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HIGHER ORDER APOSTOL-TYPE POLY-GENOCCHI POLYNOMIALS WITH PARAMETERS a, b AND c

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Abstract. In this paper, a new form of poly-Genocchi polynomials is defined by means of polylogarithm, namely, the Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c. Several properties of these polynomials are established including some recurrence relations and explicit formulas, which are used to express these higher order Apostol-type poly-Genocchi polynomials in terms of Stirling numbers of the second kind, Apostol-type Bernoulli and Frobenius polynomials of higher order. Moreover, certain differential identity is obtained that leads this new form of poly-Genocchi polynomials to be classified as Appell polynomials and, consequently, draw more properties using some theorems on Appell polynomials. Furthermore, a symmetrized generalization of this new form of poly-Genocchi polynomials that possesses a double generating function is introduced. Finally, the type 2 Apostol-poly-Genocchi polynomials with parameters a, b and c are defined using the concept of polyexponential function and several identities are derived, two of which show the connections of these polynomials with Stirling numbers of the first kind and the type 2 Apostol-type poly-Bernoulli polynomials.

1. Introduction

There are several variations of Genocchi numbers that appeared in the literature. These include the Genocchi polynomials and Genocchi polynomials of higher order, which are respectively defined by

$$\sum_{n=0}^{\infty} G_n(x)\frac{t^n}{n!} = \frac{2t}{e^t+1}e^{xt}, \quad |t| < \pi, \tag{1}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{t^n}{n!} = \left(\frac{2t}{e^t+1}\right)^k e^{xt}, \tag{2}$$

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and Apostol-Genocchi polynomials, and Apostol-Genocchi polynomials of higher order, which are respectively defined by

$$\sum_{n=0}^{\infty} G_n(x,\lambda)\frac{t^n}{n!} = \frac{2t}{\lambda e^t+1}e^{xt}, \tag{3}$$

$$\sum_{n=0}^{\infty} G_n^{(k)}(x,\lambda)\frac{t^n}{n!} = \left(\frac{2t}{\lambda e^t+1}\right)^k e^{xt}, \tag{4}$$

(see [1,2,6,7,17,18,22]). Other interesting variations of Genocchi polynomials can be found in the papers [11, 14, 16, 29], which contain remarkable results that can possibly be the basis in generating identities for a higher level generalization of Genocchi polynomials. It is also interesting to explore some other known polynomials (e.g. [4,12,13,15,26]), which are closely related to Genocchi polynomials.

Another variation of Genocchi numbers, also known as poly-Genocchi polynomials, was introduced by Kim et al. [19] using the concept of *kth* polylogarithm, denoted by $\mathrm{Li}_k(z)$, which is given by

$$\mathrm{Li}_k(z) = \sum_{n=0}^{\infty}\frac{z^n}{n^k},\ k\in\mathbb{Z}. \tag{5}$$

The poly-Genocchi polynomials were defined as follows:

$$\sum_{n=0}^{\infty} G_n^{(k)}(x)\frac{x^n}{n!} = \frac{2Li_k(1-e^t)}{e^t+1}e^{xt}. \tag{6}$$

Kim et al. [19] also defined a modified poly-Genocchi polynomials, denoted by $G_{n,2}^{(k)}(x)$, as follows:

$$\sum_{n=0}^{\infty} G_{n,2}^{(k)}(x)\frac{x^n}{n!} = \frac{Li_k(1-e^{-2t})}{e^t+1}e^{xt}, \tag{7}$$

and obtained several properties of these polynomials. Note that, when $k=1$, Equations (6) and (7) give the Genocchi polynomials in (1). That is,

$$G_n^{(1)}(x) = G_{n,2}^{(1)}(x) = G_n(x).$$

On the other hand, Kurt [23] defined two forms of generalized poly-Genocchi polynomials with parameters a, b, and c, by

$$\frac{2Li_k(1-(ab)^{-t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_n^{(k)}(x;a,b,c)\frac{x^n}{n!}, \tag{8}$$

$$\frac{2Li_k(1-(ab)^{-2t})}{a^{-t}+b^t}e^{xt} = \sum_{n=0}^{\infty} G_{n,2}^{(k)}(x;a,b,c)\frac{x^n}{n!}, \tag{9}$$

which were motivated by the definitions (6) and (7), respectively. Kurt [23] also derived several properties parallel to those of poly-Genocchi polynomials by Kim et al. [19]. Note that, when $x = 0$, (6) reduces to

$$\frac{2Li_k(1-e^t)}{e^t+1} = \sum_{n=0}^{\infty} G_n^{(k)} \frac{x^n}{n!}, \tag{10}$$

where $G_n^{(k)}$ are called the poly-Genocchi numbers.

In this paper, a new variation of poly-Genocchi polynomials with parameters a, b and c, namely, the Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c, will be investigated. Sections 2 and 3 provide the definition of this new variation of poly-Genocchi polynomials, some special cases and their relations with some Genocchi-type polynomials. Section 4 devotes its discussion on some identities that link the Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c to Appell polynomials. Section 5 focuses on the connections of these higher order Apostol-type polynomials to Stirling numbers of the second kind and different variations of higher order Bernoulli-type polynomials. Section 6 demonstrates the symmetrized generalization of these higher order Apostol-type polynomials. Section 7 introduces type 2 Apostol-poly-Genocchi polynomials with parameters a, b and c using the concept of polyexponential function [21]. Section 8 contains the conclusion of the paper.

2. Definition

Here, a new variation of poly-Genocchi polynomials, the Apostol-Type poly-Genocchi polynomials of higher order with parameters a b and c will be introduced.

Definition. The Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c, denoted by $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$, are defined as follows:

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} c^{xt}, \quad |t| < \frac{\sqrt{(\ln\lambda)^2+\pi^2}}{|\ln a + \ln b|}. \tag{11}$$

When $\alpha = 1$, (11) yields

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t} c^{xt}, \quad |t| < \frac{\sqrt{(\ln\lambda)^2+\pi^2}}{|\ln a + \ln b|}, \tag{12}$$

where $\mathcal{G}_n^{(k)}(x;\lambda,a,b,c) = \mathcal{G}_n^{(k,1)}(x;\lambda,a,b,c)$ denotes the Apostol-type poly-Genocchi polynomials with parameters a, b and c.

The following are special cases of the Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c:

1. When $c = e$, Equation (11) reduces to

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,e)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} e^{xt}. \tag{13}$$

For convenience, $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b)$ will be used to denote $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,e)$. That is,

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} e^{xt}. \tag{14}$$

2. When $k = 1$, (11) yields

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\frac{2t\ln ab}{a^{-t}+\lambda b^t}\right)^{\alpha} c^{xt}, \quad |t| < \frac{\sqrt{(\ln\lambda)^2+\pi^2}}{|\ln a+\ln b|}, \tag{15}$$

where the polynomials $\mathcal{G}_n^{(\alpha)}(x;\lambda,a,b,c) = \mathcal{G}_n^{(1,\alpha)}(x;\lambda,a,b,c)$ will be called the Apostol-type Genocchi polynomials of higher order with parameters a, b and c. When $\alpha = 1$, (15) yields

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(1)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \frac{2t\ln ab}{a^{-t}+\lambda b^t} c^{xt}, \quad |t| < \frac{\sqrt{(\ln\lambda)^2+\pi^2}}{|\ln a+\ln b|}, \tag{16}$$

where the polynomials $\mathcal{G}_n^{(1)}(x;\lambda,a,b,c)$ will be called the Apostol-type Genocchi polynomials with parameters a b and c.

3. When $a = 1, b = e$, (14) will reduce to

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,1,e)\frac{t^n}{n!} = \left(\frac{Li_k(1-e^{-2t})}{1+\lambda e^t}\right)^{\alpha} e^{xt}. \tag{17}$$

We will use the notations

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda) = \mathcal{G}_n^{(k,\alpha)}(x;\lambda,1,e) \text{ and } \mathcal{G}_n^{(k)}(x;\lambda) = \mathcal{G}_n^{(k)}(x;\lambda,1,e)$$

and call these polynomials Apostol-type poly-Genocchi polynomials of higher order and Apostol-type poly-Genocchi polynomials, respectively.

4. When $\lambda = 1$, (17) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;1)\frac{t^n}{n!} = \left(\frac{Li_k(1-e^{-2t})}{1+e^t}\right)^{\alpha} e^{xt}, \tag{18}$$

which is the higher order version of Equation (7), i.e., the higher order version of the modified poly-Genocchi polynomials of Kim et al. [19]. We will use $G_{n,2}^{(k,\alpha)}(x)$ to denote $\mathcal{G}_n^{(k,\alpha)}(x;1)$.

5. Using the fact that

$$Li_1(z) = -\ln(1-z),$$

when $k = 1$, (17) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(1,\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{2t}{1+\lambda e^t}\right)^{\alpha} e^{xt}, \tag{19}$$

and when $\lambda = 1$, (19) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(1,\alpha)}(x;1)\frac{t^n}{n!} = \left(\frac{2t}{1+e^t}\right)^{\alpha} e^{xt},$$

where $\mathcal{G}_n^{(1,\alpha)}(x;\lambda)$ and $\mathcal{G}_n^{(1,\alpha)}(x;1)$ are exactly the Genocchi polynomials $G_n^{(k)}(x)$ and Apostol-Genocchi polynomials $G_n^{(k)}(x,\lambda)$ of higher order in (4) and (2), respectively.

3. Relations with some Genocchi-type polynomials

In this section, some relations for $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ expressed in terms of some Genocchi-type polynomials will be established.

Theorem 3.1. *The Apostol-type poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the recurrence relation*

$$\mathcal{G}_n^{(k,\alpha)}(x+1;\lambda,a,b,c) = \sum_{r=0}^{n} \binom{n}{r} (\ln c)^r \mathcal{G}_{n-r}^{(k,\alpha)}(x;\lambda,a,b,c). \tag{20}$$

Proof. Equation (11) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x+1;\lambda,a,b,c)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} e^{xt\ln c}e^{t\ln c} \\
&= \left\{\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}\right\}\left\{\sum_{n=0}^{\infty}\frac{(t\ln c)^n}{n!}\right\} \\
&= \sum_{n=0}^{\infty}\sum_{r=0}^{n} \mathcal{G}_{n-r}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^{n-r}}{(n-r)!}\frac{(\ln c)^r t^r}{r!} \\
&= \sum_{n=0}^{\infty}\left\{\sum_{r=0}^{n}\binom{n}{r}(\ln c)^r \mathcal{G}_{n-r}^{(k,\alpha)}(x;\lambda,a,b,c)\right\}\frac{t^n}{n!}.
\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$ completes the proof of the theorem. ☐

Consider a special case of (14) by taking $x = 0$. This gives

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(0;\lambda,a,b)\frac{t^n}{n!} = \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha}. \tag{21}$$

We use the notation $\mathcal{G}_i^{(k,\alpha)}(\lambda,a,b) = \mathcal{G}_i^{(k,\alpha)}(0;\lambda,a,b)$ and call them the Apostol-type poly-Genocchi numbers of higher order with parameters a and b. The following theorem contains an identity that expresses $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ as polynomial in x, which involves $\mathcal{G}_i^{(k,\alpha)}(\lambda,a,b)$ as coefficients.

Theorem 3.2. *The Apostol-type poly-Genocchi polynomials of higher order with parameters a,b,c satisfy the relation*

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{i=0}^{n} \binom{n}{i} (\ln c)^{n-i} \mathcal{G}_i^{(k,\alpha)}(\lambda,a,b) x^{n-i}. \tag{22}$$

Proof. Equation (11) can be written as

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;a,b,c)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} c^{xt} = e^{xt\ln c}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(\lambda,a,b)\frac{t^n}{n!}\\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{\infty}\frac{(xt\ln c)^{n-i}}{(n-i)!}\mathcal{G}_i^{(k,\alpha)}(\lambda,a,b)\frac{t^i}{i!}\\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{\infty}\binom{n}{i}(\ln c)^{n-i}\mathcal{G}_i^{(k,\alpha)}(\lambda,a,b)x^{n-i}\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

The next identity gives the relation between $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ and $\mathcal{G}_n^{(k,\alpha)}(x;\lambda)$.

Theorem 3.3. *The Apostol-type poly-Genocchi polynomials of higher order with parameters a,b,c satisfy the relation*

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) = (\ln a+\ln b)^n \mathcal{G}_n^{(k,\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda\right). \tag{23}$$

Proof. Using (11), we have

$$\begin{aligned}\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}(1+\lambda(ab)^t)}\right)^{\alpha} e^{xt\ln c}\\ &= e^{\frac{x\ln c+\alpha\ln a}{\ln ab}t\ln ab}\left(\frac{Li_k(1-e^{-2t\ln ab})}{1+\lambda e^{t\ln ab}}\right)^{\alpha}\\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^n\mathcal{G}_n^{(k,\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda\right)\frac{t^n}{n!}.\end{aligned}$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

4. Classification as Appell polynomials

The following theorem contains a differential identity that can be used to classify Apostol-type poly-Genocchi polynomials as Appell polynomials [24,27, 28].

Theorem 4.1. *The Apostol-type poly-Genocchi polynomials with parameters a, b, c satisfy the relation*

$$\frac{d}{dx}\mathcal{G}_{n+1}^{(k,\alpha)}(x;\lambda,a,b,c) = (n+1)(\ln c)\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c). \tag{24}$$

Proof. Applying the first derivative to Equation (11), we have

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = t(\ln c)\left(\frac{Li_k(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)}\right)^{\alpha} e^{xt\ln c},$$

$$\sum_{n=0}^{\infty}\frac{d}{dx}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^{n-1}}{n!} = \sum_{n=0}^{\infty}(\ln c)\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\frac{d}{dx}\mathcal{G}_{n+1}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}(\ln c)\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$, we obtain the desired result. □

Remark 4.2. When $c = e$, Equation (24) reduces to

$$\frac{d}{dx}\mathcal{G}_{n+1}^{(k,\alpha)}(x;\lambda,a,b) = (n+1)\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b), \tag{25}$$

which is one of the properties for the polynomial to be classified as Appell polynomial.

Being classified as Appell polynomials, the generalized poly-Genocchi polynomials $\mathcal{G}_n^{(k)}(x;a,b)$ must possess the following properties

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b) = \sum_{i=0}^{n}\binom{n}{i}c_i x^{n-i},$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b) = \left(\sum_{i=0}^{\infty}\frac{c_i}{i!}D^i\right)x^n$$

for some scalar $c_i \neq 0$. It is then necessary to find the sequence $\{c_n\}$. Using (22) with $c = e$, $c_i = \mathcal{G}_i^{(k,\alpha)}(\lambda,a,b)$ which implies the following corollary.

Corollary 4.3. *The Apostol-type poly-Genocchi polynomials with parameters a, b, c satisfy the formula*

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b) = \left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_i^{(k,\alpha)}(\lambda,a,b)}{i!}D^i\right)x^n.$$

For example, when $n = 3$,

$$\mathcal{G}_3^{(k,\alpha)}(x;\lambda,a,b) = \left(\sum_{i=0}^{\infty}\frac{\mathcal{G}_i^{(k,\alpha)}(\lambda,a,b)}{i!}D^i\right)x^3$$

$$
\begin{aligned}
&= \frac{\mathcal{G}_0^{(k,\alpha)}(\lambda,a,b)}{0!}x^3 + \frac{\mathcal{G}_1^{(k,\alpha)}(\lambda,a,b)}{1!}D^1x^3 \\
&\quad + \frac{\mathcal{G}_2^{(k,\alpha)}(\lambda,a,b)}{2!}D^2x^3 + \frac{\mathcal{G}_3^{(k,\alpha)}(\lambda,a,b)}{3!}D^3x^3 \\
&= \mathcal{G}_0^{(k,\alpha)}(\lambda,a,b)x^3 + 3\mathcal{G}_1^{(k,\alpha)}(\lambda,a,b)x^2 \\
&\quad + 3\mathcal{G}_2^{(k,\alpha)}(\lambda,a,b)x + \mathcal{G}_3^{(k,\alpha)}(\lambda,a,b).
\end{aligned}
$$

The next corollary immediately follows from Equation (25) and the characterization of Appell polynomials [24, 27, 28].

Corollary 4.4. *The Apostol-type poly-Genocchi polynomials with parameters a, b, c satisfy the addition formula*

$$
\mathcal{G}_n^{(k,\alpha)}(x+y;\lambda,a,b) = \sum_{i=0}^{\infty}\binom{n}{i}\mathcal{G}_i^{(k,\alpha)}(x;\lambda,a,b)y^{n-i}. \tag{26}
$$

Taking $x = 0$ in formula (26) and using the fact $\mathcal{G}_n^{(k)}(0;\lambda,a,b) = \mathcal{G}_n^{(k)}(\lambda,a,b)$, Corollary 4.4 gives formula (22) in Theorem 3.2 with $c = e$.

An extension of this addition formula can be derived as follows:

$$
\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x+y;\lambda,a,b,c)\frac{t^n}{n!} &= \left(\frac{Li_k(1-(ab)^{-2t})}{(a^{-t}+\lambda b^t)}\right)^{\alpha} c^{xt}e^{yt\ln c} \\
&= \left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}(y\ln c)^n\frac{t^n}{n!}\right) \\
&= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}\binom{n}{i}\mathcal{G}_i^{(k,\alpha)}(x;\lambda,a,b,c)(y\ln c)^{n-i}\right)\frac{t^n}{n!}.
\end{aligned}
$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives the following theorem.

Theorem 4.5. *The Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c satisfy the addition formula*

$$
\mathcal{G}_n^{(k,\alpha)}(x+y;\lambda,a,b,c) = \sum_{i=0}^{\infty}\binom{n}{i}(\ln c)^{n-i}\mathcal{G}_i^{(k,\alpha)}(x;\lambda,a,b,c)y^{n-i}. \tag{27}
$$

By taking $x = 0$, Equation (27) exactly gives (22).

5. Connections with some special numbers and polynomials

In this section, some connections of the higher order Apostol-type poly-Genocchi polynomials $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ with other well-known special numbers and polynomials will be established.

To introduce the first connection, we define an Apostol-type poly-Bernoulli polynomials of higher order with parameters a, b and c as follows:

$$\left(\frac{Li_k(1-e^{-t})}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k,\alpha)}(x;\lambda)\frac{x^n}{n!}. \tag{28}$$

When $\alpha = 1$, (28) reduces to

$$\left(\frac{Li_k(1-e^{-t})}{\lambda e^t - 1}\right) e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k)}(x;\lambda)\frac{x^n}{n!}, \tag{29}$$

where $\mathcal{B}_n^{(k)}(x;\lambda) = \mathcal{B}_n^{(k,1)}(x;\lambda)$ denotes the Apostol-type poly-Bernoulli polynomials with parameters a, b and c. When $k = 1$, (28) gives

$$\left(\frac{t}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(1,\alpha)}(x;\lambda)\frac{x^n}{n!}, \tag{30}$$

where $\mathcal{B}_n^{(1,\alpha)}(x;\lambda) = B_n^{(\alpha)}(x;\lambda)$, the Apostol-Bernoulli polynomials of higher order in [25]. Also, when $\lambda = 1$, (28) will give

$$\left(\frac{Li_k(1-e^{-t})}{e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty} \mathcal{B}_n^{(k,\alpha)}(x;1)\frac{x^n}{n!}, \tag{31}$$

where $\mathcal{B}_n^{(k,\alpha)}(x;1) = B_n^{(k,\alpha)}(x)$, the higher order version of poly-Bernoulli polynomials of Bayad and Hamahata [3, 20]. When $\alpha = 1$, (31) reduces to the definition of poly-Bernoulli numbers and polynomials [8–10, 20].

Now, we are ready to introduce the following theorem.

Theorem 5.1. *The Apostol-type poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the relation*

$$\begin{aligned} &\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) \\ &= \sum_{j=0}^{\alpha} \binom{\alpha}{j} (-1)^j \lambda^{\alpha-j} \mathcal{B}_n^{(k,\alpha)}\left(\frac{(\alpha-j)\ln b + x\ln c + (2\alpha - j)\ln a}{2(\ln a + \ln b)}; \lambda^2\right) \\ &\qquad \times 2^n (\ln a + \ln b)^n. \end{aligned} \tag{32}$$

In particular, the Apostol-type poly-Genocchi polynomials with parameters a, b, c satisfy the relation

$$\begin{aligned} \mathcal{G}_n^{(k)}(x;\lambda,a,b,c) = \Bigg\{ &\lambda \mathcal{B}_n^{(k)}\left(\frac{\ln b + 2\ln a + x\ln c}{2(\ln a + \ln b)}; \lambda^2\right) \\ &- \mathcal{B}_n^{(k)}\left(\frac{\ln a + x\ln c}{2(\ln a + \ln b)}; \lambda^2\right)\Bigg\} 2^n (\ln a + \ln b)^n. \end{aligned} \tag{33}$$

Proof. Rewrite Equation (11) as

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$

$$= \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} e^{xt\ln c}$$
$$= \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-2t}-(\lambda b^t)^2}(a^{-t}-\lambda b^t)\right)^{\alpha} e^{xt\ln c}$$
$$= \left(\frac{Li_k(1-(ab)^{-2t})}{(1-(\lambda(ab)^t)^2)}\right)^{\alpha} (e^{-t\ln a}-\lambda e^{t\ln b})^{\alpha} e^{xt\ln c} e^{2\alpha t\ln a}$$
$$= \left(\frac{Li_k(1-e^{-2t(\ln ab)})}{-(\lambda e^{2t\ln(ab)}-1)}\right)^{\alpha} \left(e^{t(-\ln a+(x\ln c/\alpha)+2\ln a)}-\lambda e^{t(\ln b+(x\ln c/\alpha)+2\ln a)}\right)^{\alpha}.$$

Applying the Binomial Theorem yields

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$
$$= \left(\frac{Li_k(1-e^{-2t(\ln ab)})}{-(\lambda e^{2t\ln(ab)}-1)}\right)^{\alpha} \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-\lambda)^{\alpha-j} e^{jt(\ln a+(x\ln c/\alpha))} e^{(\alpha-j)t(\ln b+(x\ln c/\alpha)+2\ln a)}$$
$$= \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\lambda^{\alpha-j}\left(\frac{Li_k(1-e^{-2t(\ln ab)})}{\lambda^2 e^{2t\ln(ab)}-1}\right)^{\alpha} e^{[((\alpha-j)\ln b+x\ln c+(2\alpha-j)\ln a)/2\ln ab](2t\ln ab)}.$$

Using the definition of Apostol-type poly-Bernoulli polynomials in (28), we have

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$
$$= \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\lambda^{\alpha-j}\left\{\sum_{n=0}^{\infty}\mathcal{B}_n^{(k,\alpha)}\left(\frac{(\alpha-j)\ln b+x\ln c+(2\alpha-j)\ln a}{2(\ln a+\ln b)};\lambda^2\right)2^n(\ln ab)^n\frac{t^n}{n!}\right\}$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\lambda^{\alpha-j}\mathcal{B}_n^{(k,\alpha)}\left(\frac{(\alpha-j)\ln b+x\ln c+(2\alpha-j)\ln a}{2(\ln a+\ln b)};\lambda^2\right)2^n(\ln ab)^n\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (32). □

The next theorem contains an identity that relates the Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c to Stirling numbers of the second kind $\left\{ n \atop m \right\}$ defined in [5] by

$$\sum_{n=m}^{\infty}\left\{ n \atop m \right\}\frac{t^n}{n!} = \frac{(e^t-1)^m}{m!}. \tag{34}$$

Here, it is important to note that if $(c_0, c_1, \ldots, c_j, \ldots)$ is any sequence of numbers and l is a positive integer, then

$$\left(\sum_{j=0}^{\infty} c_j\frac{t^j}{j!}\right)^l = \prod_{i=1}^{l}\left(\sum_{n_i=0}^{\infty}\frac{c_{n_i}}{n_i!}t^{n_i}\right)$$

$$= \sum_{n=0}^{\infty} \left\{ \sum_{n_1+n_2+\cdots+n_\alpha=n} \prod_{i=1}^{l} c_{n_i} \binom{n}{n_1, n_2, \ldots, n_\alpha} \right\} \frac{t^n}{n!}, \tag{35}$$

(see [5]).

Theorem 5.2. *The Apostol-type poly-Genocchi polynomials of higher order with parameters a, b and c satisfy the relation*

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{j=0}^{n} \binom{n}{j} (\ln a + \ln b)^{n-j} \mathcal{G}_{n-j}^{(\alpha)} \left(\frac{x \ln c + \alpha \ln a}{\ln a + \ln b}; \lambda \right) d_j, \tag{36}$$

where

$$d_j = \sum_{n_1+n_2+\cdots+n_\alpha=j} \prod_{i=1}^{\alpha} c_{n_i} \binom{j}{n_1, n_2, \ldots, n_\alpha} \quad \text{and}$$

$$c_j = \sum_{m=0}^{j} (-1)^{m+1} \frac{(2\ln ab)^j m! \left\{ {j+1 \atop m+1} \right\}}{(j+1)(m+1)^{k-1}}.$$

Proof. Now, (11) can be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) \frac{t^n}{n!} \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=1}^{\infty} \frac{(1 - e^{2t \ln ab})^m}{m^k} \right)^\alpha \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=0}^{\infty} \frac{(1 - e^{2t \ln ab})^{m+1}}{(m+1)^k} \right)^\alpha \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=0}^{\infty} \frac{m!}{(m+1)^{k-1}} \frac{(1 - e^{2t \ln ab})^{m+1}}{(m+1)!} \right)^\alpha \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=0}^{\infty} \frac{(-1)^{m+1} m!}{(m+1)^{k-1}} \sum_{j=m+1}^{\infty} \left\{ {j \atop m+1} \right\} \frac{(2t \ln ab)^j}{j!} \right)^\alpha \\
&= c^{xt} \left(\frac{2t \ln ab}{a^{-t} + \lambda b^t} \right)^\alpha \left(\sum_{m=0}^{\infty} \sum_{j=m}^{\infty} \frac{(-1)^{m+1} m! \left\{ {j+1 \atop m+1} \right\}}{(j+1)(m+1)^{k-1}} \frac{(2t \ln ab)^j}{j!} \right)^\alpha .
\end{aligned}$$

Using (15), we get

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) \frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(\alpha)}(x;\lambda,a,b,c) \frac{t^n}{n!} \right) \left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!} \right)^\alpha ,$$

where

$$c_j = \sum_{m=0}^{j} (-1)^{m+1} \frac{(2\ln ab)^j m! \left\{ {j+1 \atop m+1} \right\}}{(j+1)(m+1)^{k-1}}.$$

Note that, using (35), $\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!}\right)^{\alpha}$ can be expressed as

$$\left(\sum_{j=0}^{\infty} c_j \frac{t^j}{j!}\right)^{\alpha} = \sum_{n=0}^{\infty} d_n \frac{t^n}{n!},$$

where

$$d_n = \sum_{n_1+n_2+\cdots+n_\alpha=n} \prod_{i=1}^{\alpha} c_{n_i} \binom{n}{n_1, n_2, \ldots, n_\alpha}.$$

It follows that

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \left\{ \sum_{j=0}^{n} \binom{n}{j} \mathcal{G}_{n-j}^{(\alpha)}(x;\lambda,a,b,c) d_j \right\} \frac{t^n}{n!}.$$

Comparing the coefficients and using Equation (23) complete the proof of the theorem. □

Remark 5.3. When $\alpha = 1$, $d_j = c_j$.

The identities in the following theorem are derived using the relation in (13).

Theorem 5.4. *The Apostol-type poly-Genocchi polynomials of higher order* $\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)$ *with parameters* a, b, c *satisfy the following explicit formulas*:

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k,\alpha)}(-m\ln c;\lambda,a,b)(x)^{(m)}, \tag{37}$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{m=0}^{\infty} \sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k,\alpha)}(\lambda,a,b)(x)_m, \tag{38}$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b) \sum_{l=0}^{n} \sum_{m=0}^{n-l} \binom{n}{l} \left\{ {l+s \atop s} \right\} \frac{\binom{n-l}{m}}{\binom{l+s}{s}} \mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,a,b) B_m^{(s)}(x\ln c;\lambda), \tag{39}$$

$$\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b) = \sum_{m=0}^{\infty} \frac{\binom{n}{m}}{(1-\mu)^s} \sum_{j=0}^{s} \binom{s}{j} (-\mu)^{s-j} \mathcal{G}_{n-m}^{(k,\alpha)}(j;\lambda,a,b) F_m^{(s)}(x;\mu), \tag{40}$$

where $(x)^{(n)} = x(x+1)\cdots(x+n-1)$, $(x)_n = x(x-1)\cdots(x-n+1)$, *the rising and falling factorials of* x *of degree* n *and* $F_n^{(s)}(x;\mu)$, *the Frobenius polynomials of higher order* [25], *defined by*

$$\left(\frac{1-\mu}{e^t-\mu}\right)^s e^{xt} = \sum_{n=0}^{\infty} F_n^{(s)}(x;\mu) \frac{t^n}{n!}.$$

Proof. The proof of relations (37)-(39) makes use of the definition of Stirling numbers of the second kind in (34). Using the generalized Binomial Theorem, (11) may be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} \\
&= \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} \sum_{m=0}^{\infty} \binom{x+m-1}{m}(1-e^{-t\ln c})^m \\
&= \sum_{m=0}^{\infty} (x)^{(m)} \frac{(e^{t\ln c}-1)^m}{m!} \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} e^{-mt\ln c} \\
&= \sum_{m=0}^{\infty} (x)^{(m)} \left(\sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} \frac{(t\ln c)^n}{n!}\right) \left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(-m\ln c;\lambda,a,b)\frac{t^n}{n!}\right) \\
&= \sum_{m=0}^{\infty} (x)^{(m)} \sum_{n=0}^{\infty}\sum_{l=0}^{n} \left\{ {l \atop m} \right\} (\ln c)^l \frac{t^l}{l!} \mathcal{G}_{n-l}^{(k,\alpha)}(-m\ln c;\lambda,a,b)\frac{t^{n-l}}{(n-l)!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k,\alpha)}(-m\ln c;\lambda,a,b)(x)^{(m)} \right\} \frac{t^n}{n!}.
\end{aligned}$$

Comparing coefficients completes the proof of (37). To prove (38), (11) we write as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} \\
&= \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} \sum_{m=0}^{\infty} \binom{x}{m}(e^{t\ln c}-1)^m \\
&= \sum_{m=0}^{\infty} (x)_m \frac{(e^{t\ln c}-1)^m}{m!} \left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha} \\
&= \sum_{m=0}^{\infty} (x)_m \left(\sum_{n=0}^{\infty} \left\{ {n \atop m} \right\} \frac{(t\ln c)^n}{n!}\right) \left(\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(0;\lambda,a,b)\frac{t^n}{n!}\right) \\
&= \sum_{m=0}^{\infty} (x)_m \sum_{n=0}^{\infty}\sum_{l=0}^{n} \left\{ {l \atop m} \right\} (\ln c)^l \frac{t^l}{l!} \mathcal{G}_{n-l}^{(k,\alpha)}(\lambda,a,b)\frac{t^{n-l}}{(n-l)!} \\
&= \sum_{n=0}^{\infty} \left\{ \sum_{m=0}^{\infty}\sum_{l=m}^{n} \left\{ {l \atop m} \right\} \binom{n}{l} (\ln c)^l \mathcal{G}_{n-l}^{(k,\alpha)}(\lambda,a,b)(x)_m \right\} \frac{t^n}{n!}.
\end{aligned}$$

Again, comparing coefficients completes the proof of (38). Using (30), (11) may be expressed as

$$\sum_{n=0}^{\infty} \mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$

$$= \left(\frac{(e^t-1)^s}{s!}\right)\left(\frac{t^s e^{xt\ln c}}{(e^t-1)^s}\right)\left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha}\frac{s!}{t^s}$$
$$= \left(\sum_{n=0}^{\infty}\left\{ {n+s \atop s} \right\}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{m=0}^{\infty}B_m^{(s)}(x\ln c;\lambda)\frac{t^m}{m!}\right)\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(0;\lambda,a,b)\frac{t^m}{m!}\right)\frac{s!}{t^s}$$
$$= \left(\sum_{n=0}^{\infty}\left\{ {n+s \atop s} \right\}\frac{t^{n+s}}{(n+s)!}\right)\left(\sum_{n=0}^{\infty}\sum_{m=0}^{n}\binom{n}{m}B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-m}^{(k,\alpha)}(\lambda,a,b)\frac{t^n}{n!}\right)\frac{s!}{t^s}$$
$$= \left(\sum_{n=0}^{\infty}\sum_{l=0}^{n}\left\{ {l+s \atop s} \right\}\frac{t^{l+s}}{(l+s)!}\sum_{m=0}^{n-l}\binom{n-l}{m}B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,a,b)\frac{t^{n-l}}{(n-l)!}\right)\frac{s!}{t^s}.$$

This can further be written as

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$
$$= \left(\sum_{l=0}^{\infty}\sum_{n=l}^{\infty}\sum_{m=0}^{n-l}\left\{ {l+s \atop s} \right\}\frac{l!s!}{(l+s)!}\binom{n-l}{m}B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,a,b)\frac{n!}{(n-l)!l!}\frac{t^n}{n!}\right)$$
$$= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\left\{ {l+s \atop s} \right\}\frac{\binom{n-l}{m}}{\binom{l+s}{s}}B_m^{(s)}(x\ln c;\lambda)\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,a,b)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (39). To prove relation (40), express (11) as

$$\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$
$$= \left(\frac{(1-\mu)^s}{(e^t-\mu)^s}e^{xt\ln c}\right)\left(\frac{(e^t-\mu)^s}{(1-\mu)^s}\right)\left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha}$$
$$= \frac{1}{(1-\mu)^s}\left(\sum_{n=0}^{\infty}F_n^{(s)}(x\ln c;\mu)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\left(\frac{Li_k(1-(ab)^{-2t})}{a^{-t}+\lambda b^t}\right)^{\alpha}e^{jt}\right)$$
$$= \frac{1}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\left(\sum_{n=0}^{\infty}F_n^{(s)}(x\ln c;\mu)\frac{t^n}{n!}\right)\left(\sum_{n=0}^{\infty}\mathcal{G}_n^{(k,\alpha)}(x;\lambda,a,b)\frac{t^n}{n!}\right)$$
$$= \frac{1}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\binom{n}{m}\mathcal{G}_{n-m}^{(k,\alpha)}(x;\lambda,a,b)F_m^{(s)}(x\ln c;\mu)\right)\frac{t^n}{n!}$$
$$= \sum_{n=0}^{\infty}\left(\sum_{m=0}^{n}\frac{\binom{n}{m}}{(1-\mu)^s}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\mathcal{G}_{n-m}^{(k,\alpha)}(x;\lambda,a,b)F_m^{(s)}(x\ln c;\mu)\right)\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ gives (40). $\square$

6. Symmetrized generalization

Definition. For $m, n \geq 0$, the symmetrized generalization of multi poly-Genocchi polynomials with parameters a, b and c is defined as follows:
(41)
$$\mathcal{S}_n^{(m,\alpha)}(x, y; \lambda, a, b, c) = \sum_{k=0}^{m} \binom{m}{k} \frac{\mathcal{G}_n^{(-k,\alpha)}(x; \lambda, a, b, c)}{(\ln a + \ln b)^n} \left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b} \right)^{m-k}.$$

The following theorem contains the double generating function for
$$\mathcal{S}_n^{(m,\alpha)}(x, y; \lambda, a, b, c).$$

Theorem 6.1. *For $n, m \geq 0$, we have*

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{S}_n^{(m,\alpha)}(x, y; \lambda, a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} = \frac{e^{\left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + \alpha \ln a}{\ln a + \ln b}\right)t} e^{2t}}{(1 + \lambda e^t)(e^{2t} - e^{2t+u} + e^u)}. \tag{42}$$

Proof.
$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{S}_n^{(m,\alpha)}(x, y; \lambda, a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} \\
&= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \sum_{k=0}^{m} \frac{\mathcal{G}_n^{(-k,\alpha)}(x; \lambda, a, b, c)}{(\ln a + \ln b)^n} \left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b} \right)^{m-k} \frac{t^n}{n!} \frac{u^m}{k!(m-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \sum_{m=k}^{\infty} \frac{\mathcal{G}_n^{(-k,\alpha)}(x; \lambda, a, b, c)}{(\ln a + \ln b)^n} \left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b} \right)^{m-k} \frac{t^n}{n!} \frac{u^m}{k!(m-k)!} \\
&= \sum_{n=0}^{\infty} \sum_{k=0}^{\infty} \frac{\mathcal{G}_n^{(-k,\alpha)}(x; \lambda, a, b, c)}{(\ln a + \ln b)^n} \frac{t^n}{n!} \frac{u^k}{k!} \sum_{l=0}^{\infty} \left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b} \right)^{l} \frac{u^l}{l!} \\
&= e^{\left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b}\right)u} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{\mathcal{G}_n^{(-k,\alpha)}(\lambda, a, b, c)}{(\ln a + \ln b)^n} \frac{t^n}{n!} \frac{u^k}{k!}.
\end{aligned}$$

Applying (23) yields
$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{S}_n^{(m,\alpha)}(x, y; \lambda, a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} \\
&= e^{\left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b}\right)u} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \mathcal{G}_n^{(-k,\alpha)} \left(\frac{x \ln c + \alpha \ln a}{\ln a + \ln b}; \lambda \right) \frac{t^n}{n!} \frac{u^k}{k!}.
\end{aligned}$$

Now, using (17), we obtain
$$\begin{aligned}
&\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \mathcal{S}_n^{(m,\alpha)}(x, y; \lambda, a, b, c) \frac{t^n}{n!} \frac{u^m}{m!} \\
&= e^{\left(\frac{y \ln c + \alpha \ln a}{\ln a + \ln b}\right)u} e^{\left(\frac{x \ln c + \alpha \ln a}{\ln a + \ln b}\right)t} \sum_{k=0}^{\infty} \frac{L_{i_{(-k)}}(1 - e^{-2t})}{1 + \lambda e^t} \frac{u^k}{k!}
\end{aligned}$$

$$= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}}{1+\lambda e^t}\sum_{k=0}^{\infty} L_{i_{(-k)}}(1-e^{-2t})\frac{u^k}{k!}.$$

Employing the definition of polylogarithm yields

$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\mathcal{S}_n^{(m,\alpha)}(x,y;\lambda,a,b,c)\frac{t^n}{n!}\frac{u^m}{m!}$$
$$= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}}{1+\lambda e^t}\sum_{k=0}^{\infty}\sum_{m=0}^{\infty}\frac{\left(1-e^{-2t}\right)^m}{m^{-k}}\frac{u^k}{k!}$$
$$= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}}{(1+\lambda e^t)(1-((1-e^{-2t})e^u))}$$
$$= \frac{e^{\left(\frac{y\ln c+\alpha\ln a}{\ln a+\ln b}\right)u}e^{\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b}\right)t}e^{2t}}{(1+\lambda e^t)(e^{2t}-e^{2t+u}+e^u)}. \qquad \square$$

The Apostol-type poly-Genocchi polynomials discussed above will be referred to as type 1 Apostol-poly-Genocchi polynomials. Type 2 of these polynomials are introduced in the next section.

7. Type 2 higher order Apostol-poly-Genocchi polynomials

Another variation of Genocchi polynomials is defined using the polyexponential function [21],

$$e_k(z) = \sum_{m=1}^{\infty}\frac{z^m}{(m-1)!m^k}. \tag{43}$$

Note that when $k=1$, $e_1(z)=e^z-1$. Hence, if $z=\log(1+2t)$,

$$e_1(z) = e_1(\log(1+2t)) = e^{\log(1+2t)}-1 = 2t.$$

Definition. The type 2 Apostol-poly-Genocchi polynomials of higher order with parameters a,b and c, denoted by $\mathcal{G}_{n,2}^{(k)}(x;\lambda,a,b,c)$, are defined as follows:

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\frac{e_k(\log(1+2t\ln ab))}{a^{-t}+\lambda b^t}\right)^{\alpha}c^{xt},\ |t| < \frac{\sqrt{(\ln\lambda)^2+\pi^2}}{|\ln a+\ln b|}. \tag{44}$$

The following are special cases of $\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)$:

1. When $x=0$, we use $\mathcal{G}_{n,2}^{(k,\alpha)}(\lambda,a,b)$ to denote $\mathcal{G}_{n,2}^{(k,\alpha)}(0;\lambda,a,b,c)$, the type 2 Apostol-poly-Genocchi numbers with parameters a,b. That is,

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(\lambda,a,b)\frac{t^n}{n!} = \left(\frac{e_k(\log(1+2t\ln ab))}{a^{-t}+\lambda b^t}\right)^{\alpha}. \tag{45}$$

2. When $a=1, b=c=e$, (44) yields

$$\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda)\frac{t^n}{n!} = \left(\frac{e_k(\log(1+2t))}{1+\lambda e^t}\right)^{\alpha}e^{xt}, \tag{46}$$

where the polynomials $\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda) = \mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,1,e,e)$ are called the type 2 Apostol-poly-Genocchi polynomials.

3. When $k = 1$, (44) gives

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\frac{2t\ln ab}{a^{-t}+\lambda b^t}\right)^{\alpha} c^{xt}, \tag{47}$$

where the polynomials $\mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,a,b,c) = \mathcal{G}_{n,2}^{(1,\alpha)}(x;\lambda,a,b,c)$ are called the type 2 Apostol-Genocchi polynomials with parameters a, b and c, which are related to the type 1 Apostol-Genocchi polynomials with parameters a, b and c as follows:

$$\mathcal{G}_{n,2}^{(1,\alpha)}(x;\lambda,a,b,c) = \frac{\mathcal{G}_{n}^{(1,\alpha)}(x;\lambda,a,b,c)}{\ln ab}.$$

4. When $a = 1, b = c = e$, (47) yields

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(1,\alpha)}(x;\lambda,1,e,e)\frac{t^n}{n!} = \left(\frac{2t}{1+\lambda e^t}\right)^{\alpha} e^{xt}, \tag{48}$$

where the polynomials $\mathcal{G}_{n,2}^{(1,\alpha)}(x;\lambda,1,e,e) = \mathcal{G}_{n}^{(\alpha)}(x;\lambda)$ are the type 2 Apostol-Genocchi polynomials in (2). Furthermore, when $\alpha = 1$,

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(1)}(x;\lambda)\frac{t^n}{n!} = \frac{2t}{1+\lambda e^t} e^{xt}, \tag{49}$$

where $\mathcal{G}_{n,2}(x;\lambda) = \mathcal{G}_{n,2}^{(1)}(x;\lambda)$, the type 2 Apostol-Genocchi polynomials.

Now rewrite (44) as follows:

$$\begin{aligned}\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} &= \left(\frac{e_k(\log(1+2t\ln ab))}{a^{-t}(1+\lambda(ab)^t)}\right)^{\alpha} e^{xt\ln c} \\ &= e^{\frac{x\ln c+\alpha\ln a}{\ln ab}t\ln ab}\left(\frac{e_k(\log(1+2t\ln ab))}{1+\lambda e^{t\ln ab}}\right) \\ &= \sum_{n=0}^{\infty}(\ln a+\ln b)^n \mathcal{G}_{n,2}^{(k,\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda\right)\frac{t^n}{n!},\end{aligned}$$

and comparing the coefficients yield the following theorem.

Theorem 7.1. *The type 2 Apostol-poly-Genocchi polynomials with parameters a, b and c satisfy the relation*

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c) = (\ln a+\ln b)^n \mathcal{G}_{n,2}^{(k,\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda\right). \tag{50}$$

When $k = 1$, (50) reduces to the following relation

$$\mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,a,b,c) = (\ln a+\ln b)^{n-j} \mathcal{G}_{n,2}^{(\alpha)}\left(\frac{x\ln c+\alpha\ln a}{\ln a+\ln b};\lambda\right). \tag{51}$$

The next theorem contains an identity that relates the type 2 Apostol-poly-Genocchi polynomials of higher order with parameters a, b and c to Stirling numbers of the first kind $\begin{bmatrix} n \\ m \end{bmatrix}$ defined by

$$\sum_{n=m}^{\infty} \begin{bmatrix} n \\ m \end{bmatrix} \frac{t^n}{n!} = \frac{(\log(1+t))^m}{m!}. \tag{52}$$

Theorem 7.2. *The type* 2 *Apostol-poly-Genocchi polynomials of higher order with parameters a, b and c satisfy the relation*

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x; \lambda, a, b, c) = \sum_{j=0}^{n} \binom{n}{j} (\ln a + \ln b)^{n-j} \mathcal{G}_{n-j,2}^{(\alpha)} \left(\frac{x \ln c + \alpha \ln a}{\ln a + \ln b}; \lambda \right) d_j, \tag{53}$$

where

$$d_j = \sum_{n_1+n_2+\cdots+n_\alpha = j} \prod_{i=1}^{\alpha} c_{n_i} \binom{j}{n_1, n_2, \ldots, n_\alpha} \quad \text{and} \quad c_j = \sum_{m=0}^{j} \frac{(2\ln ab)^j \begin{bmatrix} j+1 \\ m+1 \end{bmatrix}}{(j+1)(m+1)^{k-1}}.$$

Proof. Applying the definition of polyexponential function (43), (44) may be written as

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x; \lambda, a, b, c) \frac{t^n}{n!} \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=1}^{\infty} \frac{(\log(1 + 2t \ln ab))^m}{(m-1)! m^k} \right)^\alpha \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=0}^{\infty} \frac{(\log(1 + 2t \ln ab))^{m+1}}{m!(m+1)^k} \right)^\alpha \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \frac{\log(1 + 2t \ln ab))^{m+1}}{(m+1)!} \right)^\alpha .
\end{aligned}$$

This can further be written, using (52), as follows:

$$\begin{aligned}
&\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x; \lambda, a, b, c) \frac{t^n}{n!} \\
&= \frac{c^{xt}}{(a^{-t} + \lambda b^t)^\alpha} \left(\sum_{m=0}^{\infty} \frac{1}{(m+1)^{k-1}} \sum_{j=m+1}^{\infty} \begin{bmatrix} j \\ m+1 \end{bmatrix} \frac{(2t \ln ab)^j}{j!} \right)^\alpha \\
&= c^{xt} \left(\frac{2t \ln ab}{a^{-t} + \lambda b^t} \right)^\alpha \left(\sum_{m=0}^{\infty} \sum_{j=m}^{\infty} \frac{\begin{bmatrix} j+1 \\ m+1 \end{bmatrix}}{(j+1)(m+1)^{k-1}} \frac{(2t \ln ab)^j}{j!} \right)^\alpha .
\end{aligned}$$

Applying (47) yields

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \left(\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}\right)\left(\sum_{j=0}^{\infty} c_j\frac{t^j}{j!}\right)^{\alpha}, \tag{54}$$

where

$$c_j = \sum_{m=0}^{j} \frac{(2\ln ab)^j \begin{bmatrix} j+1 \\ m+1 \end{bmatrix}}{(j+1)(m+1)^{k-1}}.$$

Note that, using (35), Equation (54) can be expressed as

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} = \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\binom{n}{j}\mathcal{G}_{n-j,2}^{(\alpha)}(x;\lambda,a,b,c)d_j\right\}\frac{t^n}{n!},$$

where

$$d_j = \sum_{n_1+n_2+\cdots+n_\alpha=j}\prod_{i=1}^{\alpha} c_{n_i}\binom{j}{n_1,n_2,\ldots,n_\alpha}.$$

This immediately gives (53) by comparing the coefficients and using Equation (50). □

The next theorem shows the relationship between the type 2 Apostol-poly-Genocchi polynomials of higher order with parameters a, b and c and the type 2 Apostol-poly-Bernoulli polynomials defined as

$$\left(\frac{e_k(\log(1+t))}{\lambda e^t - 1}\right)^{\alpha} e^{xt} = \sum_{n=0}^{\infty}\mathcal{B}_{n,2}^{(k,\alpha)}(x;\lambda)\frac{x^n}{n!}. \tag{55}$$

These polynomials and those in (28) are generalizations of Bernoulli-type polynomials.

Theorem 7.3. *The type 2 Apostol-poly-Genocchi polynomials of higher order with parameters a, b, c satisfy the relation*

$$\begin{aligned} &\mathcal{G}_{n,2}^{(k)}(x;\lambda,a,b,c) \\ &= \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\lambda^{\alpha-j}\mathcal{B}_{n,2}^{(k,\alpha)}\left(\frac{(\alpha-j)\ln b + x\ln c + (2\alpha-j)\ln a}{2(\ln a+\ln b)};\lambda^2\right)2^n(\ln ab)^n. \end{aligned} \tag{56}$$

Proof. Rewrite Equation (44) as

$$\begin{aligned} &\sum_{n=0}^{\infty}\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!} \\ &= \left(\frac{e_k(\log(1+2t\ln ab))}{a^{-t}+\lambda b^t}\right)^{\alpha} e^{xt\ln c} \\ &= \left(\frac{e_k(\log(1+2t\ln ab))}{a^{-2t}-(\lambda b^t)^2}\right)^{\alpha}(a^{-t}-\lambda b^t)^{\alpha}e^{xt\ln c} \end{aligned}$$

$$= \left(\frac{e_k(\log(1+2t\ln ab))}{(1-(\lambda(ab)^t)^2)}\right)^{\alpha} (e^{-t\ln a} - \lambda e^{t\ln b})^{\alpha} e^{xt\ln c} e^{2t\alpha\ln a}$$
$$= \left(\frac{e_k(\log(1+2t\ln ab))}{-(\lambda^2 e^{2t\ln(ab)}-1)}\right)^{\alpha} \left(e^{t(-\ln a+(x\ln c/\alpha)+2\ln a)} - \lambda e^{t(\ln b+(x\ln c/\alpha)+2\ln a)}\right)^{\alpha}.$$

Applying the Binomial Theorem yields

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$
$$= \left(\frac{e_k(\log(1+2t\ln ab))}{-(\lambda^2 e^{2t\ln(ab)}-1)}\right)^{\alpha} \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-\lambda)^{\alpha-j} e^{jt(\ln a+(x\ln c/\alpha))} e^{(\alpha-j)t(\ln b+(x\ln c/\alpha)+2\ln a)}$$
$$= \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\lambda^{\alpha-j}\left(\frac{e_k(\log(1+2t\ln ab))}{\lambda^2 e^{2t\ln(ab)}-1}\right)^{\alpha} e^{[((\alpha-j)\ln b+\alpha(x\ln c/\alpha)+(2\alpha-j)\ln a)/2\ln ab](2t\ln ab)}.$$

Using the definition of type 2 Apostol-poly-Bernoulli polynomials of higher order in (55), we have

$$\sum_{n=0}^{\infty} \mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)\frac{t^n}{n!}$$
$$= \sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\lambda^{\alpha-j}\left\{\sum_{n=0}^{\infty}\mathcal{B}_{n,2}^{(k,\alpha)}\left(\frac{(\alpha-j)\ln b+x\ln c+(2\alpha-j)\ln a}{2(\ln a+\ln b)};\lambda^2\right)2^n(\ln ab)^n\frac{t^n}{n!}\right\}$$
$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{\alpha}\binom{\alpha}{j}(-1)^j\lambda^{\alpha-j}\mathcal{B}_{n,2}^{(k,\alpha)}\left(\frac{(\alpha-j)\ln b+x\ln c+(2\alpha-j)\ln a}{2(\ln a+\ln b)};\lambda^2\right)2^n(\ln ab)^n\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (56). □

Remark 7.4. It is left to the reader to prove the following identities. The proof can be done following the proof of the corollary responding identities in Sections 3, 4 and 5 for type 1 Apostol-poly-Genocchi polynomials of higher order with parameters a, b and c:

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{i=0}^{n}\binom{n}{i}(\ln c)^{n-i}\mathcal{G}_{i,2}^{(k,\alpha)}(\lambda,a,b)x^{n-i},$$
$$\mathcal{G}_{n,2}^{(k,\alpha)}(x+1;\lambda,a,b,c) = \sum_{r=0}^{n}\binom{n}{r}(\ln c)^{r}\mathcal{G}_{n-r,2}^{(k,\alpha)}(x;\lambda,a,b,c),$$
$$\frac{d}{dx}\mathcal{G}_{n+1,2}^{(k,\alpha)}(x;\lambda,a,b,c) = (n+1)(\ln c)\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c),$$
$$\mathcal{G}_{n,2}^{(k,\alpha)}(x+y;\lambda,a,b,c) = \sum_{i=0}^{\infty}\binom{n}{i}(\ln c)^{n-i}\mathcal{G}_{i,2}^{(k,\alpha)}(x;\lambda,a,b,c)y^{n-i},$$
$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c) = \sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{{l \atop m}\right\}\binom{n}{l}(\ln c)^{l}\mathcal{G}_{n-l,2}^{(k,\alpha)}(-m\ln c;\lambda,a,b)(x)^{(m)},$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b,c)=\sum_{m=0}^{\infty}\sum_{l=m}^{n}\left\{ l \atop m \right\}\binom{n}{l}(\ln c)^{l}\mathcal{G}_{n-l,2}^{(k,\alpha)}(\lambda,a,b)(x)_{m},$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b)=\sum_{l=0}^{n}\sum_{m=0}^{n-l}\binom{n}{l}\left\{ l+s \atop s \right\}\frac{\binom{n-l}{m}}{\binom{l+s}{s}}\mathcal{G}_{n-l-m}^{(k,\alpha)}(\lambda,a,b)B_{m}^{(s)}(x\ln c;\lambda),$$

$$\mathcal{G}_{n,2}^{(k,\alpha)}(x;\lambda,a,b)=\sum_{m=0}^{\infty}\frac{\binom{n}{m}}{(1-\mu)^{s}}\sum_{j=0}^{s}\binom{s}{j}(-\mu)^{s-j}\mathcal{G}_{n-m,2}^{(k,\alpha)}(j;\lambda,a,b)F_{m}^{(s)}(x;\mu).$$

8. Conclusion

Using polylogarithm Apostol-type polynomials of higher order with parameters a, b and c and a variation of poly-Genocchi polynomials, called the Apostol-type poly-Genocchi polynomials of higher order, also known as type 1 Apostol-poly-Genocchi polynomials of higher order were introduced. Some interesting properties and identities of these polynomials parallel to those of the poly-Euler polynomials and poly-Bernoulli polynomials were proved. Using a differential identity, the type 1 Apostol-poly-Genocchi polynomials were classified as Appell polynomials, which, consequently, gave some interesting relations. Moreover, these type 1 Apostol-poly-Genocchi polynomials of higher order were expressed in terms of Stirling numbers of the second kind and Apostol-type poly-Bernoulli polynomials of higher order. Furthermore, the symmetrized generalization of the type 1 Apostol-poly-Genocchi polynomials of higher order was introduced and a double generating function was established. Type 2 Apostol-poly-Genocchi polynomials of higher order with parameters a, b and c were also defined. Several identities were established, two of which showed the connections of these polynomials with Stirling numbers of the first kind and the type 2 Apostol-type poly-Bernoulli polynomials. One may try to investigate the two types of Apostol-poly-Bernoulli polynomials of higher order defined in (28) and (55), by establishing more properties and extending them to a more general form by adding three more parameters a, b and c.

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