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Weak FI-extending Modules with ACC or DCC on Essential Submodules

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Abstract. In this paper we study modules with the WFI^+-extending property. We prove that if M satisfies the WFI^+-extending, pseudo duo properties and $M/(\text{Soc}\,M)$ has finite uniform dimension then M decompose into a direct sum of a semisimple submodule and a submodule of finite uniform dimension. In particular, if M satisfies the WFI^+-extending, pseudo duo properties and ascending chain (respectively, descending chain) condition on essential submodules then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2. Moreover, we show that if M is a WFI-extending module with pseudo duo, C_2 and essential socle then the quotient ring of its endomorphism ring with Jacobson radical is a (von Neumann) regular ring. We provide several examples which illustrate our results.

1. Introduction

Assume that all rings are associative and have identity elements and all modules are unital right modules. Let R be any ring and M a right R-module. Recall that M is called *CS-module* (or *extending module, module with* C_1) if every submodule of M is essential in a direct summand of M. Equivalently, every complement in M is a direct summand of M (see [7, 10, 18]). The class of extending modules contains injective, semisimple and uniform modules (i.e., every non-zero submodule

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is essential in the module). We say that the module M has *finite uniform* (*Goldie*) *dimension* if M does not contain an infinite direct sum of non-zero submodules. It is well-known that a module M has finite uniform dimension if and only if there exists a positive integer n and uniform submodules U_i ($1 \leq i \leq n$) of M such that $U_1 \oplus U_2 \oplus \cdots \oplus U_n$ is an essential submodule of M. In this case n is an invariant of the module called the *uniform dimension* of M (see, [1, p.294 Example 2] or [18, p.81]).

Armendariz [2, Proposition 1.1] proved that a module M satisfies DCC (descending chain condition) on essential submodules if and only if $M/(\operatorname{Soc} M)$ is an Artinian module. On the other hand, Goodearl [8, Proposition 3.6] proved that the module M satisfies ACC (ascending chain condition) on essential submodules if and only if $M/(\operatorname{Soc} M)$ is a Noetherian module. It is proved in [13, Theorem 2.1] that the following statements are equivalent for a module M:

(i) M/N has finite uniform dimension for every essential submodule N of M,

(ii) every homomorphic image of $M/(\operatorname{Soc} M)$ has finite uniform dimension.

Camillo and Yousif [6, Corollary 3] proved that if M is a CS-module and $M/(\operatorname{Soc} M)$ has finite uniform dimension then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 of M and submodule M_2 with finite uniform dimension, and in this case M is a direct sum of uniform modules. They deduced in [6, Proposition 5] that if M is a CS-module then M has ACC (respectively, DCC) on essential submodules if and only if $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2 of M.

A module M is called a *weak CS-module* if, for each semisimple submodule S of M, there exists a direct summand K of M such that S is essential in K. Clearly, CS-modules are weak CS-modules. Smith [12, Corollary 2.7, Theorem 2.8] showed that the result of [6] mentioned above can be extended to weak CS-modules. A module M is called C_{11}-module if, every submodule of M has a complement which is a direct summand of M. Smith and Tercan [14, Theorem 5.2, Corollary 5.3] extended the result of [6] to modules with C_{11}^{+} (i.e., every direct summand of the module satisfies C_{11} property).

A module M is called *weak* C_{11}*-module*, denoted WC_{11}, if each of its semisimple submodules has a complement which is a direct summand. Tercan [16, Theorem 11, Corollary 12] showed that aforementioned results of [14] can be extended to WC_{11}^{+}-modules.

A module M is called *FI-extending* if every fully invariant submodule (i.e., every submodule such that the image under all endomorphisms contained in itself) is essential in a direct summand of M (see [3, 4]). Recently, a weak version of FI-extending was introduced and investigated. To this end, following [19], a module is called *Weak FI-extending* (or, WFI-extending) if, each of its semisimple fully invariant submodules is essential in a direct summand of M. If M is any module and X is any simple submodule, the sum of all submodules of M that are isomorphic to X is a submodule called the homogeneous component of M generated by X. It is well known that the socle of M is the direct sum of the various homogeneous components and any homogeneous component of socle is a fully invariant submodule of the module M. Thus, if a module is WFI-extending then any homogeneous component of socle is essential in a direct summand of the module. Note that the following implications hold for a module M:

$$\begin{array}{ccccc} CS & \Longrightarrow & C_{11} & \Longrightarrow & FI\text{-extending} \\ \Downarrow & & \Downarrow & & \Downarrow \\ WCS & \Longrightarrow & WC_{11} & \Longrightarrow & WFI\text{-extending} \end{array}$$

No other implications can be added to this table in general. To see why this is the case, please consult [19]. Note that it is an open problem to determine whether the FI-extending (and also the WFI-extending, WC_{11}, WCS) property is inherited by direct summands or not?

The purpose of this paper is to try to extend the result of [16, Theorem 11, Corollary 12] to WFI^+-extending (and so also FI^+-extending) modules. To do this, we need to add the pseudo duo condition on the class of fully invariant submodules of the module. Moreover, we also extend a result on the endomorphism ring of continuous modules to WFI-extending modules with the pseudo duo condition which yields that the quotient ring of endomorphism ring with its Jacobson radical is a (von Neumann) regular ring. For any unexplained terminology and definitions, we refer to [1, 5, 10, 18].

2. Weak FI^+-extending Modules

Let P be some module property of modules. Following [14], we shall say that a module M satisfies P^+ if every direct summand of M satisfies P. For example, if a module has injective socle then it satisfies WC_{11}^+ and hence also it satisfies WFI^+. Moreover, if R is a Dedekind domain then any R-module M with finite uniform dimension is a WFI^+-extending module (see [19]). Recall that every direct summand of a non-zero C_{11}^+-module with finite uniform dimension is a (finite) direct sum of uniform modules [14, Proposition 4.4]. However, this is not true for WFI^+-extending modules, in general. The following example clarifies the situation:

Example 2.1. Let R be a principal ideal domain. If R is not a complete discrete valuation ring then there exists an indecomposable torsion-free R-module M of rank 2 by [9, Theorem 19]. For M, $\operatorname{Soc} M = 0$. So that M satisfies WFI^+ and M_R has finite uniform dimension, namely 2. But M is not a direct sum of uniform modules.

For more examples similar to Example 2.1 (see [17, Corollary 16]). Surprisingly, Example 2.1 and [17, Corollary 16] also show that we can not replace WFI^+ with FI^+ in the former case. Since WFI-extending modules are based on the semisimple fully invariant submodules, the following companion condition works well with WFI-extending property. To this end, M is said to have *pseudo duo property* provided that any semisimple submodule of M has at least one fully invariant (in M) direct summand in its decomposition i.e., if N is a semisimple submodule of M

whenever $N = N_1 \oplus N_2$ then at least one of the N_i $(i = 1, 2)$ is a fully invariant submodule of M. Observe that any duo module clearly satisfies the pseudo duo property. However, there are several modules with the pseudo duo property which are not duo modules. In fact, any non duo module with zero socle would be an example. In particular, let $R = \begin{bmatrix} \mathbb{Z} & \mathbb{Z} \\ 0 & \mathbb{Z} \end{bmatrix}$. Then R_R is not duo. But $\operatorname{Soc} R_R = 0$. Hence R satisfies the pseudo duo property. One might wonder whether WFI^+-extending with the pseudo duo condition implies FI^+-extending (or C_{11}^+) or not. However, [19, Example 2.4] makes it clear that the aforementioned implication is not true, in general.

Theorem 2.2. *Let M be a finitely generated WFI^+-extending module with the pseudo duo property. Let N be a semisimple submodule of M such that M/N has finite uniform dimension. Then N is finitely generated.*

Proof. Let $n < \infty$ be the uniform dimension of M/N. Suppose that N is not finitely generated. Then there exist non-finitely generated submodules N_1 and N_2 such that $N = N_1 \oplus N_2$. By hypothesis, at least one of the N_i $(i = 1, 2)$ is fully invariant in M, say N_1. By WFI-extending, there exist submodules M_1, M' of M such that $M = M_1 \oplus M'$ and N_1 is essential in M_1. Then $\operatorname{Soc} M = \operatorname{Soc} M_1 \oplus \operatorname{Soc} M' = N_1 \oplus \operatorname{Soc} M'$. Hence $N = N_1 \oplus (N \cap \operatorname{Soc} M')$ by the modular law.

Now $N_2 \cong N \cap \operatorname{Soc} M'$ so the submodule $N \cap \operatorname{Soc} M'$ is not finitely generated. Repeating this argument, there exist $N_i \leq M_i \leq M$ $(2 \leq i \leq n + 1)$ such that for each $2 \leq i \leq n + 1$, N_i is not finitely generated, $M = M_1 \oplus M_2 \oplus \cdots \oplus M_{n+1}$. Let $L = N_1 \oplus N_2 \oplus \cdots \oplus N_{n+1}$. Then $M/L \cong (M_1/N_1) \oplus (M_2/N_2) \oplus \cdots \oplus (M_{n+1}/N_{n+1})$. Since M/L has finite uniform dimension then there exists $1 \leq i \leq n + 1$ such that $M_i = N_i$. But M_i is finitely generated and hence so is N_i, a contradiction. Thus N is finitely generated. □

The next example shows that WFI^+-extending property is not superfluous in Theorem 2.2.

Example 2.3. Let K be a field and V an infinite dimensional vector space over K. Let R be the trivial extension K with V i.e.,

$$R = \begin{bmatrix} K & & V \\ & \setminus & \\ 0 & & K \end{bmatrix} = \left\{ \begin{bmatrix} k & v \\ 0 & k \end{bmatrix} \mid k \in K,\ v \in V \right\}.$$

Then R is a commutative indecomposable ring with respect to the usual matrix operations. Moreover, R_R is not WFI^+-extending with the pseudo duo property and contains a semisimple submodule I such that R/I has finite uniform dimension but I is not finitely generated.

Proof. Let $I = \operatorname{Soc} R = \begin{bmatrix} 0 & V \\ 0 & 0 \end{bmatrix}$. It is straightforward to see that R_R is not WFI-extending with pseudo duo property. Define $\varphi : R \to K$ by $\varphi(\begin{bmatrix} k & v \\ 0 & k \end{bmatrix}) = k$ where

$k \in K$, $v \in V$. Then φ is an epimorphism with kernel I. Thus R/I has uniform dimension 1. Since V is infinite dimensional, I is not finitely generated. □

Corollary 2.4. *Let M be a finitely generated FI^+-extending module with the pseudo duo property. If $M/(\operatorname{Soc} M)$ has finite uniform dimension then $\operatorname{Soc} M$ is finitely generated.*

Proof. Immediate by Theorem 2.2. □

Now, let us think of general modules over arbitrary rings. Since we require both the pseudo duo property and that $M/(\operatorname{Soc} M)$ has finite uniform dimension in our next results, it would be better to clarify these conditions are independent.

Example 2.5.

(i) Let M be the free $\mathbb{Z}$-module of infinite rank i.e., $M = \bigoplus_{i=1}^{\infty} \mathbb{Z}$. Then $\operatorname{Soc} M_{\mathbb{Z}} = 0$. Hence M satisfies pseudo duo property. However, $M/(\operatorname{Soc} M) \cong M_{\mathbb{Z}}$ which has infinite uniform dimension.

(ii) Let R be a prime ring and let $M_R = (R \oplus R)_R$. Then, it is clear that $\operatorname{Soc} M = \operatorname{Soc} R \oplus \operatorname{Soc} R$ which is essential in M_R and hence $M/\operatorname{Soc} M$ has finite uniform dimension. Now, let $N = \operatorname{Soc} R \oplus \operatorname{Soc} R$. Define $f_1 : M \to M$ by $f_1(x, y) = (y, 0)$ and $f_2 : M \to M$ by $f_2(x, y) = (0, x)$. Obviously f_1, $f_2 \in \operatorname{End}(M_R)$. Let $N_1 = \operatorname{Soc} R \oplus 0$, $N_2 = 0 \oplus \operatorname{Soc} R$. So, we have $f_1(N_2) = N_1 \not\subseteq N_2$ and $f_2(N_1) = N_2 \not\subseteq N_1$. It follows that M_R does not have pseudo duo property.

The following is a key lemma for our main theorem in this section.

Lemma 2.6. *Let M be a module such that M satisfies WFI^+ and has pseudo duo property, and such that $M/\operatorname{Soc} M$ has finite uniform dimension. Suppose that $\operatorname{Soc} M$ is contained in a finitely generated submodule of M. Then M has finite uniform dimension.*

Proof. Suppose M does not have finite uniform dimension. Then $\operatorname{Soc} M$ is not finitely generated. Then there exist submodules S_1, S_2 of $\operatorname{Soc} M$ such that S_i is not finitely generated for $i = 1, 2$, and $\operatorname{Soc} M = S_1 \oplus S_2$. By the pseudo duo assumption, without loss of generality, we may assume that S_1 is fully invariant in M. By hypothesis, there exist submodules K, K' of M such that $M = K \oplus K'$, and S_1 is essential in K. By [1, Proposition 9.7, 9.119], $S_1 \oplus S_2 = \operatorname{Soc} M = S_1 \oplus \operatorname{Soc} K'$. Thus $\operatorname{Soc} K' \cong S_2$ and hence $\operatorname{Soc} K'$ is not finitely generated. Also, $\operatorname{Soc} K \oplus \operatorname{Soc} K' = \operatorname{Soc} M = S_1 \oplus \operatorname{Soc} K'$, so that $\operatorname{Soc} K \cong S_1$, and hence $\operatorname{Soc} K$ is not finitely generated. By hypothesis, there exists a finitely generated submodule N of M such that $\operatorname{Soc} M \leq N$. Suppose that $K = \operatorname{Soc} K$. Then $\operatorname{Soc} K$ is a direct summand of M and hence also a direct summand of N. It follows that $\operatorname{Soc} K$ is finitely generated which is a contradiction. Thus $K \neq \operatorname{Soc} K$. Similarly, $K' \neq \operatorname{Soc} K'$. Now, $M/\operatorname{Soc} M \cong [K/(\operatorname{Soc} K)] \oplus [K'/(\operatorname{Soc} K')]$. It follows that the modules $K/(\mathrm{Soc}\,K)$ and $K'/(\mathrm{Soc}\,K')$ each have smaller uniform dimension than $M/(\mathrm{Soc}\,M)$. By induction on the uniform dimension of $M/(\mathrm{Soc}\,M)$, we conclude that K and K' both have finite uniform dimension, and so does $M = K \oplus K'$, a contradiction. Thus M has finite uniform dimension. □

Now we have the following result which was pointed out in the introduction.

Theorem 2.7. *Let M be a WFI^+-extending module with the pseudo duo property such that $M/(\mathrm{Soc}\,M)$ has finite uniform dimension. Then M contains a semisimple submodule M_1 and a submodule M_2 with finite uniform dimension such that $M = M_1 \oplus M_2$.*

Proof. If $M = \mathrm{Soc}\,M$ then there is nothing to prove. Suppose that $M \neq \mathrm{Soc}\,M$. Let $m \in M$, $m \notin \mathrm{Soc}\,M$. Then $\mathrm{Soc}\,M = \mathrm{Soc}\,(mR) \oplus X$ for some module X of M. Now, by the pseudo duo property one of the $\mathrm{Soc}\,(mR)$ or X is fully invariant in M. First assume that $\mathrm{Soc}\,(mR)$ is fully invariant in M. By hypothesis, there exist submodules K, K' of M such that $M = K \oplus K'$ and $\mathrm{Soc}\,(mR)$ is essential in K. Hence $\mathrm{Soc}\,K = \mathrm{Soc}\,(mR) \leq mR$. By Lemma 2.6, K has finite uniform dimension. Now $K \neq \mathrm{Soc}\,K$. Otherwise, $K \leq mR$ and hence $mR = K \oplus (mR \cap K')$, by the modular law. Since $mR \cap K' \cong K + mR/mR$, $mR = K = \mathrm{Soc}\,K$. It follows that $m \in \mathrm{Soc}\,K$ and so $m \in \mathrm{Soc}\,M$ which is a contradiction.

Now assume that X is a fully invariant submodule of M. By hypothesis, $M = K \oplus K'$ and X is essential in K' where K, K' are submodules of M. So $\mathrm{Soc}\,X = \mathrm{Soc}\,K' = X$. Hence $\mathrm{Soc}\,M = \mathrm{Soc}\,(mR) \oplus X = \mathrm{Soc}\,K \oplus X$. It follows that $\mathrm{Soc}\,(mR) \cong \mathrm{Soc}\,K$ i.e., there exists an isomorphism $\alpha : \mathrm{Soc}\,K \to \mathrm{Soc}\,(mR)$. Note that $\mathrm{Soc}\,(mR) \leq mR$. So $\mathrm{Soc}\,K \leq \alpha^{-1}(m)R$. By Lemma **??**, K has finite uniform dimension. Observe that $K \neq \mathrm{Soc}\,K$. If it were $K = \mathrm{Soc}\,K$ then $\alpha^{-1}(m)R = K \oplus (\alpha^{-1}(m)R \cap K')$, by the modular law. Since $\alpha^{-1}(m)R \cap K' \cong K + \alpha^{-1}(m)R/\alpha^{-1}(m)R$, $\alpha^{-1}(m)R \cap K' = 0$. Therefore $\alpha^{-1}(m)R = K' = \mathrm{Soc}\,K'$. Hence $mR = \alpha(\mathrm{Soc}\,K') = \mathrm{Soc}\,(mR)$ which yields that $m \in \mathrm{Soc}\,(mR)$. So that $m \in \mathrm{Soc}\,M$, which is a contradiction. Now $M/(\mathrm{Soc}\,M) \cong K/(\mathrm{Soc}\,K) \oplus K'/(\mathrm{Soc}\,K')$ implies that the module $K/(\mathrm{Soc}\,K)$ has smaller uniform dimension than $M/(\mathrm{Soc}\,M)$. By induction on the uniform dimension of $M/(\mathrm{Soc}\,M)$, there exist submodules K_1, K_2 of K such that $K = K_1 \oplus K_2$, K_1 is semisimple and K_2 has finite uniform dimension. Then M is the direct sum of the semisimple submodule K_1, and the submodule $K_2 \oplus K'$, which has finite uniform dimension. □

Next we apply the former result to WFI^+-extending (and, also FI^+-extending) modules which satisfies ACC (respectively, DCC) on essential submodules.

Corollary 2.8. *Let M be a WFI^+-extending module with the pseudo duo property which satisfies ACC (respectively, DCC) on essential submodules. Then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respetively, Artinian) submodule M_2.*

Proof. We prove the result in the ACC case, the DCC case is similar. Suppose M satisfies ACC on essential submodules. By [8, Proposition 3.6], $M/(\mathrm{Soc}\,M)$ is

Noetherian. Hence by Theorem 2.7, $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and submodule M_2 with finite uniform dimension. Now $\operatorname{Soc} M = M_1 \oplus (\operatorname{Soc} M_2)$ by [1, Proposition 1.19] and hence $M/(\operatorname{Soc} M) \cong M_2/(\operatorname{Soc} M_2)$. Thus $M_2/(\operatorname{Soc} M_2)$ is Noetherian. But $\operatorname{Soc} M_2$ is Noetherian, because M_2 has finite uniform dimension. Thus M_2 is Noetherian. □

Recall that a module M is said to have SIP if the intersection of every pair of direct summands is also a direct summand (see, for example [18]). So, we have the following corollaries:

Corollary 2.9. *Let M be a WFI-extending module with the pseudo duo property which has SIP. Assume M satisfies ACC (respectively, DCC) on essential submodules. Then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2.*

Proof. By [19, Theorem 3.12], M is WFI^+-extending module. Now, Corollary 2.8 yields the result □

Corollary 2.10. *Let M be an FI-extending module with the pseudo duo property which has SIP. Assume M satisfies ACC (respectively, DCC) on essential submodules. Then $M = M_1 \oplus M_2$ for some semisimple submodule M_1 and Noetherian (respectively, Artinian) submodule M_2.*

Proof. Immediate by Corollary 2.9. □

We close this section by giving an example which illustrates that the converse of Theorem 2.7 is not true, in general.

Example 2.11. Let F be a field and $T = \frac{F[x]}{<x^4>} = \{a\bar{1} + b\bar{x} + c\bar{x}^2 + d\bar{x}^3 \mid a, b, c, d \in F \text{ and } \bar{x} = x + <x^4>\}$. Put $R = F + F\bar{x}^2 + F\bar{x}^3$ which is a subring of T. Note that R is a commutative local ring and its ideals of are 0, R, $F\bar{x}^2$, $F\bar{x}^3$, $F\bar{x}^2 \oplus F\bar{x}^3$ (see [5, Exercise 8.1.10]). Observe that $\operatorname{Soc} R = \mathrm{J}(R) = F\bar{x}^2 \oplus F\bar{x}^3$ which is essential in R. Clearly R is not WFI-extending.

Now, let $M = M_1 \oplus M_2$ be the right R-module where $M_1 = \operatorname{Soc} R$ and $M_2 = R$. So, $0 \oplus M_2$ is not WFI-extending which gives that M is not WFI^+-extending. Next, let us show that M does not satisfy the pseudo duo property. For, let $N = \operatorname{Soc} M = M_1 \oplus M_1$ and $N_1 = M_1 \oplus 0$, $N_2 = 0 \oplus M_1$. In a similar argument in Example 2.5, we have that neither N_1 nor N_2 is fully invariant in M. Therefore M does not satisfy the pseudo duo property.

3. Endomorphism Rings of Weak FI-extending Modules

In this section our concern is the endomorphism ring of weak FI-extending modules. We will use S and $\mathrm{J}(S)$ to denote the endomorphism ring of a module M and the Jacobson radical of S respectively. Further Δ will stand for the ideal $\{\alpha \in S \mid \ker \alpha \text{ is essential in } M\}$. Recall that a CS-module M is called continuous if, for each direct summand N of M and each monomorphism $\alpha : N \to M$, the submodule $\varphi(N)$ is also a direct summand of M (see [10, 18]). It was proved in

[10, Proposition 3.5] that if M is continuous, then S/Δ is a (von Neumann) regular ring and $\Delta = \mathrm{J}(S)$. This nice result was generalized to modules with C_{11} and C_2 in [15, Theorem 3.3] as well as weak C_{11} modules with C_2 and essential socle in [17, Theorem 12]. It is natural to expect that whether [10, Proposition 3.5] can be generalized to weak FI-extending modules with C_2. However, [17, Example 11] eliminates this expectation. On the other side, let M be the $\mathbb{Z}$-module $(\mathbb{Z}/\mathbb{Z}p) \oplus \mathbb{Q}$ where p is any prime integer. Then M is a WFI-extending module with C_2 (see [14]). Note that $\mathrm{Soc}\, M_{\mathbb{Z}} = (\mathbb{Z}/\mathbb{Z}p) \oplus 0$ which is not essential in M. Observe that $M_{\mathbb{Z}}$ has the pseudo duo property. So, we have the following result.

Theorem 3.1. *Let M be a WFI-extending module with the pseudo duo property, C_2 and essential socle. Then S/Δ is a (von Neumann) regular ring and $\Delta = \mathrm{J}(S)$.*

Proof. Let $\alpha \in S$. Let $K = \mathrm{Soc}(\ker \alpha)$. Then K is a direct summand of $\mathrm{Soc}\, M$. Hence $\mathrm{Soc}\, M = K \oplus X$ for some submodule X of M. By the pseudo duo property, we think of submodules K and X seperately. First assume that K is fully invariant in M. By hypothesis, there exists a complement L of K such that L is a direct summand of M (see [19, Proposition 2.3]). Then $M = L \oplus L'$ for some submodule L' of M. Since $\mathrm{Soc}\, M$ is essential in M, $\ker \alpha \cap L = 0$. It follows that $\alpha|_L$ is a monomorphism. So, by C_2, $\alpha(L)$ is a direct summand of M. Hence there exists $\beta \in S$ such that $\beta\alpha = 1_L$. Then $(\alpha - \alpha\beta\alpha)(K \oplus L) = (\alpha - \alpha\beta\alpha)(L) = 0$, and so $K \oplus L \leq \ker(\alpha - \alpha\beta\alpha)$. Since $K \oplus L$ is essential in M, $\alpha - \alpha\beta\alpha \in \Delta$. Therefore S/Δ is a (von Neumann) regular ring.

Next assume that X is a fully invariant submodule of M. By WFI-extending property, there exist direct summands L, L' of M such that $M = L \oplus L'$ and X is essential in L. Since $\mathrm{Soc}\, M$ is essential in M, $X \cap K$ is essential in $L \cap \ker \alpha$ which gives that $L \cap \ker \alpha = 0$. Therefore $\alpha|_L$ is a monomorphism. Thus $\alpha(L)$ is a direct summand of M, by C_2. Then there exists $\gamma \in S$ such that $\gamma\alpha = 1|_L$. It can be easily seen that $\ker \alpha \oplus L$ is essential in M. Now, let $W = \ker \alpha$. So, we have $(\alpha - \alpha\gamma\alpha)(W \oplus L) = (\alpha - \alpha\gamma\alpha)(L) = 0$, and so $W \oplus L \leq \ker(\alpha - \alpha\gamma\alpha)$. Since $W \oplus L$ is essential in M, $\alpha - \alpha\gamma\alpha \in \Delta$. Thus S/Δ is a von Neumann regular ring.

In any case, we have that S/Δ is a regular ring. This also proves that $\mathrm{J}(S) \leq \Delta$. Now, let $f \in \Delta$. Since $\ker f \cap \ker(1 - f) = 0$ and $\ker f$ is essential in M, $\ker(1 - f) = 0$. Hence $(1 - f)M$ is a direct summand of M, by C_2. However, $(1 - f)M$ is essential in M since $\ker f \leq (1 - f)M$. Thus $(1 - f)M = M$, and therefore $1 - f$ is a unit in S. It follows that $f \in \mathrm{J}(S)$, and hence $\Delta = \mathrm{J}(S)$. □

Corollary 3.2. *Let M be an FI-extending module with the pseudo duo property, C_2, and essential socle. Then S/Δ is a (von Neumann) regular ring and $\Delta = \mathrm{J}(S)$.*

Proof. By Theorem 3.1. □

Corollary 3.3. *Let M be a right nonsingular module with the pseudo duo property and essential socle. If M is WFI-extending (or FI-extending) with C_2, then S is a regular ring.*

Proof. Let $f \in \Delta$ and $W = \ker f$. Then, for any $x \in M$, $N = \{r \in R \mid xr \in W\}$ is an essential right ideal of R. Now $f(x)N = 0$. Since M is nonsingular, $f(x) = 0$, and since x was arbitrary $f = 0$ (see [20, Lemma 1.3]). It follows that $\Delta = 0$. Hence the result follows by the Theorem 3.1. □

Note that there are commutative, local rings R such that $\mathrm{Soc}\, R = \mathrm{J}^2$ is simple essential in R. These rings have all the stated properties in Theorem 3.1. For, such rings, see [11, Example 2.6].

In the sense of construction certain examples, Corollary 3.3 is a useful tool. For example, let R be any domain which satisfies C_2 condition (i.e., division ring). Let M be the right R-module R. Then, it is easy to see that M has all of the assumptions of Corollary 3.3 except M has essential socle. However, $S = \mathrm{End}\,(M_R)$ is not (von Neumann) regular. Thus the condition essential socle in Theorem 3.1 is not superfluous.

Furthermore, the next example shows that the pseudo duo assumption in Theorem 3.1 is not unnecessary either.

Example 3.4. Let R be any local Kasch ring such that $\mathrm{J}\,(R) = \mathrm{Soc}\, R$ is simple and essential in R (see [11, Example 2.5]). Now, let M be the right R-module $R \oplus R$. Observe that $\mathrm{Soc}\, M = (\mathrm{Soc}\, R) \oplus (\mathrm{Soc}\, R)$ is essential in M. It is easy to check that M does not have the pseudo duo property. Moreover, M is a WFI-extending (actually FI-extending) module by [19, Theorem 2.8]. Since R is a Kasch ring, it has C_2 property [11, Proposition 1.46]. It is well-known that being right Kasch is Morita invariant which yields that $M_2(R) = \begin{bmatrix} R & R \\ R & R \end{bmatrix}$ has C_2 condition. By [11, Theorem 7.16], M_R satisfies C_2 property. But $S = \mathrm{End}\,(M_R) \cong M_2(R)$ and $S/\mathrm{J}\,(S)$ is not a (von Neumann) regular ring.

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