X,0)\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\mathrm{d}X\\
&\quad\left.-\int_0^T\int_{\mathbb{R}^N}|u_t(X,t)|^2\vartheta^2\left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}}\right)\mathrm{d}X\mathrm{d}t\right|
\end{aligned}$$

$$\leq \frac{1}{2}\Big(\|U(T)\|_H^2 + \|U_0\|_H^2 \Big) + \int_0^T \int_{\mathbb{R}^N} |u_t(X,t)|^2 \vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) \mathrm{d}X\mathrm{d}t = I_3,$$

$$\int_0^T \int_{\mathbb{R}^N} f(X,u)u\vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) \mathrm{d}X\mathrm{d}t$$
$$\leq T \int_{\mathbb{R}^N} g_1(X)\vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) \mathrm{d}X = I_4.$$

Then (19) becomes

(20)
$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \gamma(X)\vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) u^2 \mathrm{d}X\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^N} \vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) |\nabla_{\alpha,\beta} u|^2 \mathrm{d}X\mathrm{d}t \\ &\leq I_1 + I_2 + I_3 + I_4. \end{aligned}$$

Applying the result of Lemma 2.7 and (15), we have

(21)
$$I_2 + I_3 \leq C_{10}.$$

Applying Hölder's inequality and Sobolev inequality, we have

(22)
$$|I_1| \leq \frac{C_{11}T}{R}.$$

From (7), we obtain

(23)
$$I_4 \leq C_\vartheta T \int_{|X|_{\alpha,\beta} \geq R} g_1(X)\mathrm{d}X.$$

From (21), (22) and (23), we obtain

(24)
$$\begin{aligned} &\int_0^T \int_{\mathbb{R}^N} \gamma(X)\vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) u^2 \mathrm{d}X\mathrm{d}t \\ &+ \int_0^T \int_{\mathbb{R}^N} \vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) |\nabla_{\alpha,\beta} u|^2 \mathrm{d}X\mathrm{d}t \\ &\leq C_{10} + C_\vartheta T \int_{|X|_{\alpha,\beta} \geq R} g_1(X)\mathrm{d}X + \frac{C_{11}T}{R}. \end{aligned}$$

On the other hand, since $\vartheta \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) u(X,t) \in \mathbb{S}_1^2(\mathbb{R}^N)$, from Lemma 2.2, we have

(25)
$$\int_{\mathbb{R}^N} \gamma(X)\vartheta^2 \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) u^2 \mathrm{d}X + \int_{\mathbb{R}^N} \left| \nabla_{\alpha,\beta} \left(\vartheta \left(\frac{|X|_{\alpha,\beta}^{2(1+\alpha+\beta)}}{R^{2(1+\alpha+\beta)}} \right) u \right) \right|^2 \mathrm{d}X$$

$$\geq \widehat{C}_1\left\{\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|^{2(1+\alpha+\beta)}_{\alpha,\beta}}{R^{2(1+\alpha+\beta)}}\right)u^2\mathrm{d}X+\int_{\mathbb{R}^N}\left|\nabla_{\alpha,\beta}\left(\vartheta\left(\frac{|X|^{2(1+\alpha+\beta)}_{\alpha,\beta}}{R^{2(1+\alpha+\beta)}}\right)u\right)\right|^2\mathrm{d}X\right\}.$$

We get

$$\int_{\mathbb{R}^N}\left|\nabla_{\alpha,\beta}\left(\vartheta\left(\frac{|X|^{2(1+\alpha+\beta)}_{\alpha,\beta}}{R^{2(1+\alpha+\beta)}}\right)u\right)\right|^2\mathrm{d}X \tag{26}$$
$$\leq 2\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|^{2(1+\alpha+\beta)}_{\alpha,\beta}}{R^{2(1+\alpha+\beta)}}\right)|\nabla_{\alpha,\beta}u|^2\mathrm{d}X+\frac{C_{11}}{R^2}.$$

From (24), (25) and (26), we obtain

$$\int_0^T\int_{\mathbb{R}^N}\vartheta^2\left(\frac{|X|^{2(1+\alpha+\beta)}_{\alpha,\beta}}{R^{2(1+\alpha+\beta)}}\right)u^2\mathrm{d}X\mathrm{d}t \tag{27}$$
$$+\int_0^T\int_{\mathbb{R}^N}\left|\nabla_{\alpha,\beta}\left(\vartheta\left(\frac{|X|^{2(1+\alpha+\beta)}_{\alpha,\beta}}{R^{2(1+\alpha+\beta)}}\right)u\right)\right|^2\mathrm{d}X\mathrm{d}t$$
$$\leq C_{12}+C_{13}T\int_{|X|_{\alpha,\beta}\geq R}g_1(X)\mathrm{d}X+\frac{C_{14}T}{R}+\frac{C_{15}T}{R^2}.$$

From (27) and (15), we have

$$\lim_{T,R\to+\infty}\frac{1}{T}\int_0^T\int_{|X|_{\alpha,\beta}\geq R}\Big(|u(X,t)|^2+|u_t(X,t)|^2+|\nabla_{\alpha,\beta}u(X,t)|^2\Big)\mathrm{d}X\mathrm{d}t=0.$$
□

Lemma 3.2. *Assume* i), ii)*, and $U_n \rightharpoonup U$ in H. Then for every $t \geq 0$*

$$S(t)U_n \rightharpoonup S(t)U \ \text{in}\ H. \tag{28}$$

Proof. The proof is similar to the one of Lemma 1 in [10]. We therefore omit the details. □

Lemma 3.3. *Assume* i), ii) *and let $\{U_n\}_{n=1}^{\infty}$ be weakly convergent to U in H. Then*

$$\lim_{T\to+\infty}\limsup_{n\to+\infty}\|S(T)U_n-S(T)U\|_H=0. \tag{29}$$

Proof. Let $S(t)U_n=(u_n(t),u_{nt}(t))$ be the solution of the problem (9)-(10) with initial data $U_n=(u_n(0),u_{nt}(0))$ and $S(t)U_0=(u(t),u_t(t))$ be the solution with the initial data $U_0=(u(0),u_t(0))$. Lemma 2.7 implies

$$\sup_{t\geq 0,n\geq 0}\|S(t)U_n\|_H\leq C_{16}. \tag{30}$$

Taking an inner product of (1) by $u_t+\frac{\lambda}{2}u$ in $L^2(\mathbb{R}^N)$ and integrating over $[0,T]$ with respect to time t yield

$$\frac{\lambda}{2}\left\{\int_0^T\Big[E(u(t),u_t(t))-\int_{\mathbb{R}^N}f(X,u(X,t))u(X,t)\mathrm{d}X\Big]\mathrm{d}t\right\} \tag{31}$$

$$
\begin{aligned}
&= \frac{1}{2}E(u(0), u_t(0)) - \frac{1}{2}E(u(T), u_t(T)) \\
&\quad + \frac{\lambda}{2}\int_{\mathbb{R}^N} u(X,0)u_t(X,0)\mathrm{d}X + \int_{\mathbb{R}^N} F(X, u(X,T))\mathrm{d}X \\
&\quad + \frac{\lambda^2}{4}\|u(0)\|^2_{L^2(\mathbb{R}^N)} - \frac{\lambda^2}{4}\|u(T)\|^2_{L^2(\mathbb{R}^N)} + \frac{\lambda-1}{2}\int_0^T\int_{\mathbb{R}^N} |u_t|^2\mathrm{d}X\mathrm{d}t \\
&\quad - \frac{\lambda}{2}\int_{\mathbb{R}^N} u(X,T)u_t(X,T)\mathrm{d}X - \int_{\mathbb{R}^N} F(X, u(X,0))\mathrm{d}X.
\end{aligned}
$$

Hence

$$
\left|\int_0^T \left[E(u(t), u_t(t)) - \int_{\mathbb{R}^N} f(X,u)u\mathrm{d}X\right]\mathrm{d}t\right| \le C_{17}. \tag{32}
$$

Similarly to the case for (32), we also have

$$
\left|\int_0^T \left[E(u_n(t), u_{nt}(t)) - \int_{\mathbb{R}^N} f(X,u_n)u_n\mathrm{d}X\right]\mathrm{d}t\right| \le C_{18}. \tag{33}
$$

Multiplying (1) by u_t and integrating over $[t,T]\times\mathbb{R}^N$ we have

$$
\begin{aligned}
&\frac{1}{2}E(u(T), u_t(T)) - \int_{\mathbb{R}^N} F(X, u(X,T))\mathrm{d}X + \lambda\int_t^T \|u_\tau\|^2_{L^2(\mathbb{R}^N)}\mathrm{d}\tau \\
&= \frac{1}{2}E(u(t), u_t(t)) - \int_{\mathbb{R}^N} F(X, u(X,t))\mathrm{d}X,\ 0\le t\le T.
\end{aligned} \tag{34}
$$

From (32) and (34), we have

$$
\begin{aligned}
&\frac{1}{2}E(u(T), u_t(T)) - \int_{\mathbb{R}^N} F(X, u(X,T))\mathrm{d}X \\
&\quad + \frac{\lambda}{T}\int_0^T\int_t^T \|u_\tau\|^2_{L^2(\mathbb{R}^N)}\mathrm{d}\tau\mathrm{d}t \\
&= \frac{1}{T}\int_0^T \left[\frac{1}{2}E(u(t), u_t(t)) - \int_{\mathbb{R}^N} F(X, u(X,t))\mathrm{d}X\right]\mathrm{d}t \\
&\ge \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\Big(-F(X,u(X,t)) + \frac{1}{2}f(X,u)u\Big)\mathrm{d}X\mathrm{d}t - \frac{C_{17}}{2T}.
\end{aligned} \tag{35}
$$

From (33) and (34), we have

$$
\begin{aligned}
&\frac{1}{2}E(u_n(T), u_{nt}(T)) - \int_{\mathbb{R}^N} F(X, u_n(X,T))\mathrm{d}X \\
&\quad + \frac{\lambda}{T}\int_0^T\int_t^T \|u_{n\tau}\|^2_{L^2(\mathbb{R}^N)}\mathrm{d}\tau\mathrm{d}t \\
&\le \frac{1}{T}\int_0^T\int_{\mathbb{R}^N}\Big(-F(X,u_n(X,t)) + \frac{1}{2}f(X,u_n)u_n\Big)\mathrm{d}X\mathrm{d}t + \frac{C_{18}}{2T}.
\end{aligned} \tag{36}
$$

From (35) and (36), it follows that

$$\begin{aligned}
&\frac{1}{2}E(u_n(T),u_{nt}(T)) - \int_{\mathbb{R}^N} F(X,u_n(X,T))\mathrm{d}X \\
&+ \frac{\lambda}{T}\int_0^T \int_t^T \|u_{n\tau}\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}\tau\mathrm{d}t \\
\le\ & \frac{1}{T}\int_0^T \int_{\mathbb{R}^N} \Big(F(X,u(X,t)) - F(X,u_n(X,t))\Big)\mathrm{d}X\mathrm{d}t \\
&+ \frac{1}{T}\int_0^T \int_{\mathbb{R}^N} \frac{1}{2}\Big(f(X,u_n)u_n - f(X,u)u\Big)\mathrm{d}X\mathrm{d}t + \frac{1}{2}E(u(T),u_t(T)) \\
&- \int_{\mathbb{R}^N} F(X,u(X,T))\mathrm{d}X + \frac{\lambda}{T}\int_0^T \int_t^T \|u_\tau\|^2_{L^2(\mathbb{R}^N)}\,\mathrm{d}\tau\mathrm{d}t + \frac{C_{17}+C_{18}}{2T}.
\end{aligned}$$

Similarly Lemma 3.4 in [18], we have for any arbitrary $\epsilon > 0$, there exists T_0 such that

$$\limsup_{n\to+\infty} E(u_n(T),u_{nt}(T)) \le E(u(T),u_t(T)) + \epsilon \quad \text{for all} \quad T \ge T_0. \tag{37}$$

From Lemma 2.2 and Lemma 3.2, we have the inequality

$$\begin{aligned}
&\limsup_{n\to+\infty} \|S(T)U_n - S(T)U_0\|_H^2 \\
&\le C_{19}\limsup_{n\to+\infty} E(u_n(T)-u(T),u_{nt}(T)-u_t(T)) \\
&= C_{19}\limsup_{n\to+\infty}\Bigg\{ \int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u_n|^2\mathrm{d}X - 2\int_{\mathbb{R}^N} \nabla_{\alpha,\beta}u_n\cdot\nabla_{\alpha,\beta}u_n\mathrm{d}X \\
&\quad + \int_{\mathbb{R}^N} |\nabla_{\alpha,\beta}u|^2\mathrm{d}X + \int_{\mathbb{R}^N} \gamma(X)|u_n(T)|^2\mathrm{d}X \\
&\quad - 2\int_{\mathbb{R}^N} \gamma(X)u_n(T)u(T)\mathrm{d}X + \int_{\mathbb{R}^N} \gamma(X)|u(T)|^2\mathrm{d}X \\
&\quad + \int_{\mathbb{R}^N} |u_{nt}(T)|^2\mathrm{d}X - 2\int_{\mathbb{R}^N} u_{nt}(T)u_t(T)^2\mathrm{d}X + \int_{\mathbb{R}^N} |u_t(T)|^2\mathrm{d}X\Bigg\} \\
&= C_{19}\left(\limsup_{n\to+\infty} E(u_n(T),u_{nt}(T)) - E(u(T),u_t(T))\right).
\end{aligned} \tag{38}$$

From (37), (38) yields (29). $\square$

We then formulate the main result of the section.

Theorem 3.4. *Assume* i) *and* ii)*. Then the semigroup* $\{S(t)\}_{t\ge0}$ *associated with the problem* (9)-(10) *is asymptotically compact in the phase space* H*, i.e., let* $\{U_n\}_{n=1}^\infty$ *be a bounded sequence in* H *and a time sequence* $\{t_n\}_{n=1}^\infty$ *such that* $t_n \to +\infty$ *as* $n \to +\infty$*, then* $\{S(t_n)U_n\}_{n=1}^\infty$ *is precompact in* H*.*

Proof. The proof is similar to the one of Theorem 3.5 in [18]. We therefore omit the details. □

From Theorem 3.4, we have:

Theorem 3.5. *Assume* i) *and* ii). *Then, the semigroup generated by the problem* (9)-(10) *possesses a global attractor in H which is a compact invariant subset that attracts every bounded set of H with respect to the norm topology.*

Example 3.6. Consider the following problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2} + \lambda \dfrac{\partial u}{\partial t} + u = P_{\frac{1}{2},\frac{1}{2}} u + f(X,u) \text{ for } X = (x,y,z) \in \mathbb{R}^3,\ t > 0, \\ u(x,y,z,0) = u_0(x,y,z),\ \dfrac{\partial u}{\partial t}(x,y,z,0) = u_1(x,y,z) \text{ for } (x,y,z) \in \mathbb{R}^3, \end{cases} \tag{39}$$

where $u_0(x,y,z) \in \mathbb{S}_1^2(\mathbb{R}^3), u_1(x,y,z) \in L^2(\mathbb{R}^3)$, λ is a positive constant,

$$P_{\frac{1}{2},\frac{1}{2}} u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + |x|\,|y|\,\frac{\partial^2 u}{\partial z^2},$$

and

$$f(X,u) = \begin{cases} \frac{-u(1-u)}{|X|^8+1} & \text{for } u < 1, |X|^2 = x^2 + y^2 + z^2, \\ \frac{u(1-u)}{|X|^8+1} & \text{for } u \geq 1. \end{cases}$$

We obtain

$$F(X,u) = \begin{cases} \frac{-1}{|X|^8+1}\left(\frac{u^2}{2} - \frac{u^3}{3}\right) & \text{for } u < 1, \\ \frac{1}{|X|^8+1}\left(\frac{u^2}{2} - \frac{u^3}{3} - \frac{1}{3}\right) & \text{for } u \geq 1. \end{cases}$$

Obviously $f(X,u)u \leq 0, F(X,u) \leq 0$ for all $X \in \mathbb{R}^3, u \in \mathbb{S}_1^2(\mathbb{R}^3)$.

It is easily checked that the function $f(X,u)$ satisfies the conditions (f_1)-(f_3) in which we can take

$$N_{\frac{1}{2},\frac{1}{2}} = 4,\ \rho = 2,\ h(X) = 0,\ C_1 = 1,\ C_2 = 1,$$
$$g(X) = \frac{1}{|X|^8+1},\ g_1(X) = g_2(X) \equiv 0.$$

Applying Theorem 3.5 we conclude that the semigroup generated by the problem (39) possesses a global attractor in $\mathbb{S}_1^2(\mathbb{R}^3) \times L^2(\mathbb{R}^3)$.

4. Gradient system and global attractor

In the section, we will show that the dynamical system $(H, S(t))$ associated with the problem (1)-(2) has a strict Lyapunov function Φ on the whole phase space H, and consequently is a gradient system. Considering the asymptotic compactness of the system and applying the results in [6], we prove the existence of a global attractor for $(H, S(t))$ without assuming dissipativity in the explicit form.

Firstly, let us recall some relevant concepts and two important abstract results in [6].

Definition. Let $\mathbb{Y} \subset \mathbb{X}$ be a positive invariant set of a dynamical system $(\mathbb{X}, S(t))$.

• The continuous functional $\Phi(y)$ defined on $\mathbb{Y}$ is said to be the Lyapunov function for $(\mathbb{X}, S(t))$ on $\mathbb{Y}$ if and only if the function $t \to \Phi(S(t)y)$ is a nonincreasing function for any $y \in \mathbb{Y}$.

• The Lyapunov function $\Phi(y)$ is said to be strict on $\mathbb{Y}$ if and only if for all $t \geq 0$ and some $y \in \mathbb{Y}$, the equation $\Phi(S(t)y) = \Phi(y)$ implies that $S(t)y = y$ for all $t \geq 0$, i.e., y is a stationary point of $\mathbb{X}, S(t)$.

• The dynamical system $(\mathbb{X}, S(t))$ is said to be gradient if and only if there exists a strict Lyapunov function for $(\mathbb{X}, S(t))$ on whole phase space $\mathbb{X}$.

Theorem 4.1. *Assume that $(\mathbb{X}, S(t))$ is a gradient system which, moreover, is asymptotically compact. Assume that Lyapunov function $\Phi(u)$ associated with the system is bounded from above on any bounded subset of $\mathbb{X}$ and the set $\Phi_R = \{u : \Phi(u) \leq R\}$ is bounded for any R. If the set of stationary points*

$$\mathcal{N} = \Big\{u : S(t)u = u\Big\} \text{ for all } t \geq 0$$

is bounded in $\mathbb{X}$, then $(\mathbb{X}, S(t))$ possesses a compact global attractor.

Theorem 4.2. *Let a dynamical system $(\mathbb{X}, S(t))$ possess a compact global attractor on $\mathcal{A}$. Then $\mathcal{A} = \mathcal{M}^u(\mathcal{N})$, where $\mathcal{M}^u(\mathcal{N})$ is the unstable manifold defined as*

$$\mathcal{M}^u(\mathcal{N}) = \Big\{v \in \mathbb{X} : u(0) = v,\ \lim_{t\to-\infty} dist(u(t), \mathcal{N}) = 0, u(t) \text{ is a full trajectory}\Big\}.$$

Moreover, $\mathcal{A}$ consists of all full trajectories $\Gamma = \{u(t) : t \in \mathbb{R}\}$ such that

$$\lim_{t\to-\infty} dist(u(t), \mathcal{N}) = \lim_{t\to+\infty} dist(u(t), \mathcal{N}) = 0.$$

Now we give the main conclusion for the present paper. Define the set of stationary points of $H, S(t)$ and its unstable manifold as

$$\mathcal{N}_0 = \Big\{(u, 0) \in H : -P_{\alpha,\beta}u + \gamma(X)u = f(X, u), X \in \mathbb{R}^N\Big\}$$

and

$$\mathcal{M}^u(\mathcal{N}_0) = \Big\{(u_0, u_1) \in H : U(0) = (u_0, u_1), \lim_{t\to-\infty} dist(U(t), \mathcal{N}_0) = 0,$$
$$U(t) \text{ is a full trajectory}\Big\}$$

respectively. From i) and ii) applying Theorem 3.5, we have $\mathcal{N}_0 \neq \emptyset$.

Theorem 4.3. *Dynamical system $(H, S(t))$ associated with the problem* (1)-(2) *possesses a compact connected global attractor $\mathcal{A}$. Moreover, $\mathcal{A} = \mathcal{M}^u(\mathcal{N}_0)$ and it consists of all full trajectories $\Gamma = \{U(t) : t \in \mathbb{R}\}$ such that*

$$\lim_{t\to-\infty} dist(U(t), \mathcal{N}_0) = 0, \quad \lim_{t\to\infty} dist(U(t), \mathcal{N}_0) = 0.$$

Proof. Let us define a functional Φ on H as follows:

$$\Phi(U) = \frac{1}{2}\left(\|u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \gamma(X)u^2 \mathrm{d}X + \|v\|^2_{L^2(\mathbb{R}^N)} \right) - \int_{\mathbb{R}^N} F(X, u(X,t))\mathrm{d}X. \tag{40}$$

Then from (1) we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\Phi(S(t)U_0) &= \frac{\mathrm{d}}{\mathrm{d}t}\Big[\frac{1}{2}E(u(t), u_t(t)) - \int_{\mathbb{R}^N} F(X, u(X,t))\mathrm{d}X\Big] \\ &= -\lambda \|u_t\|^2_{L^2(\mathbb{R}^N)} \leq 0,\end{aligned}$$

which means

$$\Phi(S(t)U_0) + \lambda \int_0^t \|u_\tau(\tau)\|^2_{L^2(\mathbb{R}^N)} \mathrm{d}\tau = \Phi(U_0), \ \forall U_0 \in H,$$

and if $\Phi(S(t)U_0) = \Phi(U_0)$ for all $t > 0$ and for some U_0, it entails

$$\lambda \int_0^t \|u_\tau(\tau)\|^2_{L^2(\mathbb{R}^N)} \mathrm{d}\tau = 0, \quad \forall t \geq 0.$$

This is possible only if $S(t)U = (u_0, 0)$ is a stationary point. So $(H, S(t))$ admits a strict Lyapunov function Φ. It is easy to show that for any $U \in H$

$$\begin{aligned}\Phi(U) &\geq C_{20} \|U\|^2_H - \|g_2\|_{L^1(\mathbb{R}^N)}, \\ \Phi(U) &\leq C_{21} \|U\|^2_H + \|g_2\|_{L^1(\mathbb{R}^N)}.\end{aligned}$$

So $\Phi(U)$ is bounded from above on any bounded subset of H and the set $\Phi_R = \{U : \Phi(U) \leq R\}$ is bounded for any R. At last, we shall show that $\mathcal{N}_0$ is bounded in H. Taking an inner product of the equation $-P_{\alpha,\beta}u + \gamma(X)u = f(X,u)$ with $u(X)$ in $L^2(\mathbb{R}^N)$ yields

$$\|\nabla_{\alpha,\beta}u\|^2_{L^2(\mathbb{R}^N)} + \int_{\mathbb{R}^N} \gamma(X)|u(X)|^2 \mathrm{d}X \leq \|g_1\|_{L^1(\mathbb{R}^N)},$$

which implies

$$\|u\|^2_{\mathbb{S}^2_1(\mathbb{R}^N)} \leq C_{22}.$$

From Theorem 3.5, Theorem 4.1 and Theorem 4.2, we get the result. ☐

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