



Original Article

Inverse method to obtain reactivity in nuclear reactors with P1 point reactor kinetics model using matrix formulation

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ABSTRACT

The aim of this work considers a second order point reactor kinetics model based on the P1 approximation of transport theory, called in this work as P1 point reactor model. The P1 point reactor model implicitly considers the time derivative of the neutron source which has not been thus considered previously. The inverse method to calculate the reactivity in nuclear reactors -chosen because its high accuracy- Matrix Formulation. The numerical results shown that the Matrix Formulation for the reactivity estimation constitutes a method with insignificant calculation errors.

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1. Introduction

The essential process at the core of a nuclear reactor is nuclear fission. When heavy atoms like Uranium 235 are bombarded with thermal neutrons, a chain reaction is produced, releasing a great amount of energy that is later converted into electric energy [1].

The equations of point kinetics express the temporal variation of the neutron density and the concentration of the delayed neutron precursors. In practice, it is convenient to maintain control of the reactor by determining the reactivity, depending upon the neutron density shape, for example, in a power excursion this is exponential and the reactivity value may be incorrect, ignoring the real value. For this reason, start recalling there are different methods for classical reactivity calculation [2–5] that implement the point kinetic equations. Likewise, there are methods in the literature that solve for reactivity, such as the Lagrange polynomial method [6] and another method that do not use the historic of the neutron density [7]. In a recent publication, the best accuracy is obtained with the matrix formulation method [8].

In the last few years, the point kinetic equations have been modified according to the needs of a specific type of reactor. A

fractional point kinetic model was derived that consist in three terms between derivatives of order non-integer [9]. After, the transport equation was used, considering the time variation of the neutron density current, achieving a modified point-kinetic equation [10]. These works led to the modified reactivity was calculated as a function of the classical reactivity [11], employing the derivative method [7]. In that same year, a work was published where the point kinetic equation was modeled using the neutron telegraph equation [12]. Finally, based on the neutron telegraph equation, the modified reactivity was calculated again using the derivative method [13].

The aim of this work is to present, for the first time, a matrix formulation method [8] for the calculation of the modified reactivity with an inverse method that considers a second-order point reactor kinetics model, based on the P1 approximation of transport theory, called in this work P1 point reactor model. We show that the matrix formulation method is simple, reliable and precise. The only existing method, known as the derivative method, may have drawbacks as it is only valid for seven forms of neutron density [7], while the matrix formulation method does not have that limitation [8].

This work is organized as follows: the second section shows how to obtain the modified point kinetic equation, and how its inverse is used to calculate the P1 point kinetic reactivity. The third section presents the matrix method [8] used to solve the P1 point

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kinetic reactivity. In the fourth section, we show the maximum differences of the P1 point kinetic reactivity and the classical reactivity, using the matrix formulation method. In the last section some results are presented.

2. Theoretical considerations

2.1. Preliminaries

It is common to find that the point reactor model of neutron kinetics is obtained from the one-speed diffusion equation, e.g., Ref. [1]. In this sense, all neutrons are considered to have the same energy, which is known as one-speed approximation or one-group of energy approximation. The derivation of the neutron flux $\varphi(\mathbf{r}, t)$ is obtained by considering that the rate of change of the neutron density in a given volume is the difference between the rate of neutrons produced and lost due to absorption or leakage in that given volume:

$$\frac{1}{v} \frac{\partial \varphi(\mathbf{r}, t)}{\partial t} + \nabla \cdot \mathbf{J}(\mathbf{r}, t) + \sum_a(\mathbf{r}) \varphi(\mathbf{r}, t) = S(\mathbf{r}, t) \quad (1)$$

where v is neutron velocity, $\sum_a(\mathbf{r})$ is the macroscopic absorption cross section. The relation between $\varphi(\mathbf{r}, t)$ and $\mathbf{J}(\mathbf{r}, t)$ is given by

$$\mathbf{J}(\mathbf{r}, t) = -D(\mathbf{r}) \nabla \varphi(\mathbf{r}, t) \quad (2)$$

This equation was unfortunately referred to as Fick's Law: the behavior in the neutrons transport is a different phenomenon than molecular transport, besides of abysmal differences in space-time scales and mean free trajectory. The diffusion coefficient $D(\mathbf{r})$ is defined as:

$$D(\mathbf{r}) = \frac{1}{3 \Sigma_{tr}(\mathbf{r})} \quad (3)$$

The macroscopic transport cross section $\Sigma_{tr}(\mathbf{r})$ is given by

$$\Sigma_{tr}(\mathbf{r}) = \Sigma_t(\mathbf{r}) - \bar{\mu}_0 \Sigma_s(\mathbf{r}) \quad (4)$$

where $\Sigma_t(\mathbf{r})$ is the macroscopic total cross section, $\Sigma_s(\mathbf{r})$ is the macroscopic scattering cross section, and $\bar{\mu}_0$ is the averaging scattering angle cosine.

Replacing Eq. (2) into Eq. (1), leads to

$$\frac{1}{v} \frac{\partial \varphi(\mathbf{r}, t)}{\partial t} + \sum_a(\mathbf{r}) \varphi(\mathbf{r}, t) = S(\mathbf{r}, t) + D(\mathbf{r}) \nabla^2 \varphi(\mathbf{r}, t) \quad (5)$$

which is known as the *diffusion equation*. Now, by applying the procedure given in Ref. [1], the point kinetics equations can be obtained:

$$\frac{dn(t)}{dt} = \frac{\rho(t) - \beta}{\Lambda} n(t) + \sum_{i=1}^m \lambda_i c_i(t) \quad (6)$$

where $n(t)$ is the neutron density, $\rho(t)$ is the reactivity, β is the total effective fraction for delayed neutron precursors, λ_i is the decay constant of the i th group of delayed neutron precursors and $c_i(t)$ is the concentration of the i th group of delayed neutron precursors, given by the radioactive disintegration law, which is discussed in detail in the following section. In order to obtain this equation it was necessary to consider a homogeneous and uniform reactor – like a batch reactor, so that the diffusion coefficient is not a function of the position \mathbf{r} .

2.2. P1 point reactor model

The current density for the one-speed model is given by [1].

$$\begin{aligned} \frac{1}{v} \frac{\partial \mathbf{J}}{\partial t} + \nabla \cdot \int_{4\pi} \hat{\Omega} \hat{\Omega} \phi(\mathbf{r}, \hat{\Omega}, t) d\hat{\Omega} + \Sigma_t(\mathbf{r}) \mathbf{J}(\mathbf{r}, t) \\ = \bar{\mu}_0 \Sigma_s(\mathbf{r}) \mathbf{J}(\mathbf{r}, t) + S_1(\mathbf{r}, t) \end{aligned} \quad (7)$$

where $\bar{\mu}_0$ is the averaging scattering angle cosine, and $\phi(\mathbf{r}, \hat{\Omega}, t)$ is the angular flux, which is approximated by:

$$\phi(\mathbf{r}, \hat{\Omega}, t) \cong \frac{1}{4\pi} \varphi(\mathbf{r}, t) + \frac{3}{4\pi} \mathbf{J}(\mathbf{r}, t) \cdot \hat{\Omega} \quad (8)$$

Then, the current vector can be written as

$$\frac{1}{v} \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} + \frac{1}{3} \nabla \varphi(\mathbf{r}, t) + \sum_{tr}(\mathbf{r}) \mathbf{J}(\mathbf{r}, t) = S_1(\mathbf{r}, t) \quad (9)$$

Eqs. (1) and (9) are known as P1 equations, where Eq. (8) is equivalent to expanding the angular flux in Legendre polynomials. Now, considering that the neutron source term is isotropic, i.e., $S_1(\mathbf{r}, t) = 0$, this equation can be rewritten as:

$$\tau(\mathbf{r}) \frac{\partial \mathbf{J}(\mathbf{r}, t)}{\partial t} + \mathbf{J}(\mathbf{r}, t) = -D(\mathbf{r}) \nabla \varphi(\mathbf{r}, t) \quad (10)$$

where the term $\tau(\mathbf{r})$ can be defined as *relaxation time*, given by

$$\tau(\mathbf{r}) = \frac{3D(\mathbf{r})}{v} \quad (11)$$

Eq. (10) considers memory effects [9], and Eq. (2) considers that the phenomena is instantaneous.

If we apply the divergence operator ($\nabla \cdot$) to Eq. (10), it leads to

$$\left(\tau \frac{\partial}{\partial t} + 1 \right) \nabla \cdot \mathbf{J}(\mathbf{r}, t) = -D \nabla^2 \varphi(\mathbf{r}, t) \quad (12)$$

for a homogeneous reactor where $\tau, D \neq f(\mathbf{r})$. Now, using Eq. (1), we can determine the term $\nabla \cdot \mathbf{J}(\mathbf{r}, t)$:

$$\nabla \cdot \mathbf{J}(\mathbf{r}, t) = -\frac{1}{v} \frac{\partial \varphi(\mathbf{r}, t)}{\partial t} - \sum_a \varphi(\mathbf{r}, t) + S(\mathbf{r}, t) \quad (13)$$

Substituting this equation into Eq. (12):

$$\left(\tau \frac{\partial}{\partial t} + 1 \right) \left(-\frac{1}{v} \frac{\partial \varphi(\mathbf{r}, t)}{\partial t} - \sum_a \varphi(\mathbf{r}, t) + S(\mathbf{r}, t) \right) = -D \nabla^2 \varphi(\mathbf{r}, t) \quad (14)$$

performing the indicated operations and grouping terms, we obtain:

$$\begin{aligned} \frac{\tau}{v} \frac{\partial^2 \varphi(\mathbf{r}, t)}{\partial t^2} + \left[\tau \sum_a + \frac{1}{v} \right] \frac{\partial \varphi(\mathbf{r}, t)}{\partial t} + \sum_a \varphi(\mathbf{r}, t) = S(\mathbf{r}, t) + D \nabla^2 \varphi(\mathbf{r}, t) \\ + \tau \frac{\partial S(\mathbf{r}, t)}{\partial t} \end{aligned} \quad (15)$$

which is a hyperbolic PDE. When $\tau = 0$ it can be observed that the PDE is converted into a parabolic one, and the diffusion equation is recovered.

The neutrons produced $S(\mathbf{r}, t)$ includes instantaneous and delayed neutrons:

$$S(\mathbf{r}, t) = (1 - \beta)k_{\infty}\Sigma_a\varphi(\mathbf{r}, t) + \sum_{i=1}^m \lambda_i C_i(\mathbf{r}, t) \quad (16)$$

where β is the total effective fraction of delayed neutrons which, k_{∞} is the infinite multiplication factor, λ_i is the decay constant of the i th precursor group. The concentration of the i th precursor group $C_i(\mathbf{r}, t)$ is given by:

$$\frac{\partial C_i(\mathbf{r}, t)}{\partial t} = \beta_i k_{\infty} \sum_a \varphi(\mathbf{r}, t) - \lambda_i C_i(\mathbf{r}, t) \quad i = 1, 2, \dots, m \quad (17)$$

where β_i is the fraction of delayed neutrons of i th precursor group.

According with Hayasaka and Takeda [14], Eq. (15) was previously described by Beckurts and Wirtz [15] and ASH [16]. However, the concept of relaxation time τ was not treated as such in those works. On the other hand, in the work of Niederauer [17], the relaxation time τ was defined as *finite transport time* and the symbol used was τ_0 .

Applying variable separation for the solution $\varphi(\mathbf{r}, t) = vn(t)\psi(\mathbf{r})$, where $\psi(\mathbf{r})$ correspond to the eigen function of the Helmholtz equation. This separation the variables is also applied to the neutron source and the concentration of the i th precursor group: $S(\mathbf{r}, t) = s(t)\psi(\mathbf{r})$ and $C(\mathbf{r}, t) = c(t)\psi(\mathbf{r})$, respectively. It is important to note that lowercase is used for variables which depend only on time. A three dimensional reactor is governed by the Helmholtz equation: $\nabla^2\psi(\mathbf{r}) = -B_g^2\psi(\mathbf{r})$, where B_g is a parameter called *geometrical buckling*. The space eigen functions are determined as the solution to the eigenvalue problem $\nabla^2\psi_{\eta}(\mathbf{r}) + B_{\eta}^2\psi_{\eta}(\mathbf{r}) = 0$. The parameter geometrical buckling is characterized by its dependence on the geometry of the reactor and is the eigenvalue when $\eta = 1$. Then, with these fundamental considerations and the well-mixed reactor assumption, the *P1 point reactor model* is obtained:

$$\tau \frac{d^2 n(t)}{dt^2} + (\tau v \Sigma_a + 1) \frac{dn(t)}{dt} = \frac{\rho(t) - \beta}{\Lambda} n(t) + \sum_{i=1}^m \lambda_i c_i(t) + \tau \frac{ds(t)}{dt} \quad (18)$$

$$s(t) = \frac{(1 - \beta)}{\Lambda} n(t) + \sum_{i=1}^m \lambda_i c_i(t) \quad (19)$$

$$\frac{dc_i(t)}{dt} = \frac{\beta_i}{\Lambda} n(t) - \lambda_i c_i(t) \quad i = 1, 2, \dots, m \quad (20)$$

Some definitions involved in these equations are the *reactivity* $\rho = (k - 1)k^{-1}$, where k is the *effective multiplication factor*; the *neutron generation time* $\Lambda = P_{nl}(k v \Sigma_a)^{-1}$, and the nonleakage probability given by $P_{nl} = (1 + L^2 B_g^2)^{-1}$ where L is the *diffusion length* ($D \Sigma_a^{-1}$).

Unlike Eq. (18), Niederauer in 1967 expresses the equation in terms of the time derivative of the concentration of the precursors group [17], and not in terms of the time derivative of the neutron source. Other more recent works such as [10–13] are essentially based on Niederauer's work [17]. We can observe on Eq. (18) that if $\tau \rightarrow 0$, this expression becomes the conventional point kinetic equation given by Eq. (6).

Next, we show the inverse point kinetic equation method, with the objective of reaching a formulation of the temporal reactivity form.

3. Calculating the reactivity from the inverse kinetics

Since reactivity is an important parameter in a nuclear reactor, it is important to know it with the highest precision possible, therefore in this section, we deduce the reactivity for the classical point kinetic Eq. (6) and the reactivity for the *P1 point reactor model*, given by Eq. (18).

From Eq. (6) we obtain the classical reactivity expression, which is related to the neutron density in the following way:

$$\rho_{class} = \beta + \frac{\Lambda}{n(t)} \frac{dn(t)}{dt} - \frac{\Lambda}{n(t)} \sum_{i=1}^m \lambda_i c_i(t) \quad (21)$$

This equation allows to calculate the reactivity when knowing the neutron density and the concentration of precursors, which has been used in different works e.g. Ref. [1–8]. In this work ρ_{class} is called the *classical reactivity*.

The expression for the *P1 point kinetic reactivity* ρ_{p1} which is obtained from Eq. (18), considers the neutron source given by Eq. (19):

$$\rho_{p1} = \beta + \frac{\Lambda}{n(t)} \left[\tau \frac{d^2 n(t)}{dt^2} + (\tau v \Sigma_a + 1) \frac{dn(t)}{dt} - \sum_{i=1}^m \lambda_i c_i(t) - \tau \frac{ds(t)}{dt} \right] \quad (22)$$

which can be re-written in terms of ρ_{class} :

$$\rho_{p1} = \rho_{class} + \frac{\Lambda \tau}{n(t)} \left[\frac{d^2 n(t)}{dt^2} + v \Sigma_a \frac{dn(t)}{dt} - \frac{ds(t)}{dt} \right] \quad (23)$$

where

$$\frac{ds(t)}{dt} = \frac{(1 - \beta)}{\Lambda} \frac{dn(t)}{dt} + \frac{n(t)}{\Lambda} \sum_{i=1}^m \lambda_i \beta_i - \sum_{i=1}^m \lambda_i^2 c_i(t) \quad (24)$$

This equation is obtained from the time derivative of Eq. (19) and the result is combined with Eq. (20).

The classical reactivity ρ_{class} given by Eq. (21) and the *P1 point reactor reactivity* ρ_{p1} given by Eq. (23), need the value of precursors concentration. The exact shape of the precursor concentration $c_i(t)$, is solved using Eq. (20) through the integrating factor and having into account the initial condition $c_i(0) = \beta_i n(0) / \lambda_i \Lambda$. We get:

$$c_i(t) = \frac{\beta_i}{\Lambda} \left[\frac{e^{-\lambda_i t}}{\lambda_i} n(0) + \int_0^t e^{\lambda_i(t-t')} n(t') dt' \right] \quad (25)$$

where $n(0)$ is the initial condition for the neutron density, which marks the starting point in the simulations ($t = 0$). For some shapes of the neutron density $n(t)$, the integral in Eq. (25) can be solved analytically; which means that we can compare with our proposed method.

We observe on Eq. (23) that the *P1 point reactor kinetic reactivity* ρ_{p1} includes the classical reactivity in Eq. (21), therefore we can define this reactivity as:

$$\rho(t)_{p1} = \rho(t)_{class} + \tau \rho_{nc}(t) \quad (26)$$

where the term $\rho_{nc}(t)$ is a function of the derivative with respect to time of the neutron source:

$$\rho_{nc}(t) = \frac{\Lambda}{n(t)} \left[\frac{d^2 n(t)}{dt^2} + v \Sigma_a \frac{dn(t)}{dt} - \frac{ds(t)}{dt} \right] \quad (27)$$

The criticality condition of the reactor is fulfilled with the initial condition at $t = 0$, so that, $\rho_{p1}(0) = \rho_{class}(0) = 0$.

4. Numerical method to solve the classical reactivity ρ_{class} and P1 reactivity ρ_{p1}

The methods that can be used to calculate the classical reactivity ρ_{class} and the P1 point reactor kinetic reactivity ρ_{p1} should be as accurate and with the lowest computational cost. A method reported in the literature that meet these characteristics is the matrix formulation method [8].

In this work, we use the point kinetics equation given by Eqs. (6) and (20) to obtain the precursor concentration $c_i(t)$ with the matrix formulation method [8] knowing the form of the neutron density, which is known through the detection systems of nuclear power plants (e.g., power range neutron monitor, local power range monitor and average power range monitor). So, by substituting this value in Eq. (21) and Eq. (24), we can solve for ρ_{class} and ρ_{p1} .

In order to present the numerical method, we rewrite Eqs. (6) and (20) including the initial conditions:

$$\frac{dn(t)}{dt} = \frac{\rho - \beta}{\Lambda} n(t) + \sum_{i=1}^m \lambda_i c_i(t) \tag{28}$$

$$\frac{dc_i(t)}{dt} = \frac{\beta_i}{\Lambda} n(t) - \lambda_i c_i(t); \quad i = 1, 2, \dots, m \tag{29}$$

$$n(t=0) = n(0) \tag{30}$$

$$c_i(t=0) = \frac{\beta_i}{\Lambda \lambda_i} n(0) \tag{31}$$

When the matrix formulation method is applied to these equations it reduces them one homogeneous first order differential equation, such that:

$$\frac{dX(t)}{dt} = Q(t)X(t), \quad X(0) = x_0 \tag{32}$$

where $\frac{dX(t)}{dt}$ and $X(t)$ are vector functions of $m + 1$ dimensions, with their respective initial condition $n(0)$ at $X(0)$ and its matrix function $Q(t)$ of $(m + 1) \times (m + 1)$ dimensions:

$$\begin{aligned} \frac{dX(t)}{dt} &= \frac{d}{dt} \begin{bmatrix} n(t) \\ c_1(t) \\ c_2(t) \\ \vdots \\ c_m(t) \end{bmatrix}, \quad X(t) = \begin{bmatrix} n(t) \\ c_1(t) \\ c_2(t) \\ \vdots \\ c_m(t) \end{bmatrix}, \quad X(0) \\ &= n(0) \begin{bmatrix} 1 \\ \beta_1(\Lambda\lambda_1)^{-1} \\ \beta_2(\Lambda\lambda_2)^{-1} \\ \vdots \\ \beta_m(\Lambda\lambda_m)^{-1} \end{bmatrix} \end{aligned} \tag{33}$$

$$Q(t) = \begin{bmatrix} n'(t)n(t)^{-1} & 0 & 0 & \dots & 0 \\ \beta_1\Lambda^{-1} & -\lambda_1 & 0 & \dots & 0 \\ \beta_2\Lambda^{-1} & 0 & -\lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \beta_m\Lambda^{-1} & 0 & 0 & \dots & -\lambda_m \end{bmatrix} \tag{34}$$

It can be verified that by multiplying the first row of the matrix $Q(t)$ given by Eq. (34) with the column vector $X(t)$ given by Eq. (33), the time derivative of the density of the neutron population is obtained, which agrees with Eq. (32), that is: $\frac{dn(t)}{dt} = \frac{n'(t)}{n(t)} * n(t) = \frac{dn(t)}{dt}$. By performing the other multiplications, it is possible to verify that the equations of the precursor concentration given by Eq. (29) can be obtained. The homogenous system with the matrix function $Q(t)$ given by Eq. (32) is an initial value problem that can be solved for any time using the exponential matrix. The solution for Eq. (32) is such that:

$$X_{k+1} = X_k e^{\int_{t_k}^{t_{k+1}} Q(t) dt} \tag{35}$$

where X_k is the value of the vector function at a time t_k and X_{k+1} is the value for a later time t_{k+1} . The term in the exponential matrix is solved by integrating each element of the matrix, thus creating a new matrix $M_k = \int_{t_k}^{t_{k+1}} Q(t) dt$. Then:

$$M_k = \begin{bmatrix} \text{Ln} \begin{bmatrix} n_{k+1} \\ n_k \end{bmatrix} & 0 & 0 & \dots & 0 \\ \frac{\beta_1}{\Lambda} h & -\lambda_1 h & 0 & \dots & 0 \\ \frac{\beta_2}{\Lambda} h & 0 & -\lambda_2 h & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \frac{\beta_m}{\Lambda} h & 0 & 0 & \dots & -\lambda_m h \end{bmatrix} \tag{36}$$

where h is the time step given by $h = t_{k+1} - t_k$ and Ln is the natural logarithm.

Developing this expression, we get to the solution:

$$X_{k+1} = R_k e^{D_k} R_k^{-1} X_k \tag{37}$$

where R_k is a matrix which columns are the eigenvectors of M_k , D_k is a diagonal matrix where the elements on its diagonal are the eigenvalues of M_k . This solution is computed for every time step using the matrix and vector multiplications R_k , D_k and R_k^{-1} .

To reduce the computational cost, the matrices are calculated analytically: $M_k U_m = \omega_m U_m$, where M_k is a lower triangular matrix, ω_m are the eigenvalues, and U_m are eigenvectors $m = 1, 2, \dots, k + 1$, with which the matrix is built $R_k = [U_1 \ U_2 \ \dots \ U_{m+1}]$. Then, the diagonal matrix D_k is defined by:

$$D_k = \text{diag} \left[\text{Ln} \begin{bmatrix} n_{k+1} \\ n_k \end{bmatrix} \quad -\lambda_1 h \quad \dots \quad \lambda_m h \right] \tag{38}$$

The solution for the elements u_{ij} of the matrix R_k and the elements v_{ij} of the inverse of the eigenvector matrix R_k are calculated

from the expression $R_k R_k^{-1} = I$. All are computed in each time step as follows:

$$u_{ij} = u_{ij} = \delta_{ij}, \text{ for } i, j = 1, 2, \dots, m + 1 \quad (39)$$

when $j = 1$ we obtain:

$$u_{i1} = -v_{i1} = \frac{\beta_{i-1} h}{\Lambda \left(\lambda_{i-1} h + \Lambda n \left[\frac{n_{k+1}}{n_k} \right] \right)} \text{ for } i = 2, 3, \dots, m + 1 \quad (40)$$

The approximation for the delayed neutron precursor concentrations using Eq. (37) will be replaced into Eq. (21) for the calculation of classical reactivity ρ_{class} and into Eq. (23) for the calculation of the *P1 point reactor kinetic reactivity* ρ_{p1} .

5. Results

Next, the numerical results for the calculation of the *P1 point kinetic reactivity* ρ_{p1} from Eq. (23) are presented using the method presented on the previous section. All numerical computations are performed for different simulation times $t(s)$, $\Lambda = 2 \times 10^{-5} s$, $v = 5 \times 10^5 \text{ cm/s}$, $\Sigma_a = 1.39 \times 10^{-2} \text{ cm}^{-1}$, $\tau = 6 \times 10^{-5} s$. The maximum difference in reactivity between two variables X and Y will be denoted as $\text{Max}|X - Y|$, being $|X - Y|$ the absolute value of the difference between X and Y . The computation is realized using the proposed method given by Eq. (21) and Eq. (23) for classical reactivity (ρ_{class}) and the *P1 point reactor kinetic reactivity* (ρ_{p1}), respectively. And the reference method is given by the analytical solution of these equations and in order to differentiate them they are denoted as $\rho_{class.ref}$ and $\rho_{p1.ref}$. For the different numerical computations, the dominant term in Eq. (23) is associated with the precursor concentration term. As there are two terms, the term with the highest domain being the term associated with the classical reactivity. Assuming that the form for neutron density is known $n(t)$, therefore the first and second derivatives are known and calculated analytically, i.e., they are not calculated numerically, ensuring that the comparison is made under the same conditions, depending exclusively on precision of the method and its validity according to the shape of the neutron density. Six groups of delayed neutron precursors ($m = 6$) were considered and also the nuclear parameters shown on Table 1 which are typical for a classical reactivity.

In order to study the accuracy in the calculation of reactivity, the following numerical computations were performed considering the case of a neutron density represented by $n(t) = e^{wt}$, with different values for w , and time steps of $h = 1s$ and $h = 1.5s$ respectively.

Tables 2 and 3 show the maximum difference in pcm between classical reactivity (ρ_{class}) -using the matrix formulation method-

and the reference reactivity ($\rho_{class.ref}$) calculated exactly by Eq. (21), also shown, the maximum difference in pcm between the *P1 point reactor kinetic reactivity* (ρ_{p1}) using the matrix formulation method and the *reference P1 point reactor kinetic reactivity* ($\rho_{p1.ref}$) calculated exactly using Eqs. 23–25. The source term ($ds(t)/dt$) has an important effect on the results obtained since the absence of a neutron source for this form of neutron density implies a maximum difference of $1.83 \times 10^{-2} \text{ pcm}$.

In order to analyze the differences between the classic reactivity and the *P1 point kinetic reactivity*, the following numerical computations are performed. Table 4 shows values obtained for the maximum difference between the classical reactivity and the *P1 point kinetic reactivity* for a time step of $h = 1s$. The differences may increase depending on the value w , for $w = 52.80352s^{-1}$ the maximum difference is close to 270 pcm with a source term and 44.4 pcm without a source term. Fig. 1 shows the reactivity obtained by an exponential form $n(t) = e^{wt}$ for the neutron density with $w = 0.12353$ and $h = 1s$, where the maximum difference is close to 0.6 pcm with a source term and 0.1 pcm without a source term. Fig. 2 shows the reactivity obtained with the same exponential shape for the neutron density, but with a higher value of $w = 11.6442$ and a time step $h = 0.1s$, the maximum difference reached a value of 59.6 pcm with a source term and 9.7 pcm without a source term. Apparently, this difference is not as important in the time domain, however, in the development of control strategies, the source term is crucial in the stability of the system as previously demonstrated [18].

The differences presented occur from a time Δt very close to the initial time value where the reactor is critical, from equation (26) it is obtained $\rho_{p1}(\Delta t) \approx \rho_{class}(\Delta t) + \tau \rho_{nc}(\Delta t)$, the classic reactivity is given by $\rho_{class}(\Delta t) \approx \Lambda w$.

Now, for the case without a source $\rho_{p1}(\Delta t) \approx \Lambda w + \tau[\Lambda w^2 + \Lambda v \Sigma_a w]$ and with a source $\rho_{p1}(\Delta t) \approx \Lambda w + \tau[\Lambda w^2 + \Lambda v \Sigma_a w - (1 - \beta)w]$, considering the value of $w = 11.6442$ it is obtained $\rho_{class}(\Delta t) \approx \Lambda w = 23.29 \text{ pcm}$, $\rho_{p1}(\Delta t) \approx 23.29 + 9.73 = 33.02 \text{ pcm}$ and $\rho_{p1}(\Delta t) \approx 23.29 - 59.65 = -36.36 \text{ pcm}$ for the cases of no source and source, respectively, the differences $|\rho_{p1}(\Delta t) - \rho_{class}(\Delta t)| \approx \tau \rho_{nc}(\Delta t)$ are 9.73 pcm and 59.65 pcm for the case without a source and the case with a source, respectively. These differences found within a time Δt , keeps almost constant during the whole simulation time.

It is important to point out that these differences in the calculation of reactivity may still increase, since the *P1 point kinetic reactivity* is proportional to the relaxation time τ given in Eq. (26). The contribution for the case considered in this article is a factor of 6, whether or not the source term is considered. Such high differences are due to the exponential shape of the neutron density, the changes of the first and second derivatives divided respectively by the neutron density can increase significantly.

Another numerical computation is to consider the linear form for the neutron density. Table 5 shows the maximum differences in reactivity for a neutron density described by $n(t) = a + bt$, with a time step of $h = 3s$, $t = 1000 \text{ s}$, the parameter $a = 50$ and different values for b . For the classical and *P1 point kinetic reactivity* form we observe that the matrix formulation method maintains the same precision for the different values of b , the precision for the classical and *P1 point kinetic reactivities* is 10^{-13} . Again, when the neutron source term is considered for this form of neutron density, the same maximum difference of $1.83 \times 10^{-2} \text{ pcm}$ is obtained.

In the next numerical computation the neutron density is given by $n(t) = a + bt^3$, where $t = 10 \text{ s}$, $h = 0.1s$ and $a = 1$, the results are shown on Table 6 for a given value of b . There are changes for both methods (the classical and *P1 point kinetic reactivity*). When $b = (0.0127)^4/4$, the matrix formulation maintains a precision of $1.22 \times 10^{-7} \text{ pcm}$. With the source term, it provides an equal

Table 1
Kinetics parameters for the simulation.

| Groups | $\lambda_i (s^{-1})$ | β_i |
|----------------------|-----------------------|-----------|
| 1 | 0.0127 | 0.000266 |
| 2 | 0.0317 | 0.001491 |
| 3 | 0.115 | 0.001316 |
| 4 | 0.311 | 0.002849 |
| 5 | 1.400 | 0.000896 |
| 6 | 3.870 | 0.000182 |
| $\Lambda (s)$ | 2.0×10^{-5} | |
| β | 0.007 | |
| $\Sigma_a (cm^{-1})$ | 1.39×10^{-2} | |
| $D (cm)$ | 10 | |
| $v (cm/s)$ | 5×10^5 | |
| $\tau (s)$ | 6×10^{-5} | |

Table 2
Maximum differences in reactivity in pcm for $n(t) = \exp(wt)$ with $h = 1s$.

| w (s^{-1}) | t (s) | $Max \rho_{class} - \rho_{class_ref} $ (pcm) | | $Max \rho_{P1} - \rho_{P1_ref} $ (pcm) | |
|------------------|---------|------------------------------------------------|--|------------------------------------------|------------------------|
| | | MF | | MF With source term $ds(t)/dt$ | MF Without source term |
| 0.00243 | 1000 | 5.19×10^{-13} | | 5.19×10^{-13} | 1.83×10^{-2} |
| 0.01046 | 800 | 6.96×10^{-13} | | 6.96×10^{-13} | 1.82×10^{-2} |
| 0.02817 | 600 | 9.38×10^{-13} | | 9.38×10^{-13} | 1.80×10^{-2} |
| 0.12353 | 50 | 2.84×10^{-13} | | 2.84×10^{-13} | 1.70×10^{-2} |
| 1.00847 | 100 | 7.96×10^{-14} | | 7.96×10^{-14} | 1.08×10^{-2} |
| 1.023 | 100 | 6.82×10^{-13} | | 6.82×10^{-13} | 1.08×10^{-2} |
| 1.5 | 10 | 0 | | 0 | 8.83×10^{-3} |
| 2.345 | 80 | 7.96×10^{-13} | | 7.96×10^{-13} | 6.66×10^{-3} |
| 11.6442 | 10 | 0 | | 0 | 2.00×10^{-3} |
| 52.80352 | 10 | 0 | | 0 | 5.16×10^{-4} |

Table 3
Maximum differences in reactivity in pcm for $n(t) = \exp(wt)$ with $h = 1.5s$.

| w (s^{-1}) | t (s) | $Max \rho_{class} - \rho_{class_ref} $ (pcm) | | $Max \rho_{P1} - \rho_{P1_ref} $ (pcm) | |
|------------------|---------|------------------------------------------------|--|------------------------------------------|------------------------|
| | | MF | | MF With source term $ds(t)/dt$ | MF Without source term |
| 0.00243 | 1000 | 6.08×10^{-13} | | 6.08×10^{-13} | 1.82×10^{-2} |
| 0.01046 | 800 | 6.96×10^{-13} | | 6.96×10^{-13} | 1.81×10^{-2} |
| 0.02817 | 600 | 8.81×10^{-13} | | 8.81×10^{-13} | 1.79×10^{-2} |
| 0.12353 | 50 | 3.41×10^{-13} | | 3.41×10^{-13} | 1.67×10^{-2} |
| 1.00847 | 100 | 6.82×10^{-13} | | 6.82×10^{-13} | 9.94×10^{-3} |
| 1.023 | 100 | 4.55×10^{-13} | | 4.55×10^{-13} | 9.88×10^{-3} |
| 1.5 | 10 | 1.14×10^{-13} | | 1.14×10^{-13} | 8.11×10^{-3} |
| 2.345 | 80 | 3.41×10^{-13} | | 3.41×10^{-13} | 6.25×10^{-3} |
| 11.6442 | 10 | 0 | | 0 | 2.00×10^{-3} |
| 52.80352 | 10 | 0 | | 0 | 5.16×10^{-4} |

Table 4
Maximum differences between classical reactivity and $P1$ point kinetic reactivity in pcm for $n(t) = \exp(wt)$ with $h = 1s$.

| w (s^{-1}) | t (s) | $Max \rho_{P1} - \rho_{class} $ (pcm) | |
|------------------|---------|----------------------------------------|------------------------|
| | | MF With source term $ds(t)/dt$ | MF Without source term |
| 0.00243 | 1000 | 1.26×10^{-2} | 2.03×10^{-3} |
| 0.01046 | 800 | 5.40×10^{-2} | 8.72×10^{-3} |
| 0.02817 | 600 | 1.45×10^{-1} | 2.35×10^{-2} |
| 0.12353 | 50 | 6.36×10^{-1} | 1.03×10^{-1} |
| 1.00847 | 100 | 5.18×10^0 | 8.41×10^{-1} |
| 1.023 | 100 | 5.25×10^0 | 8.53×10^{-1} |
| 1.5 | 10 | 7.70×10^0 | 1.25×10^0 |
| 2.345 | 80 | 1.20×10^1 | 1.96×10^0 |
| 11.6442 | 10 | 5.97×10^1 | 9.73×10^0 |
| 52.80352 | 10 | 2.70×10^2 | 4.44×10^1 |

difference to the numerical computations previously performed when the matrix formulation is used. Again, when the term of the neutron source is considered, the same difference of $1.83 \times 10^{-2} pcm$ is obtained. We conclude that the formulation method (FM) presents high precision for these type of neutron densities.

For a neutron population density represented by $n(t) = a + bt^4$, when $a = 1$, $h = 3s$ and $t = 1800 s$. Table 7 shows that the matrix formulation has goods results. Fig. 3 shows the difference between calculating the classical reactivity represented by Eq. (21) using the matrix formulation method given by Eq. (37) and the reference method which is the analytical solution of classical reactivity and the $P1$ point kinetic reactivity, assuming the form of the neutron density is known. Now, without the source term the maximum difference obtained is equivalent to the previous numerical computations of $1.83 \times 10^{-2} pcm$ when the matrix formulation (MF) is used. It can be inferred that the classical reactivity term is dominant over the $P1$ point kinetic reactivity term.

Table 8 shows the results obtained in the calculation of

reactivity with the neutron density described by the hyperbolic function $n(t) = a + \cosh(kt)$ and $n(t) = a + \sinh(kt)$, with $h = 5s$ and $h = 20s$. For the matrix formulation (MF), the classical reactivity and $P1$ point kinetic reactivity show better precision. This makes evident that for a density of the proposed form, the more precise method to calculate the classical reactivity, as well as for $P1$ point kinetic reactivity, is the matrix formulation (MF). However, without source term it provides an almost constant difference with respect to the numerical computations presented previously.

According to the previous results, the matrix formulation (MF) has the same error order for different neutron density shapes. The matrix formulation is also very precise in calculating the maximum differences for the $P1$ point kinetic reactivity.

In practice, the neutron density is known since it is provided by the different detectors. This signal can be simulated as a step function with some jump given by a Gaussian distribution with a mean μ and a standard deviation, σ , if the jumps are given with much standard deviation, the method may have drawbacks and

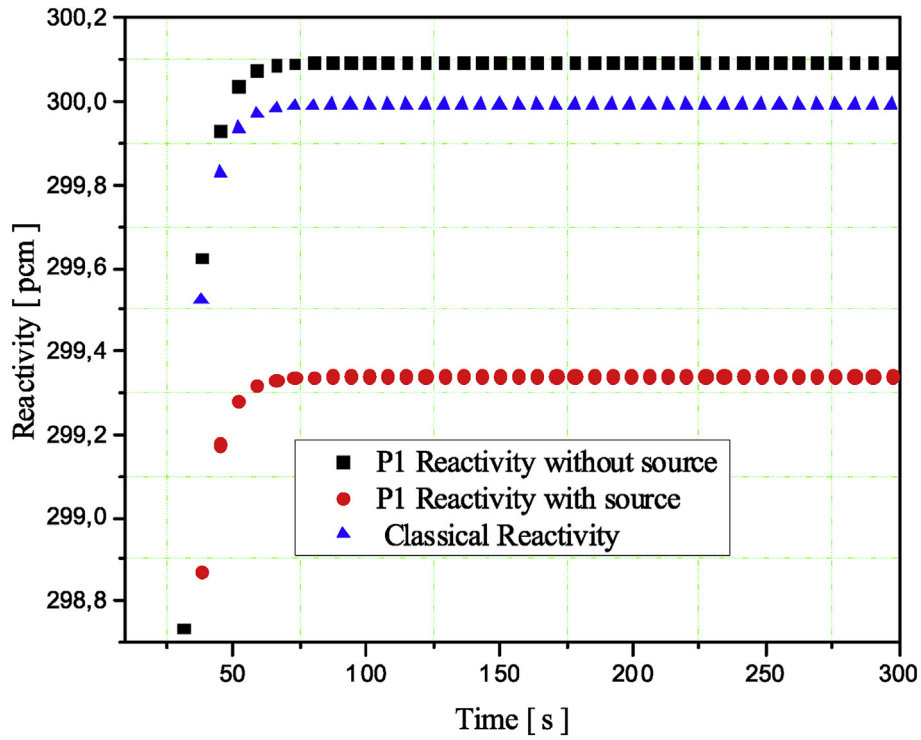


Fig. 1. Comparison of between classical reactivity and the *P1* point kinetic reactivity for $n(t) = \exp(0.12353t)$ with $h = 1s$.

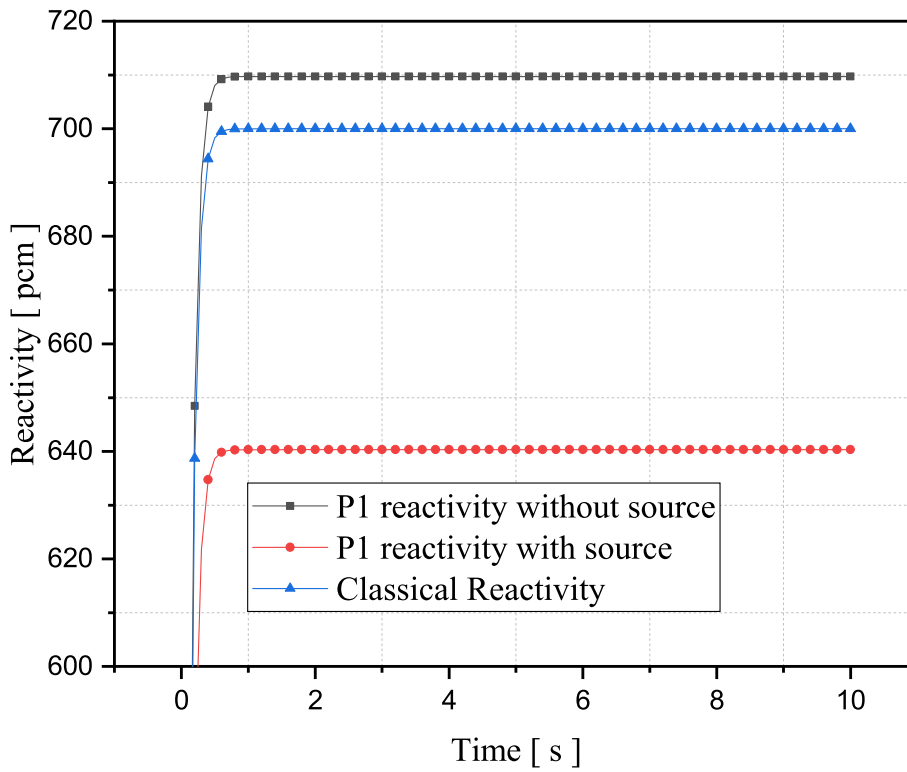


Fig. 2. Comparison of between classical reactivity and the *P1* point kinetic reactivity for $n(t) = \exp(11.6442t)$ with $h = 0.1s$.

some filter should be used first to decrease these fluctuations in neutron density.

6. Conclusions

A new method to calculate reactivity from the approximation based on the *P1* point reactor model is introduced in this work. This

Table 5

Maximum differences in reactivity in pcm for $n(t) = a + bt$ with $a = 50$ and $t = 10s$, different values for the factor band $h = 3s$.

| b | $Max \rho_{class} - \rho_{class_ref} (pcm)$ | $Max \rho_{p1} - \rho_{p1_ref} (pcm)$ | Without source term |
|-----------------|------------------------------------------------|------------------------------------------|-----------------------|
| | MF | MF With source term $ds(t)/dt$ | |
| $0.0127^5 / 9$ | 7.81×10^{-13} | 7.81×10^{-13} | 1.83×10^{-2} |
| $0.0127^4 / 40$ | 6.07×10^{-13} | 6.07×10^{-13} | 1.83×10^{-2} |
| $0.0127^4 / 4$ | 3.47×10^{-13} | 3.47×10^{-13} | 1.83×10^{-2} |

Table 6

Maximum differences in reactivity in pcm for $n(t) = a + bt^3$ with $a = 1$ and $t = 10s$, different values for the factor b and $h=3s$.

| b | $Max \rho_{class} - \rho_{class_ref} (pcm)$ | $Max \rho_{p1} - \rho_{p1_ref} (pcm)$ | MF Without source term |
|-----------------|------------------------------------------------|------------------------------------------|------------------------|
| | MF | MF With source term $ds(t)/dt$ | |
| $0.0127^5 / 9$ | 6.87×10^{-10} | 6.87×10^{-10} | 1.83×10^{-2} |
| $0.0127^4 / 40$ | 1.22×10^{-8} | 1.22×10^{-8} | 1.83×10^{-2} |
| $0.0127^4 / 4$ | 1.22×10^{-7} | 1.22×10^{-7} | 1.83×10^{-2} |

Table 7

Maximum differences in reactivity in pcm for $n(t) = a + bt^4$ with $a = 1$, $t = 1800s$ different values for the factor b and $h=3s$.

| b | $Max \rho_{class} - \rho_{class_ref} (pcm)$ | $Max \rho_{p1} - \rho_{p1_ref} (pcm)$ | MF Without source term |
|-----------------|------------------------------------------------|------------------------------------------|------------------------|
| | MF | MF With source term $ds(t)/dt$ | |
| $0.0127^5 / 9$ | 1.02×10^{-2} | 1.02×10^{-2} | 2.83×10^{-2} |
| $0.0127^4 / 40$ | 4.05×10^{-2} | 4.05×10^{-2} | 5.86×10^{-2} |
| $0.0127^4 / 4$ | 1.20×10^{-1} | 1.20×10^{-1} | 1.38×10^{-1} |

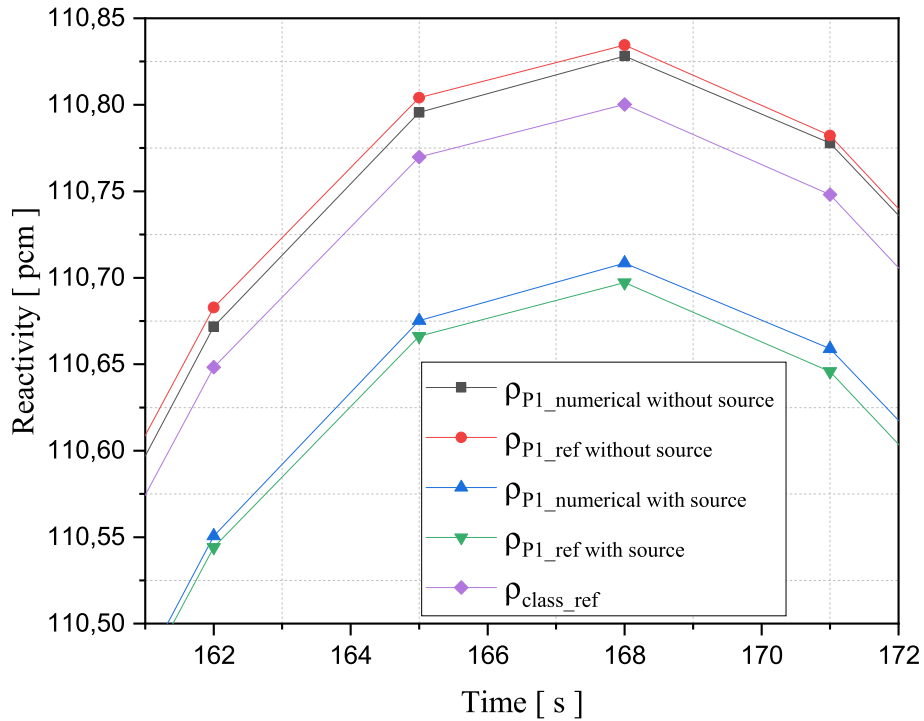


Fig. 3. Comparison in reactivity for $n(t) = a + bt^4$ with $a = 1$, $b = 0.0127^4 / 4$, $t = 600s$ and $h = 3s$.

new form is presented as a function of classical reactivity and a non-classical term that depends on the product between the time of relaxation and the time derivative of the neutron source. To perform the numerical simulations, the equations were solved

using the matrix formulation method due to its high precision for different forms of neutron density. The numerical results show that there can be large differences if the shape of the neutron density is exponential or if the relaxation time increases.

Table 8

Maximum differences of the reactivity in pcm for neutron densities with hyperbolic form with different time steps.

| $n(t)$ | a | $k (s^{-1})$ | $t(s)$ | $h(s)$ | $Max \rho_{class} - \rho_{class_ref} (pcm)$ | |
|-----------------|-----|-----------------------|--------|--------|------------------------------------------------|--------------------------------|
| | | | | | MF | MF With source term $ds(t)/dt$ |
| a + cosh(kt) | 100 | $\pi/180$ | 180 | 5 | 3.21×10^{-2} | 5.32×10^{-2} |
| a + sinh(kt) | 100 | 1.27×10^{-3} | 10,000 | 20 | 6.70×10^{-3} | 7.14×10^{-2} |
| | | | | | | 4.21×10^{-2} |

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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