J. Korean Math. Soc. **58** (2021), No. 5, pp. 1227–1237
https://doi.org/10.4134/JKMS.j200538
pISSN: 0304-9914 / eISSN: 2234-3008

ON THE RATIO OF BIOMASS TO TOTAL CARRYING CAPACITY IN HIGH DIMENSIONS

Junyoung Heo and Yeonho Kim

Abstract. This paper is concerned with a reaction-diffusion logistic model. In [17], Lou observed that a heterogeneous environment with diffusion makes the total biomass greater than the total carrying capacity. Regarding the ratio of biomass to carrying capacity, Ni [10] raised a conjecture that the ratio has a upper bound depending only on the spatial dimension. For the one-dimensional case, Bai, He, and Li [1] proved that the optimal upper bound is 3. Recently, Inoue and Kuto [13] showed that the supremum of the ratio is infinity when the domain is a multi-dimensional ball. In this paper, we generalized the result of [13] to an arbitrary smooth bounded domain in $\mathbb{R}^n, n \geq 2$. We use the sub-solution and super-solution method. The idea of the proof is essentially the same as the proof of [13] but we have improved the construction of sub-solutions. This is the complete answer to the conjecture of Ni.

1. Introduction

This paper is concerned with a reaction-diffusion logistic model. After the pioneering work of Skellam [25], many studies on reaction-diffusion logistic model have been investigated (see [2–5,7–15,17–24,26] and the references therein). To analyze the effects of diffusion and spatial heterogeneity on the total biomass of single species, Lou [17] considered the following problem:

$$(1)\qquad \begin{cases} \frac{\partial u}{\partial t} = d\Delta u + u(m(x) - u) & \text{in } \Omega \times (0, \infty), \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega \times (0, \infty), \\ u(x, 0) = u_0(x) & \text{in } \Omega, \end{cases}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and ν is the outward unit normal vector on $\partial\Omega$. Here, the diffusion rate d is positive, the function $u(x,t)$ represents the density of the species, and $m(x)$ is the local intrinsic growth rate or carrying capacity. The Neumann boundary condition means that no

Received September 23, 2020; Revised December 29, 2020; Accepted January 15, 2021.
2010 *Mathematics Subject Classification.* 35B09, 35B30, 35Q92.
Key words and phrases. Logistic model, spatial heterogeneity, total biomass.

individuals can move across the boundary $\partial\Omega$. We assume that the function $m(x)$ satisfies the following condition:

(M) $m(x) \in L^\infty(\Omega), m(x) \geq 0$, and $m \not\equiv$ constant on $\bar{\Omega}$.

Biologically, non-constant $m(x)$ indicates the heterogeneous environment which gives many different properties from the homogeneous environment case. It is well known that the problem (1) has a positive equilibrium $\theta_{d,m}$ for any $d > 0$ and $m(x)$ satisfying the condition (M). Moreover, the function $\theta_{d,m}$ is in $W^{2,p}(\Omega)$ for every $p \geq 1$ and it is the unique positive solution of the stationary problem:

$$\begin{cases} d\Delta\theta + \theta(m(x) - \theta) = 0 & \text{in } \Omega, \\ \frac{\partial\theta}{\partial\nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

For proofs of the above facts, refer to [3]. The following property was first observed by Lou [17]:

$$\int_\Omega \theta_{d,m}(x)\, dx > \int_\Omega m(x)\, dx \tag{3}$$

holds for any $d > 0$ and any m satisfying (M). Indeed, using $\theta_{d,m}^{-1}$ as a test function of (2), we obtain

$$\int_\Omega (m - \theta_{d,m})\, dx = -d \int_\Omega \frac{|\nabla\theta_{d,m}|^2}{\theta_{d,m}^2}\, dx < 0$$

since m is a non-constant function. The strict inequality means that the heterogeneous environment with diffusion makes the total biomass greater than the total carrying capacity. Define the supremum of the ratio as

$$E(m) = \sup_{d>0} \frac{\int_\Omega \theta_{d,m}(x)\, dx}{\int_\Omega m(x)\, dx}.$$

Then, from (3), we can see that $E(m) > 1$ for any m satisfying (M). Regarding the upper bound, W.-M. Ni raised a following conjecture.

Conjecture 1.1 ([14], Section 2.1)**.** *Assume that m satisfies the condition* (M)*. Then there exists a constant $C(n)$ depending only on n such that $E(m) \leq C(n)$, and $C(1) = 3$.*

For the one-dimensional case, Bai, He, and Li [1] validated the conjecture. They used ordinary differential equation techniques to show $C(1) = 3$ and then obtained the optimality of 3 by choosing

$$d_\epsilon = \sqrt{\epsilon} \quad \text{and} \quad m_\epsilon(x) = \begin{cases} 0, & \text{if } x \in [0, 1-\epsilon], \\ \frac{1}{\epsilon}, & \text{if } x \in (1-\epsilon, 1]. \end{cases}$$

In the high dimensional case, Inoue and Kuto [13] recently showed that there is no upper bound when the domain Ω is a ball. They used the uniqueness of the solution of (2) and the sub-solution and super-solution method. The authors set

$$d_\epsilon = \frac{c_1}{\epsilon^{n-2}} \quad \text{and} \quad m_\epsilon(x) = \begin{cases} \frac{1}{\epsilon^n}, & \text{if } x \in \overline{B_\epsilon}, \\ 0, & \text{if } x \in B_1 \setminus \overline{B_\epsilon}, \end{cases}$$

and then constructed the L^1 unbounded sequence of sub-solutions. Here, c_1 is some positive constant and $B_r = \{x \in \mathbb{R}^n \mid |x| < r\}$. In this paper, we generalized the result of [13] to all smooth bounded domains in $\mathbb{R}^n, n \geq 2$. Our main theorem is stated as follows.

Theorem 1.2. *Assume that $n \geq 2$. Let Ω be a smooth bounded domain in $\mathbb{R}^n$. Then*

$$\sup\{E(m) \mid m \textit{ satisfies condition } (\mathrm{M})\} = \infty.$$

The idea of the proof is essentially the same as the proof of [13], but we have improved the construction of sub-solutions. In Section 2, we prove Theorem 1.2 and in Section 3, we give some concluding remarks about the ratio $E(m)$ and related problems.

2. Proof of Theorem 1.2

First, we present the well-known sub-solution and super-solution method. Consider the following nonlinear elliptic problem:

$$\begin{cases} d\Delta u + p(x,u) = 0 & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where Ω is a smooth bounded domain in $\mathbb{R}^n$ and ν is the outer unit normal vector on $\partial\Omega$. We assume that the function $p(x,t) : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function, that is, p is measurable in $x \in \Omega$ and continuous in $t \in \mathbb{R}$.

Definition ([6], Section 4.2)**.** A function u is called a *sub-solution* of (4) if $u \in W^{1,2}(\Omega)$, $p(\cdot, u(\cdot)) \in L^2(\Omega)$ and

$$\int_\Omega d\,\nabla u \cdot \nabla\varphi - p(x,u)\varphi\, dx \leq 0$$

for any $\varphi \in W^{1,2}(\Omega), \varphi \geq 0$. It is called a *super-solution* if the above inequality is reversed.

Then, the following result is standard.

Proposition 2.1 ([6], Theorem 4.12)**.** *Suppose that v and w are sub- and super-solution of* (4)*, respectively, and $v \leq w$ a.e. in Ω. Suppose further that there exists a function $k_1 \in L^2(\Omega)$ such that*

$$|p(x,t)| \leq k_1(x) \tag{5}$$

for a.e. $x \in \Omega$ and all $t \in [v(x), w(x)]$. Then (4) *has a weak solution u satisfying $v \leq u \leq w$ a.e. in Ω.*

Now, using the above proposition, we prove Theorem 1.2. The main part of the proof is the construction of sub-solutions whose $L^1(\Omega)$-norm diverge.

Proof of Theorem 1.2. Without loss of generality, we may assume that $0 \in \Omega$. Then there is a small $r_0 > 0$ such that $B_{r_0} \subset \Omega$. Choose small ϵ in $(0, r_0)$ and let

$$d_\epsilon = \frac{c_1}{\epsilon^{n-2}} \quad \text{and} \quad m_\epsilon = \begin{cases} \frac{1}{\epsilon^n}, & \text{if } x \in \overline{B_\epsilon}, \\ \frac{c_2}{e r_0^n}, & \text{if } x \in \Omega \setminus \overline{B_\epsilon}, \end{cases}$$

where $c_1, c_2 > 0$ are small constants independent of ϵ, which will be determined later. Define functions $\overline{\theta_\epsilon}, \underline{\theta_\epsilon} : \Omega \to \mathbb{R}$ by

$$\overline{\theta_\epsilon} \equiv \frac{1}{\epsilon^n} \quad \text{and} \quad \underline{\theta_\epsilon} = \begin{cases} \frac{c_2}{\epsilon^n} e^{-\frac{|x|^n}{\epsilon^n}}, & \text{if } x \in \overline{B_\epsilon}, \\ \frac{c_2}{e|x|^n}, & \text{if } x \in \overline{B_{r_0}} \setminus \overline{B_\epsilon}, \\ \frac{c_2}{e r_0^n}, & \text{if } x \in \Omega \setminus \overline{B_{r_0}}. \end{cases}$$

Since

$$d_\epsilon \Delta \overline{\theta_\epsilon} + \overline{\theta_\epsilon}(m_\epsilon(x) - \overline{\theta_\epsilon}) = \begin{cases} 0, & \text{if } x \in \overline{B_\epsilon}, \\ \frac{1}{\epsilon^n}\left(\frac{c_2}{e r_0^n} - \frac{1}{\epsilon^n}\right), & \text{if } x \in \Omega \setminus \overline{B_\epsilon}, \end{cases}$$

and $\partial_\nu \overline{\theta_\epsilon} = 0$ on $\partial\Omega$, $\overline{\theta_\epsilon}$ is a super-solution of (2) if $c_2 \le e$.

In what follows, we will prove that $\underline{\theta_\epsilon}$ is a sub-solution of (2). We have to show that

$$\int_\Omega d_\epsilon \nabla \underline{\theta_\epsilon} \cdot \nabla \varphi - \underline{\theta_\epsilon}(m_\epsilon(x) - \underline{\theta_\epsilon})\varphi \, dx \le 0 \tag{6}$$

for any $\varphi \in W^{1,2}(\Omega)$ with $\varphi \ge 0$. By the density theorem, we may assume that $\varphi \in C_c^\infty(\Omega)$. We divide Ω into three parts,

$$A_1 = \Omega \setminus \overline{B_{r_0}}, \quad A_2 = B_{r_0} \setminus \overline{B_\epsilon}, \quad \text{and} \quad A_3 = B_\epsilon,$$

and denote the restrictions of $\underline{\theta_\epsilon}$ to A_i by

$$\theta_i = \underline{\theta_\epsilon}|_{A_i} \quad \text{for} \quad i = 1, 2, 3.$$

Then θ_i is in $C^2(\overline{A_i})$ for each $i = 1, 2, 3$ if we extend θ_i continuously on ∂A_i. Applying Green's formula to each region $\overline{A_i}$, we obtain

$$\begin{aligned} &\int_\Omega \left[-d_\epsilon \nabla \underline{\theta_\epsilon} \cdot \nabla \varphi + \underline{\theta_\epsilon}(m_\epsilon(x) - \underline{\theta_\epsilon})\varphi\right] dx \\ &= \sum_{i=1}^3 \int_{A_i} \left[-d_\epsilon \nabla \theta_i \cdot \nabla \varphi + \theta_i(m_\epsilon(x) - \theta_i)\varphi\right] dx \\ &= \sum_{i=1}^3 \left[\int_{A_i} \left[d_\epsilon \Delta \theta_i + \theta_i(m_\epsilon(x) - \theta_i)\right] \varphi \, dx - d_\epsilon \int_{\partial A_i} \frac{\partial \theta_i}{\partial \nu_i} \varphi \, dS\right] \end{aligned}$$

$$= \int_{\Omega\setminus(\partial B_\epsilon\cup\partial B_{r_0})} \left[d_\epsilon \Delta\underline{\theta_\epsilon} + \underline{\theta_\epsilon}(m_\epsilon(x) - \underline{\theta_\epsilon})\right] \varphi \, dx - d_\epsilon \sum_{i=1}^{3} \int_{\partial A_i} \frac{\partial\theta_i}{\partial\nu_i}\varphi \, dS,$$

where ν_i denotes the outer unit normal vector on ∂A_i, for $i = 1, 2, 3$. Note that $\underline{\theta_\epsilon} \in C^1(\overline{\Omega} \setminus \partial B_{r_0})$. Indeed, since $\underline{\theta_\epsilon}$ is radial,

$$\frac{c_2}{\epsilon^n} e^{-\frac{r^n}{\epsilon^n}}\Big|_{r=\epsilon} = \frac{c_2}{er^n}\Big|_{r=\epsilon} = \frac{c_2}{e\epsilon^n},$$

and

$$\frac{\partial}{\partial r}\left(\frac{c_2}{\epsilon^n} e^{-\frac{r^n}{\epsilon^n}}\right)\Big|_{r=\epsilon} = \frac{\partial}{\partial r}\left(\frac{c_2}{er^n}\right)\Big|_{r=\epsilon} = -\frac{nc_2}{e\epsilon^{n+1}},$$

so we obtain the C^1 regularity of $\underline{\theta_\epsilon}$ on ∂B_ϵ. Then, we have

$$\begin{aligned}
&\int_\Omega \left[-d_\epsilon \nabla\underline{\theta_\epsilon} \cdot \nabla\varphi + \underline{\theta_\epsilon}(m_\epsilon(x) - \underline{\theta_\epsilon})\varphi\right] \, dx \\
&= \int_{\Omega\setminus(\partial B_\epsilon\cup\partial B_{r_0})} [d_\epsilon \Delta\underline{\theta_\epsilon} + \underline{\theta_\epsilon}(m_\epsilon(x) - \underline{\theta_\epsilon})]\varphi \, dx \\
&\quad - d_\epsilon \int_{\partial B_{r_0}} \frac{\partial\theta_1}{\partial\nu_1}\varphi \, dS \; - d_\epsilon \int_{\partial B_{r_0}} \frac{\partial\theta_2}{\partial\nu_2}\varphi \, dS.
\end{aligned}$$

By the definition of $\underline{\theta_\epsilon}$, we immediately get

$$\int_{\partial B_{r_0}} \frac{\partial\theta_1}{\partial\nu_1}\varphi \, dS = 0 \quad \text{and} \quad \int_{\partial B_{r_0}} \frac{\partial\theta_2}{\partial\nu_2}\varphi \, dS \le 0$$

for any $\varphi \in W^{1,2}(\Omega)$ with $\varphi \ge 0$. Thus, to prove the inequality (6), it suffices to show that

$$d_\epsilon \Delta\underline{\theta_\epsilon} + \underline{\theta_\epsilon}(m_\epsilon(x) - \underline{\theta_\epsilon}) \ge 0 \quad \text{in } \Omega \setminus (\partial B_\epsilon \cup \partial B_{r_0}). \tag{7}$$

From now on, with some abuse of notation, we regard functions $m_\epsilon, \underline{\theta_\epsilon}$ as functions of one variable and write $m_\epsilon(r) = m_\epsilon(x)$ and $\underline{\theta_\epsilon}(r) = \underline{\theta_\epsilon}(x)$. That is,

$$m_\epsilon(r) = \begin{cases} \frac{1}{\epsilon^n}, & \text{if } 0 \le r \le \epsilon, \\ \frac{c_2}{er_0^n}, & \text{if } \epsilon < r, \end{cases}$$

and

$$\underline{\theta_\epsilon}(r) = \begin{cases} \frac{c_2}{\epsilon^n} e^{-\frac{r^n}{\epsilon^n}}, & \text{if } 0 \le r \le \epsilon, \\ \frac{c_2}{er^n}, & \text{if } \epsilon < r \le r_0, \\ \frac{c_2}{er_0^n}, & \text{if } r_0 < r. \end{cases}$$

A simple calculation shows that

$$\underline{\theta_\epsilon'}(r) = \begin{cases} -\frac{nc_2 r^{n-1}}{\epsilon^{2n}} e^{-\frac{r^n}{\epsilon^n}}, & \text{if } 0 \le r \le \epsilon, \\ -\frac{nc_2}{er^{n+1}}, & \text{if } \epsilon < r < r_0, \\ 0, & \text{if } r_0 < r, \end{cases}$$

and

$$\underline{\theta_\epsilon}''(r) = \begin{cases} \frac{n(n-1)c_2 r^{n-2}}{\epsilon^{2n}} \left(\frac{nr^n}{(n-1)\epsilon^n} - 1\right) e^{-\frac{r^n}{\epsilon^n}}, & \text{if } 0 \le r < \epsilon, \\ \frac{n(n+1)c_2}{er^{n+2}}, & \text{if } \epsilon < r < r_0, \\ 0, & \text{if } r_0 < r, \end{cases}$$

where $'$ means d/dr. Then, the inequality (7) is equivalent to

$$d_\epsilon \left(\underline{\theta_\epsilon}'' + \frac{n-1}{r}\underline{\theta_\epsilon}'\right) + \underline{\theta_\epsilon}(m(r) - \underline{\theta_\epsilon}) \ge 0, \tag{8}$$

where $r \in \{r = |x| > 0 \mid x \in \Omega\backslash(\partial B_\epsilon \cup \partial B_{r_0})\}$. Note that the assumption $n \ge 2$ implies that $\underline{\theta_\epsilon}'(0) = 0$. The following part of the proof is same as that of [13], but for the convenience of readers we repeat the proof here. A straightforward calculation gives

$$\begin{aligned} &d_\epsilon \left(\underline{\theta_\epsilon}'' + \frac{n-1}{r}\underline{\theta_\epsilon}'\right) + \underline{\theta_\epsilon}(m(r) - \underline{\theta_\epsilon}) \\ &= e^{-\frac{r^n}{\epsilon^n}} \left(\frac{n^2 c_1 c_2 r^{2n-2}}{\epsilon^{4n-2}} - \frac{2n(n-1)c_1 c_2 r^{n-2}}{\epsilon^{3n-2}} + \frac{c_2}{\epsilon^{2n}} - \frac{c_2^2}{\epsilon^{2n}} e^{-\frac{r^n}{\epsilon^n}}\right) \\ &\ge \frac{c_2}{\epsilon^{2n}} - \frac{2n(n-1)c_1 c_2}{\epsilon^{2n}} - \frac{c_2^2}{\epsilon^{2n}} \\ &= \frac{c_2}{\epsilon^{2n}}(1 - 2n(n-1)c_1 - c_2) \end{aligned}$$

for $r \in (0, \epsilon)$, and

$$\begin{aligned} &d_\epsilon \left(\underline{\theta_\epsilon}'' + \frac{n-1}{r}\underline{\theta_\epsilon}'\right) + \underline{\theta_\epsilon}(m(r) - \underline{\theta_\epsilon}) \\ &= \frac{2nc_1 c_2}{e\epsilon^{n-2} r^{n+2}} + \frac{c_2^2}{e^2 r^n r_0^n} - \frac{c_2^2}{e^2 r^{2n}} \\ &\ge \frac{2nc_1 c_2}{e\epsilon^{n-2} r^{n+2}} - \frac{c_2^2}{e^2 r^{2n}} \\ &\ge \frac{c_2}{er^{n+2}} \left(\frac{2nc_1}{\epsilon^{n-2}} - \frac{c_2}{e\epsilon^{n-2}}\right) \end{aligned}$$

for $r \in (\epsilon, r_0)$. Note that, if $r > r_0$,

$$d_\epsilon \left(\underline{\theta_\epsilon}'' + \frac{n-1}{r}\underline{\theta_\epsilon}'\right) + \underline{\theta_\epsilon}(m(r) - \underline{\theta_\epsilon}) = 0.$$

Thus, (8) holds for $c_1, c_2 > 0$ satisfying

$$1 - 2n(n-1)c_1 - c_2 \ge 0 \quad \text{and} \quad 2nc_1 - \frac{c_2}{e} \ge 0. \tag{9}$$

It is easily checked that if (c_1, c_2) are in the triangle $T \subset \mathbb{R}^2$ whose vertices are

$$(0,0), \ \left(\frac{1}{2n(e+n-1)}, \frac{e}{e+n-1}\right), \ \left(\frac{1}{2n(n-1)}, 0\right),$$

then (9) holds. Therefore, $\underline{\theta_\epsilon}$ is a sub-solution of (2) if $(c_1, c_2) \in T$.

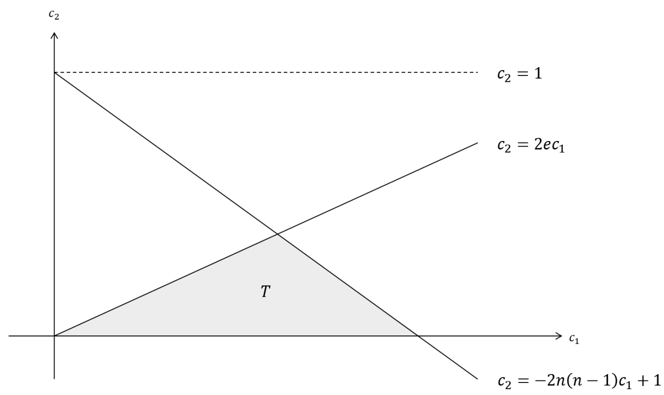

FIGURE 1. Region T : $\underline{\theta_\epsilon}$ is a sub-solution if $(c_1, c_2) \in T$

Fix a pair of constants $(c_1, c_2) \in T$. Then $c_2 < 1$, which implies that $\overline{\theta_\epsilon}$ is a super-solution and $\underline{\theta_\epsilon} < \overline{\theta_\epsilon}$. Since $0 < \underline{\theta_\epsilon}$ and $\overline{\theta_\epsilon} \equiv 1/\epsilon^n$, the condition (5) immediately follows from the fact that $m_\epsilon(x) \in L^\infty(\Omega)$. Then, there is a weak solution θ_ϵ of (2) satisfying

$$\underline{\theta_\epsilon} \leq \theta_\epsilon \leq \overline{\theta_\epsilon},$$

by Proposition 2.1. Since the weak solution of (2) is unique, we get $\theta_\epsilon = \theta_{d_\epsilon, m_\epsilon}$. On the other hand, by a direct calculation, we have

$$\begin{aligned}
\|\underline{\theta_\epsilon}\|_{L^1(\Omega)} &\geq \|\underline{\theta_\epsilon}\|_{L^1(B_{r_0} \setminus \overline{B_\epsilon})} \\
&\geq \omega_n \int_\epsilon^{r_0} \underline{\theta_\epsilon}(r) r^{n-1}\, dr \\
&\geq \frac{c_2 \omega_n}{e} \int_\epsilon^{r_0} \frac{1}{r}\, dr \\
&= \frac{c_2 \omega_n}{e} (\log r_0 - \log \epsilon),
\end{aligned}$$

and

$$\|m_\epsilon\|_{L^1(\Omega)} = |B_1| + \frac{c_2}{e r_0^n} |\Omega \setminus \overline{B_\epsilon}|,$$

where ω_n is the surface area of the unit sphere in $\mathbb{R}^n$ and $|\cdot|$ denotes the n-dimensional volume. Hence, there is a constant $C = C(n)$ depending only on n, such that

$$\frac{\|\underline{\theta_\epsilon}\|_{L^1(\Omega)}}{\|m_\epsilon\|_{L^1(\Omega)}} \geq C|\log \epsilon|.$$

Then, we obtain

$$\lim_{\epsilon \to 0} E(m_\epsilon) \geq \lim_{\epsilon \to 0} \frac{\|\theta_{d_\epsilon, m_\epsilon}\|_{L^1(\Omega)}}{\|m_\epsilon\|_{L^1(\Omega)}}$$

$$
\begin{aligned}
&\geq \lim_{\epsilon\to 0} \frac{\|\theta_\epsilon\|_{L^1(\Omega)}}{\|m_\epsilon\|_{L^1(\Omega)}}\\
&\geq \lim_{\epsilon\to 0} C|\log\epsilon|\\
&= \infty,
\end{aligned}
$$

which completes the proof since every m_ϵ satisfies the condition (M). $\square$

3. Concluding remarks

In the study of the Lotka-Volterra competition model, it is known that the ratio $E(m)$ plays an important role in the dynamics of the system. Consider the following two-species competition model:

$$
\begin{cases}
\frac{\partial U}{\partial t} = d_1\Delta U + U(m(x) - U - cV) & \text{in } \Omega\times(0,\infty),\\
\frac{\partial V}{\partial t} = d_2\Delta V + V(m(x) - bU - V) & \text{in } \Omega\times(0,\infty),\\
\frac{\partial U}{\partial \nu} = \frac{\partial V}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,\infty),\\
U(x,0) = U_0(x),\ \ V(x,0) = V_0(x) & \text{in } \Omega,
\end{cases} \tag{10}
$$

with $0 < b, c < 1$, where b and c represent interspecific competition coefficients. Define

$$\Sigma_U = \{(d_1, d_2) \in (0,\infty)\times(0,\infty) \mid (\theta_{d_1,m}, 0) \text{ is linearly stable}\}.$$

In [17], Lou proved the following:

Theorem 3.1. *Assume that m satisfies the condition* (M)*. Then,*

$$\Sigma_U \neq \emptyset \quad \textit{if and only if} \quad b > \frac{1}{E(m)}.$$

Moreover, there exists $c_* = c_*(b,\Omega,m) \in (0,1]$ such that if $c \in (0, c_*)$ and $(d_1,d_2) \in \Sigma_U$, then $(\theta_{d_1,m}, 0)$ is globally asymptotically stable.

In addition, Lou [17] conjectured that the theorem above holds for any $c \in (0,1)$. Lam and Ni [16] proved Lou's conjecture for small c, without the dependence of $b \in (0,1)$. More recently, He and Ni [10] completely solved the conjecture. In [10], the authors assumed that $b, c > 0$ are in the range $bc \leq 1$ and analyzed the global dynamics of system (10). The results on the global dynamics are divided according to the values of b and c. Here, the value $1/E(m)$ was used as the threshold. For example,

(i) if $b, c \in (0, 1/E(m)]$, then for all $d_1, d_2 > 0$, (10) has a unique coexistence steady state that is globally asymptotically stable,

(ii) if $b \in (1/E(m), 1]$ and $c \in (0, 1/E(m)]$, then either $(\theta_{d_1,m}, 0)$ is globally asymptotically stable or (10) has a unique coexistence steady state that is globally asymptotically stable, refer to Theorem 1.1 in [10] for more details. Thus, their result tells us that "diffusion-driven exclusion" phenomenon occurs only when the competition coefficient b or c is larger than the threshold value $1/E(m)$.

Our result shows that, for $n \geq 2$, resource concentration sends the value $E(m)$ to ∞. It can be interpreted as the more resources are concentrated in one place, the more likely extinction will occur despite weaker competition between the two species. It is biologically reasonable because the concentration of resources accelerates competition.

Very recently, Mazari and Ruiz-Balet [21] observed a fragmentation phenomenon for small diffusion rates. Roughly, this phenomenon means that in order to increase the total biomass, the smaller the diffusion rate, the more resources must be fragmented. They dealt with the typical domain n-dimensional box,

$$\Omega = \prod_{i=1}^{n}[0, 1].$$

On this domain, for given resource m and the corresponding solution $\theta_{d,m}$, the authors constructed fragmented resources $\{m_k\}_{k=1}^{\infty}$ such that

$$\int_\Omega \theta_{\frac{d}{4^k}, m_k}\, dx = \int_\Omega \theta_{d,m}\, dx, \tag{11}$$

where $k = 1, 2, 3, \ldots$. For the precise construction and the proof, see [21]. Note that, in our main theorem, we chose

$$d_\epsilon = \frac{c_1}{\epsilon^{n-2}},$$

which goes to ∞ as $\epsilon \to 0$ if $n \geq 3$. Thus, if Ω is a box, we can reselect $\{m_\epsilon\}$ with a fixed diffusion rate such that

$$E(m_\epsilon) \to \infty,$$

using (11). From this fact, it can be seen that the large diffusion rate is probably not a crucial factor in sending the ratio $E(m)$ to ∞.

Additionally, it is natural to consider the problem of maximizing the total biomass when the diffusion rate d is fixed and the amount of resources is limited. This problem was suggested by Lou [18]. In [18], he mentioned without the proof that the total biomass is unbounded for high dimensional habitats if m is in the class

$$\{m \in L^\infty(\Omega) \mid m \geq 0, \int_\Omega m\, dx = m_0|\Omega|\},$$

where $m_0 > 0$ is a given real number. It can be supported by our main theorem (or by Theorem 2.2 in [13]) for $n = 2$, and by the above remark for $n \geq 3$. Accordingly, for this problem, the following smaller class was introduced in [18]:

$$\mathcal{M} = \{m \in L^\infty(\Omega) \mid 0 \leq m \leq \kappa, \int_\Omega m\, dx = m_0|\Omega|\},$$

where $\kappa > 0$ and $m_0 \in (0, \kappa)$. This class was used in [4, 20–22], and in [4], it is proved that there exists a maximizer of total biomass on $\mathcal{M}$. However, regarding qualitative properties of the maximizer, very few things are known, see [20–22]. The next possible step will be the study of this direction and we hope to discuss this issue in the future work.

Acknowledgement. This work was supported by 2020 Long-Term KAIST Undergraduate Research Program under the guidance of Professor Jaeyoung Byeon. The second author was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MIST) (No. NRF-2019R1A5A1028324).

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Junyoung Heo
Department of Mathematical Sciences
KAIST
Daejeon 34141, Korea

Yeonho Kim
Department of Mathematical Sciences
KAIST
Daejeon 34141, Korea
Email address: yho0922@kaist.ac.kr