KYUNGPOOK Math. J. 61(2021), 279-308
https://doi.org/10.5666/KMJ.2021.61.2.279
pISSN 1225-6951 eISSN 0454-8124

Generalized ψ-Geraghty-Zamfirescu Contraction Pairs in b-metric Spaces

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Abstract. The purpose of this paper is to introduce a class of contractive pairs of mappings satisfying a Zamfirescu-type inequality, but controlled with altering distance functions and with parameters satisfying the so-called Geraghty condition in the framework of b-metric spaces. For this class of mappings we prove the existence of points of coincidence, the convergence and stability of the Jungck, Jungck-Mann and Jungck-Ishikawa iterative processes and the existence and uniqueness of its common fixed points.

1. Motivation

In 1922, S. Banach [4] established his famous and fundamental result in the metric fixed point theory as follows:

Theorem 1.1.(Banach Contraction Principle) *Let* (M, d) *be a complete metric space and let* $S : M \longrightarrow M$ *be a Banach contraction, that is,* S *satisfies that there exists* $\alpha \in (0, 1)$ *such that*

$$d(Sx, Sy) \leq \alpha\, d(x, y) \tag{z_1}$$

for all $x, y \in M$. *Then,* S *has a unique fixed point in* M.

Notice that Banach's contractions are continuous mappings, so, in the spirit to extend the BCP, in 1968, R. Kannan [11] introduced a new class of contractive mappings admitting discontinuous functions, as follows.

* Corresponding Author.
Received September 13, 2020; revised January 15, 2021; accepted January 19, 2021.
2020 Mathematics Subject Classification: 47H09, 47H10, 47J25.
Key words and phrases: coincidence points, Jungck-type iterative schemes, convergence, stability, common fixed points.

Theorem 1.2.(Kannan, [11]) *Let* $(M,\, d)$ *be a complete metric space and let* $S : M \longrightarrow M$ *be a Kannan contraction, that is,* S *satisfies that there exists* $\beta \in [0,\, 1/2)$ *such that*

$$d(Sx,\, Sy) \le \beta[d(x,\, Sx) + d(y,\, Sy)] \qquad (z_2)$$

for all $x,\, y \in M$*. Then,* S *has a unique fixed point in* M*.*

In the same fashion, in 1972, S. K. Chatterjea [6] introduced a similar type of contractive condition.

Theorem 1.3.(Chatterjea, [6]) *Let* $(M,\, d)$ *be a complete metric space and let* $S : M \longrightarrow M$ *be a Chatterjea contraction, that is,* S *satisfies that there exists* $\gamma \in [0,\, 1/2)$ *such that*

$$d(Sx,\, Sy) \le \gamma[d(x,\, Ty) + d(y,\, Tx)] \qquad (z_3)$$

for all $x,\, y \in M$*. Then,* S *has a unique fixed points in* M*.*

Remark 1.1. Conditions (z_1), (z_2) and (z_3) are independent to each other. (See, B.E. Rhoades [19]).

In 1972, T. Zamfirescu [23] combine conditions (z_1), (z_2) and (z_3) to obtain a general fixed point result that include the Banach, Kannan and Chatterjea contraction mappings.

Theorem 1.4.(Zamfirescu, [23]) *Let* $(M,\, d)$ *be a complete metric space and let* $S : M \longrightarrow M$ *be a Zamfirescu operator, that is,* S *satisfies that there exist real numbers* α, β *and* γ *satisfying* $0 \le \alpha < 1$ *and* $0 \le \beta, \gamma < 1/2$ *such that, for all* $x,\, y \in M$*, at least one of the following inequalities is satisfied:*

- (z_1) $d(Sx,\, Sy) \le \alpha\, d(x,\, y)$,
- (z_2) $d(Sx,\, Sy) \le \beta\, [d(x,\, Sx) + d(y,\, Sy)]$,
- (z_3) $d(Sx,\, Sy) \le \gamma\, [d(x,\, Sy) + d(y,\, Sx)]$.

Then, S *has a unique fixed point in* M*.*

Notice that conditions (z_1), (z_2) and (z_3) can be written in the following equivalent form:

$$d(Sx,\, Sy) \le h \max\left\{d(x,\, y),\ \frac{d(x,\, Sx) + d(y,\, Sy)}{2},\ \frac{d(x,\, Sy) + d(y,\, Sx)}{2}\right\}$$

for all $x,\, y \in M$ and $0 \le h < 1$.

On the other hand, in 1976, G. Jungck [9] extend the Banach fixed point principle by using a pair of commuting mappings for which he prove the existence and uniqueness of its common fixed points.

Theorem 1.5.(Jungck, [9]) *Let* S_1, S_2 *be two self-maps on a complete metric space,* (M, d) *such that,*

- (J_1) (S_1, S_2) *is a commuting pair,*

(J_2) *S_2 is continuous,*

(J_3) $S_1(M) \subset S_2(M)$,

(J_4) *there is $a \in [0, 1)$ such that*

$$d(S_1x, S_1y) \leq ad(S_2x, S_2y)$$

for all $x, y \in M$.

Then, S_1 and S_2 have a unique common fixed point $z_0 \in M$.

These classical results have been generalized and extended in several ways, including by defining the contractive maps in more general spaces and controlling the inequality contraction with external functions as well. The purpose of this paper is to define a class of pair of mappings in the fashion of Zamfirescu [23], Jungck [9] and Geraghty [7], in the setting of b-metric spaces, to study conditions for the existence of a point of coincidence, to analyze the convergence and stability of Jungck, Jungck-Mann and Jungck-Ishikawa iteration processes and, finally, to prove the existence of its common fixed points.

2. Auxiliary Results in b-metric Spaces

In this section we recall some concepts, results and properties of the b-metric spaces that will be useful in the sequel.

Definition 2.1. Let M be a nonempty set and $s \geq 1$ be a given real number. A function

$$\rho : M \times M \to \mathbb{R}_+ := [0, +\infty)$$

is said to be a b-metric if and only if for all x, y, $z \in M$ the following conditions hold:

(ρ_1) $\rho(x, y) = 0$ if only if $x = y$.

(ρ_2) $\rho(x, y) = \rho(y, x)$.

(ρ_3) $\rho(x, z) \leq s(\rho(x, y) + \rho(y, z))$.

The pair (M, ρ) is called a b-metric space (also known as a quasimetric space) and the real number $s \geq 1$ is called the coefficient of (M, ρ).

From Definition 2.1 it is clear that the class of b-metric spaces is larger than the class of usual metric spaces, thus the b-metric spaces extend the usual metric spaces, since a b-metric is a usual metric space when $s = 1$, but the converse is not true.

Example 2.1.

(1) Let (M, d) be a metric space. For $p > 1$, the function $\varphi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ defined by $\varphi(t) = t^p$, $t \in \mathbb{R}_+$ is convex, then we can define a continuous b-metric with coefficient $s = 2^{p-1}$ as:

$$\rho(x, y) = \varphi(d(x, y)) = (d(x, y))^p$$

for all $x, y \in M$. It is easy to show that ρ does not hold the triangle inequality, so (M, ρ) is not a metric space. In particular, let $M = \mathbb{R}$ and let $d(x, y) = |x - y|$ be the usual metric on $\mathbb{R}$. Then $\rho(x, y) = (x - y)^2$ is a b-metric continuous on $\mathbb{R}$, but it is not a metric on $\mathbb{R}$.

(2) The set $L^p[0, 1]$, $(0 < p < 1)$ of all real functions $x(t)$, $t \in [0, 1]$ is defined by

$$L^p[0, 1] = \left\{ x(t) \ : \ \int_0^1 |x(t)|^p dt < \infty \right\}.$$

We define a b-metric on $L^p[0, 1]$ with coefficient $s = 2^{1/p-1}$, by

$$\rho(x, y) = \left(\int_0^1 |x(t) - y(t)|^p dt \right)^{1/p},$$

for all $x, y \in L^p[0, 1]$.

Now, we present the basic concepts concerning to the convergence of sequences, Cauchy sequences and complete b-metric spaces.

Definition 2.2. Let (M, ρ) be a b-metric space with $s \geq 1$. A sequence (x_n) in M is called:

(1) *b-convergent* if there exists $x \in M$ such that $\lim_{n\to\infty} \rho(x_n, x) = 0$. In this case we write $\lim_{n\to\infty} x_n = x$ or $x_n \to x$ as $n \to \infty$.

(2) *b-Cauchy sequence* in M if $\lim_{n,m\to\infty} \rho(x_n, x_m) = 0$, that is, given $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n, m > n_0$, we have $\rho(x_n, x_m) < \varepsilon$. If every b-Cauchy sequence in M is convergent, then (M, ρ) is said to be a complete b-metric space.

Proposition 2.1. *Let (M, ρ) be a b-metric space with $s \geq 1$. The following assertions hold:*

(i) *Any b-convergent sequence has a unique limit.*

(ii) *The subsequences of a b-convergent sequence are also convergent to the limit of the original sequence.*

(iii) *Every sequence which is b-convergent is also a b-Cauchy sequence.*

The examples of b-metric spaces given above show that there exist certain b-metric functions that are continuous, but in general a b-metric is not continuous in all its variable (see, [8]).

On the other hand, A. Aghajani, M. Abbas and J.R. Rushan [3] proved the following result about b-convergent sequences

Proposition 2.2. *Let (M, ρ) be a b-metric space with $s \geq 1$, and suppose that (x_n) and (y_n) are sequences such that $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$. Then, we have,*

$$\frac{1}{s^2}\rho(x, y) \leq \lim_{n\to\infty} \inf \rho(x_n, y_n) \leq \lim_{n\to\infty} \sup \rho(x_n, y_n) \leq s^2\rho(x, y).$$

In particular, if $x = y$, then $\lim\limits_{n\to\infty} \rho(x_n, y_n) = 0$. Moreover, for each $z \in M$ we have

$$\frac{1}{s}\rho(x, z) \leq \lim_{n\to\infty} \inf \rho(x_n, z) \leq \lim_{n\to\infty} \sup \rho(x_n, z) \leq s\rho(x, z).$$

The following result about b-Cauchy sequences will be useful to establish our convergence results.

Proposition 2.3.([17]) *Let (M, ρ) be a b-metric space with $s \geq 1$. Let (x_n) be a sequence in M such that*

$$\lim_{n\to\infty} \rho(x_n, x_{n+1}) = 0.$$

If (x_n) is not a b-Cauchy sequence in M, then there exist $\varepsilon > 0$ and sequences of positive integers $(n(k))$ and $(m(k))$ with $n(k) > m(k) > k > 0$ such that

$$\rho(x_{m(k)}, x_{n(k)}) \geq \varepsilon \quad and \quad \rho(y_{m(k)}, y_{n(k)-1}) < \varepsilon,$$

and

$$\begin{aligned}
\varepsilon &\leq \lim_{k\to\infty} \sup \rho(x_{m(k)}, x_{n(k)}) \leq \varepsilon s,\\
\frac{\varepsilon}{s^2} &\leq \lim_{k\to\infty} \sup \rho(x_{m(k)}, x_{n(k)-1}) \leq \varepsilon,\\
\frac{\varepsilon}{s^2} &\leq \lim_{k\to\infty} \sup \rho(x_{m(k)-1}, x_{n(k)-1}) \leq \varepsilon s,\\
\frac{\varepsilon}{s} &\leq \lim_{k\to\infty} \sup \rho(x_{m(k)-1}, x_{n(k)}) \leq \varepsilon s^2.
\end{aligned}$$

Now, we present some definitions and properties for a pair of mappings that are useful to establish our common fixed point results.

Definition 2.3. Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ be two selfmappings. By $C(S, T)$ we denote the set of coincidence points of S and T, that is,

$$C(S, T) = \{x \in M \ : \ Sx = Tx\},$$

and by $PC(S, T)$, the set of points of coincidence (POC) of S and T, that is,

$$PC(S, T) = \{z \in M \ : \ z = Sx = Tx, \quad \text{for some} \quad x \in M\}.$$

Definition 2.4. Let (M, ρ) be a b-metric space with $s \geq 1$. The mappings $S, T : M \to M$ are said to

(1) be *compatible* [10], if and only if

$$\lim_{n\to\infty} \rho(STx_n, TSx_n) = 0,$$

whenever (x_n) is a sequence in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \quad \text{for some} \quad t \in M.$$

(2) be *non-compatible*, if there exists at least a sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \quad \text{for some} \quad t \in M.$$

but $\lim_{n\to\infty} \rho(STx_n, TSx_n)$ is either non zero or non-existent.

(3) be *weakly compatible* [11], if for all $x \in M$, $Sx = Tx$ implies that $STx = TSx$, that is, if for $x \in C(S, T)$ implies that $Sx \in C(S, T)$.

(4) Satisfy the *b-property* (EA) [1], if there exists a sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = t, \quad \text{for some} \quad t \in M.$$

(5) Satisfy the *b-limit range property* with respect to T, (in short b-CLR_T-property) [21], if there exists (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tt, \quad \text{for some} \quad t \in M.$$

Remark 2.1.

(1) If S and T are compatible selfmappings, then S and T are weakly compatible.

(2) If S and T are non-compatible, then S and T satisfy the b-property (EA).

(3) Weak compatibility and the b-property (EA) are independent to each other.

(4) It may be noted that the b-CLR_T-property avoid the requirement of the condition of closedness of the range of the involved mappings.

3. The Class of Generalized ψ-Geraghty-Zamfirescu Contraction Pairs and Its POC

In this section, we introduce the class of generalized ψ-Geraghty-Zamfirescu contraction pairs. To do this, we will use altering distance functions and Geraghty-type operators for a pair of mappings. Also, for these pair of maps we study the existence and uniqueness of its points of coincidence.

We recall that in 1973, M. Geraghty [7] introduced a condition (now called Geraghty's property) which essentially replace the constant parameters of any contractive inequality by functions satisfying some properties. This approach had been used to proof fixed point theorems that generalize the BCP. For its use in the setting of b-metric spaces, we consider the class of functions $\mathcal{B}_s$, where $\beta \in \mathcal{B}_s$ if $\beta : \mathbb{R}_+ \longrightarrow [0,\ 1/s)$, $(s \geq 1)$ and satisfies the Geraghty type property:

$$\lim_{n\to\infty} \beta(t_n) = \frac{1}{s}, \quad \text{implies that} \quad \lim_{n\to\infty} t_n = 0, \tag{3.1}$$

for any $(t_n) \subset \mathbb{R}_+$. (Cf. [7]).

On the other hand, in 1984, M. S. Khan, M. Swalech and S. Sessa [12] extended the BCP by controlling the contractive inequality with an external function $\psi : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ (called an altering distance function) which satisfies that

(i) ψ is monotonic increasing function,

(ii) ψ is a continuous mapping,

(iii) $\psi(t) = 0$, if and only if $t = 0$.

By Ψ we denote the set of all altering distance functions.

Definition 3.1. Let $(M,\ \rho)$ be a b-metric space with $s \geq 1$. Two selfmappings $S, T : M \longrightarrow M$ are called *generalized ψ-Geraghty-Zamfirescu contraction mappings*, if there exist $\psi \in \Psi$ and $\beta \in \mathcal{B}_s$ such that

$$\psi[s^2 \rho(Sx,\ Sy)] \leq \beta[\psi(N(x,\ y))]\psi(N(x,\ y)) \tag{3.2}$$

for all $x,\ y \in M$, where

$$N(x,\ y) = \max\left\{\rho(Tx,\ Ty),\ \frac{\rho(Sx,\ Tx) + \rho(Sy,\ Ty)}{2},\ \frac{\rho(Sx,\ Ty) + \rho(Sy,\ Tx)}{2s}\right\}.$$

Remark 3.1. Since functions belonging to $\mathcal{B}_s$ are strictly smaller than $1/s$, for some $s \geq 1$, then the expression $\beta(\psi(N(x,\ y)))$ in (3.2) can be estimated from above as follows:

$$\beta[\psi(N(x,\ y))] < 1/s, \quad \text{for all} \quad x,\ y \in M.$$

Example 3.1. Let us consider the b-metric space $([0,4],\ \rho)$, where the b-metric ρ is given by $\rho(x,y) = (x-y)^2$. From Example 2.1, we know that the coefficient in this case is $s = 2$. Let us consider the mappings $S, T : [0,4] \longrightarrow [0,4]$ given by the formulas

$$Sx = \begin{cases} 1, & x = 0 \\ 0, & x \in (0,4] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 1, & x = 0 \\ 4, & x \in (0,4] \end{cases}.$$

We show that S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings with $\psi(t) = t^2$ and $\beta(t) = \frac{t+1}{t+2}$. In fact:

$$\rho(Sx, Sy) = \begin{cases} 1, & x = 0,\ y \in (0,4] \\ 1, & y = 0,\ x \in (0,4] \\ 0, & x, y \in (0,4] \end{cases}, \qquad \rho(Tx, Ty) = \begin{cases} 9, & x = 0,\ y \in (0,4] \\ 9, & y = 0,\ x \in (0,4] \\ 0, & x, y \in (0,4] \end{cases}$$

$$\rho(Sx, Tx) = \begin{cases} 0, & x = 0 \\ 16, & x \in (0,4] \end{cases}, \qquad \rho(Sy, Ty) = \begin{cases} 0, & x = 0 \\ 16, & x \in (0,4] \end{cases}$$

$$\rho(Sx, Ty) = \begin{cases} 0, & x = y = 0 \\ 9, & x = 0,\ y \in (0,4] \\ 1, & y = 0,\ x \in (0,4] \\ 16, & x, y \in (0,4] \end{cases}, \qquad \rho(Sy, Tx) = \begin{cases} 0, & x = y = 0 \\ 1, & x = 0,\ y \in (0,4] \\ 9, & y = 0,\ x \in (0,4] \\ 16, & x, y \in (0,4] \end{cases}.$$

Thus, we have that

$$\frac{\rho(Sx, Tx) + \rho(Sy, Ty)}{2} = \begin{cases} 0, & x = y = 0 \\ 8, & x = 0,\ y \in (0,4] \\ 8, & y = 0,\ x \in (0,4] \\ 16, & x, y \in (0,4] \end{cases}$$

and

$$\frac{\rho(Sx, Ty) + \rho(Sy, Tx)}{4} = \begin{cases} 0, & x = y = 0 \\ 5/2, & x = 0,\ y \in (0,4] \\ 5/2, & y = 0,\ x \in (0,4] \\ 8, & x, y \in (0,4] \end{cases}.$$

Hence, we obtain

$$N(x, y) = \begin{cases} 9, & x = y = 0 \\ 9, & x = 0,\ y \in (0,4] \\ 9, & y = 0,\ x \in (0,4] \\ 16, & x, y \in (0,4] \end{cases}.$$

Now, notice that

$$\psi(s^2 \rho(Sx, Sy)) = \begin{cases} 16, & x = 0,\ y \in (0,4] \\ 16, & y = 0,\ x \in (0,4] \\ 0, & \text{otherwise} \end{cases}$$

and

$$\beta(\psi(N(x,y)))\psi(N(x,y)) = \begin{cases} \frac{81\cdot 82}{83}, & x=0,\ y\in(0,4] \\ \frac{81\cdot 82}{83}, & y=0,\ x\in(0,4] \\ >0, & \text{otherwise} \end{cases}.$$

Therefore, S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings.

In the next result we prove that any ψ-Geraghty-Zamfirescu pair cannot has more than one point of coincidence.

Proposition 3.1. *Let $(M,\ \rho)$ be a b-metric space with $s \geq 1$ and let $S,T : M \longrightarrow M$ be two selfmappings. Assume that S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings. If S and T have a POC in M, then it is unique.*

Proof. Let $z \in M$ be a POC of S and T. Then, there exists $u \in M$ such that $Su = Tu = z$. Suppose that for some $v \in M$ we have $Sv = Tv = w$ and $w \neq z$. Now, from inequality (3.2) we have

$$\psi[\rho(z,w)] \leq \psi[s^2\rho(z,w)] = \psi[s^2\rho(Su,Sv)] \leq \beta[\psi(N(u,v))]\psi[N(u,v)] \tag{3.3}$$

where,

$$\begin{aligned} N(u,v) &= \max\left\{\rho(Tu,Tv), \frac{\rho(Su,Tu)+\rho(Sv,Tv)}{2}, \frac{\rho(Su,Tv)+\rho(Sv,Tu)}{2s}\right\} \\ &= \max\left\{\rho(z,w), \frac{\rho(z,z)+\rho(w,w)}{2}, \frac{\rho(z,w)+\rho(z,w)}{2s}\right\} \\ &= \max\left\{\rho(z,w), 0, \frac{\rho(z,w)}{s}\right\} = \rho(z,w). \end{aligned} \tag{3.4}$$

Using (3.4) in (3.3), we obtain

$$\begin{aligned} \psi[\rho(z,w)] \leq \psi[s^2\rho(z,w)] &\leq \beta[\psi(\rho(z,w))]\psi(\rho(z,w)) \\ &< \frac{1}{s}\psi(\rho(z,w)) \leq \psi(\rho(z,w)) \end{aligned}$$

which is a contradiction. Hence $z = w$. □

Now, we prove that generalized ψ-Geraghty-Zamfirescu contraction mappings have a point of coincidence if they satisfy the b-property (EA).

Proposition 3.2. *Let $(M,\ \rho)$ be a b-metric space with $s \geq 1$ and let $S,T : M \longrightarrow M$ be two selfmappings. Assume that S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings satisfying the b-property (EA). If $S(M) \subset T(M)$ (or $T(M) \subset S(M)$) and $T(M)$ (resp. $S(M)$) is closed, then (S,T) has a unique point of coincidence.*

Proof. Let us assume $S(M) \subset T(M)$ and $T(M)$ closed. Since (S,T) satisfy the b-property (EA), there exists a sequence $(x_n) \subset M$ such that

$$\lim_{n\to\infty} Sx_n = Tx_n = t \in T(M).$$

Then, $t = Tu$ for some $u \in M$. We will prove that $Tu = Su = t$. Let us suppose that $Su \neq Tu$. From inequality (3.2), we have

$$\begin{aligned} \psi[\rho(Su,Tu)] &\leq \psi[s^2\rho(Su,Tu)] \\ &\leq \limsup_{n\to\infty} \psi[s^2\rho(Su,Tx_n)] \\ &= \limsup_{n\to\infty} \psi[s^2\rho(Su,Sx_n)] \\ &\leq \limsup_{n\to\infty} \beta[\psi(N(u,x_n))] \limsup_{n\to\infty} \psi(N(u,x_n)) \end{aligned} \tag{3.5}$$

where,

$$N(u,x_n) = \max\left\{\rho(Tu,Tx_n), \frac{\rho(Su,Tu)+\rho(Sx_n,Tx_n)}{2}, \frac{\rho(Su,Tx_n)+\rho(Sx_n,Tu)}{2s}\right\}.$$

Taking upper limit as $k \to \infty$ and from Proposition 2.2, we have

$$\begin{aligned} &\limsup_{n\to\infty} N(u,x_n) \\ &\leq \max\left\{s\rho(Tu,Tu), \frac{\rho(Su,Tu)+s^2\rho(Tu,Tu)}{2}, \frac{s\rho(Su,Tu)+s^2\rho(Tu,Tu)}{2s}\right\} \\ &= \max\left\{0, \frac{\rho(Su,Tu)}{2}, \frac{\rho(Su,Tu)}{2}\right\} = \frac{\rho(Su,Tu)}{2}. \end{aligned}$$

Thus, by taking upper limit as $k \to \infty$ and using (3.5) we obtain

$$\begin{aligned} \psi[\rho(Su,Tu)] \leq \psi[s^2\rho(Su,Tu)] &\leq \beta\left[\psi\left(\frac{\rho(Su,Tu)}{2}\right)\right]\psi\left(\frac{\rho(Su,Tu)}{2}\right) \\ &< \frac{1}{s}\psi\left[\frac{\rho(Su,Tu)}{2}\right] \leq \psi\left(\frac{\rho(Su,Tu)}{2}\right) \end{aligned}$$

which is a contradiction. Hence $Su = Tu$.

The uniqueness of the POC comes from Propostion 3.1. The case $T(M) \subset S(M)$ and $S(M)$ closed is proved with similarity. □

The conditions given in Proposition 3.2 to guarantee the existence of a POC for generalized ψ-Geraghty-Zamfirescu mappings are not necessary, as Example 3.1 shows, since $S0 = T0 = 1$, but $S([0,4]) = \{0,1\}$ and $T([0,4]) = \{1,4\}$. That means, neither $S([0,4]) \subset T([0,4])$ nor $T([0,4]) \subset S([0,4])$. In the next example we show that the b-property (EA) cannot be removed in Proposition 3.2.

Example 3.2. Let us consider the b-metric space $([0,2],\rho)$, where the b-metric function is given by $\rho(x,y) = |x-y|$, the usual distance on $\mathbb{R}$. Let us consider the mappings $S,T : [0,2] \longrightarrow [0,2]$ defined as:

$$Sx = \begin{cases} 1, & x \in [0,1) \\ 0, & x \in [1,2] \end{cases} \quad \text{and} \quad Tx = \begin{cases} 0, & x \in [0,1) \\ 2, & x = 1 \\ 1, & x \in (1,2] \end{cases}.$$

For these functions $PC(S,T) = \{\emptyset\}$, $S([0,4]) = \{0,1\} \subset T([0,4]) = \{0,1,2\}$ and $T([0,4])$ is closed. We show that S and T are ψ-Geraghty-Zamfirescu mappings with $\psi(t) = t^2$ and $\beta(t) = \frac{t+1}{2t+1}$. In fact,

$$\rho(Sx,Sy) = \begin{cases} 0, & x,y \in [0,1) \\ 1, & x=1,\ y \in [0,1) \\ 0, & x=1,\ y \in (1,2] \\ 1, & y=1,\ x \in [0,1) \\ 0, & y=1,\ x \in (1,2] \\ 0, & x,y \in (1,2] \end{cases}, \qquad \rho(Tx,Ty) = \begin{cases} 0, & x,y \in [0,1) \\ 2, & x=1,\ y \in [0,1) \\ 1, & x=1,\ y \in (1,2] \\ 2, & y=1,\ x \in [0,1) \\ 1, & y=1,\ x \in (1,2] \\ 0, & x,y \in (1,2] \end{cases},$$

$$\frac{\rho(Sx,Tx)+\rho(Sy,Ty)}{2} = \begin{cases} 1, & x,y \in [0,1) \\ 3/2, & x=1,\ y \in [0,1) \\ 3/2, & x=1,\ y \in (1,2] \\ 3/2, & y=1,\ x \in [0,1) \\ 3/2, & y=1,\ x \in (1,2] \\ 1, & x,y \in (1,2] \end{cases},$$

$$\frac{\rho(Sx,Ty)+\rho(Sy,Tx)}{2} = \begin{cases} 1, & x,y \in [0,1) \\ 1, & x=1,\ y \in [0,1) \\ 3/2, & x=1,\ y \in (1,2] \\ 1/2, & y=1,\ x \in [0,1) \\ 1, & y=1,\ x \in (1,2] \\ 1, & x,y \in (1,2] \end{cases}.$$

Thus, we obtain

$$N(x,y) = \begin{cases} 1, & x,y \in [0,1) \\ 2, & x=1,\ y \in [0,1) \\ 3/2, & x=1,\ y \in (1,2] \\ 2, & y=1,\ x \in [0,1) \\ 3/2, & y=1,\ x \in (1,2] \\ 1, & x,y \in (1,2] \end{cases}.$$

Finally, we compute the following quantities:

$$\psi(\rho(Sx,Sy)) = \begin{cases} 0, & x,y \in [0,1) \\ 1, & x=1,\ y \in [0,1) \\ 0, & x=1,\ y \in (1,2] \\ 1, & y=1,\ x \in [0,1) \\ 0, & y=1,\ x \in (1,2] \\ 0, & x,y \in (1,2] \end{cases}.$$

and

$$\beta(\psi(N(x,y)))\psi(N(x,y)) = \begin{cases} 2/3, & x,y \in [0,1) \\ 20/9, & x=1,\ y\in[0,1) \\ 117/88, & x=1,\ y\in(1,2] \\ 20/9, & y=1,\ x\in[0,1) \\ 117/88, & y=1,\ x\in(1,2] \\ 2/3, & x,y\in(1,2] \end{cases}.$$

Which allow us to conclude that S and T are generalized ψ-Geraghty-Zamfirescu mappings.

On the other hand, notice that any sequence $(x_n) \subset [0,1)$ satisfy that

$$\lim_{n\to\infty} Sx_n = 1 \quad \text{and} \quad \lim_{n\to\infty} Tx_n = 0,$$

whereas, if $(x_n) \subset (1,2]$

$$\lim_{n\to\infty} Sx_n = 0 \quad \text{and} \quad \lim_{n\to\infty} Tx_n = 1.$$

For the case when (x_n) converges to 1 but with their terms (for n sufficiently large) oscillating between the set $[0,1)$ and $(1,2]$, the limits $\lim_{n\to\infty} Sx_n$ and $\lim_{n\to\infty} Tx_n$ does not exist since we can extract subsequences $(x_{n(k)})$ and $(x_{n(l)})$ of (x_n) such that

$$\lim_{k\to\infty} Sx_{n(k)} = 0 \quad \text{and} \quad \lim_{l\to\infty} Sx_{n(l)} = 1,$$

as well as,

$$\lim_{k\to\infty} Tx_{n(k)} = 1 \quad \text{and} \quad \lim_{l\to\infty} Tx_{n(l)} = 0.$$

Therefore, the mappings S and T does not satisfy the b-property (EA).

Remark 3.2. Notice, from Remark 2.1, that we can drop the condition $T(M)$ ($S(M)$) be closed in Proposition 3.2, if we assume that (T,S) satisfy the b-CLR_T-property (resp. b-CLR_S-property).

4. Convergence and Stability of Some Iteration Schemes for Generalized ψ-Geraghty-Zamfirescu Pairs

In this section, we will prove the b-convergence and the (S,T)-stability of the Jungck, Jungck-Mann and Jungck-Ishikawa iterative schemes for ψ-Geraghty-Zamfirescu pairs in the setting of b-metric spaces. To pose the Jungck-Mann and Jungck-Ishikawa iterations we endowed the b-metric space $(M,\ \rho)$ with a convex structure.

4.1. Convergence of the Jungck Iteration

Let $(M,\ \rho)$ be a b-metric space and $S, T : N \longrightarrow M$ are two nonself mappings on a subset N of M such that $S(N) \subset T(N)$, where $T(N)$ is a complete subspace of M. For any $x_0 \in N$, the Jungck iterative sequence is defined by the formula

$$y_n = Sx_n = Tx_{n+1}, \quad n = 0, 1, \ldots \tag{4.1}$$

We are going to prove that for generalized ψ-Geraghty-Zamfirescu pairs, the Jungck iteration defines a b-Cauchy sequence.

Proposition 4.1. *Let S, T be two selfmaps of a b-metric space $(M,\ \rho)$ with $s \geq 1$, such that $SM \subset TM$. Assume that $(S,\ T)$ satisfies condition* (3.2)*. Then, for each $x_0 \in M$, the Jungck sequence* (4.1) *satisfies:*

(i) $\lim\limits_{n\to\infty} \rho(y_n,\ y_{n+1}) = 0$, *and*

(ii) (y_n) *is a b-Cauchy sequence in M.*

Proof. To prove (i), let $x_0 \in M$ be an arbitrary point, using the condition $SM \subset TM$, we construct the Jungck sequence defined by

$$y_n = Sx_n = Tx_{n+1}, \quad n = 0, 1, \ldots$$

From condition (3.2), we have

$$\begin{aligned} \psi(s^2\rho(y_n,\ y_{n+1})) &= \psi[s^2\rho(Sx_n,\ Sx_{n+1})] \\ &\leq \beta[\psi(\rho(N(x_n,\ x_{n+1})))]\psi[N(x_n,\ x_{n+1})], \end{aligned} \tag{4.2}$$

where:

$$\begin{aligned} N(x_n,\ x_{n+1}) = \max\Bigg\{&\rho(Tx_n,\ Tx_{n+1}),\ \frac{\rho(Sx_n,\ Tx_n) + \rho(Sx_{n+1},\ Tx_{n+1})}{2}, \\ &\frac{\rho(Sx_n,\ Tx_{n+1}) + \rho(Sx_{n+1},\ Tx_n)}{2s}\Bigg\} \\ = \max\Bigg\{&\rho(y_{n-1},\ y_n),\ \frac{\rho(y_n,\ y_{n-1}) + \rho(y_{n+1},\ y_n)}{2}, \\ &\frac{\rho(y_n,\ y_n) + \rho(y_{n+1},\ y_{n-1})}{2s}\Bigg\}. \end{aligned}$$

Since,

$$\frac{\rho(y_{n-1},\ y_n) + \rho(y_n,\ y_{n+1})}{2} \leq \max\{\rho(y_{n-1},\ y_n),\ \rho(y_n,\ y_{n+1})\}$$

and

$$\frac{\rho(y_{n-1},\ y_{n+1})}{2s} \leq \frac{s}{2s}(\rho(y_{n-1},\ y_n)+\rho(y_n,\ y_{n+1})) \leq \max\{\rho(y_{n-1},\ y_n),\ \rho(y_n,\ y_{n+1})\}.$$

Then, we conclude

$$N(x_n,\ x_{n+1}) \leq \max\{\rho(y_{n-1},\ y_n),\ \rho(y_n,\ y_{n+1})\}.$$

If $N(x_n,\ x_{n+1}) \leq \rho(y_n,\ y_{n+1})$, using (4.2) we obtain

$$\begin{aligned}\psi[\rho(y_n,\ y_{n+1})] &\leq \psi[s^2\rho(y_n,\ y_{n+1})] \leq \beta[\psi(\rho(y_n,\ y_{n+1}))]\psi(\rho(y_n,\ y_{n+1})) \\ &< \frac{1}{s}\psi(\rho(y_n,\ y_{n+1})) < \psi(\rho(y_n,\ y_{n+1})).\end{aligned}$$

Since $\psi \in \Psi$, it follows that

$$\rho(y_n,\ y_{n+1}) < \rho(y_n,\ y_{n+1}),$$

which is a contradiction. Therefore, $N(x_n,\ x_{n+1}) \leq \rho(y_{n-1},\ y_n)$. Using again (4.2), we get

$$\begin{aligned}\psi[\rho(y_n,\ y_{n+1})] &\leq \psi[s^2\rho(y_n,\ y_{n+1})] \leq \beta[\psi(\rho(y_{n-1},\ y_n))]\psi[\rho(y_{n-1},\ y_n)] \\ &< \frac{1}{s}\psi(\rho(y_{n-1},\ y_n)) < \psi(\rho(y_{n-1},\ y_n)).\end{aligned} \tag{4.3}$$

Thus, $\rho(y_n,\ y_{n+1}) < \rho(y_{n-1},\ y_n)$, which means that $(\rho(y_n,\ y_{n+1}))_n$ is a nondecreasing sequence of non-negative real numbers. Therefore, there exists $L \geq 0$ such that

$$\lim_{n\to\infty} \rho(y_n,\ y_{n+1}) = L.$$

If we assume that $L > 0$, from (4.3) we have

$$\begin{aligned}\psi(L) = \limsup_{n\to\infty} \psi(\rho(y_n, y_{n+1})) &\leq \limsup_{n\to\infty} \psi[s^2\rho(y_n, y_{n+1})] \\ &\leq \limsup_{n\to\infty} \beta[\psi(\rho(y_{n-1}, y_n))] \limsup_{n\to\infty} \psi[\rho(y_{n-1}, y_n)].\end{aligned}$$

It follows that

$$\psi(L) \leq \limsup_{n\to\infty} \beta[\psi(\rho(y_{n-1}, y_n))]\psi(L).$$

On the other hand, notice that

$$\frac{1}{s} \leq 1 = \frac{\psi(L)}{\psi(L)} \leq \limsup_{n\to\infty} \beta[\psi(\rho(y_{n-1}, y_n))] < 1/s.$$

Therefore, $\lim_{n\to\infty} \beta[\psi(\rho(y_{n-1}, y_n))] = 1/s$. This implies that $\lim_{n\to\infty} \psi(\rho(y_{n-1}, y_n)) = 0$ and since $\psi \in \Psi$, we conclude that $L = \lim_{n\to\infty} \rho(y_{n-1}, y_n) = 0$, which is a contradiction. Therefore,

$$\lim_{n\to\infty} \rho(y_n, y_{n+1}) = 0.$$

Now, we want to show that (y_n) is a b-Cauchy sequence in M. We assume the contrary, that is, (y_n) is a not a b-Cauchy sequence. By Proposition 2.3, there exist an $\varepsilon > 0$ and two sequences $n(k)$ and $m(k)$ with $n(k) > m(k) > k > 0$ such that

$$\rho(y_{m(k)}, y_{n(k)}) \geq 0 \quad \text{and} \quad \rho(y_{m(k)}, y_{n(k)-1}) < \varepsilon.$$

From (3.2), we have

$$\begin{aligned} \psi[\rho(y_{m(k)}, y_{n(k)})] &\leq \psi[s^2 \rho(y_{m(k)}, y_{n(k)})] \\ &= \psi[s^2 \rho(Sx_{m(k)}, Sx_{n(k)})] \\ &\leq \beta[\psi(N(x_{m(k)}, x_{n(k)}))]\psi[N(x_{m(k)}, x_{n(k)})]. \end{aligned} \tag{4.4}$$

Where,

$$\begin{aligned} N(x_{m(k)}, x_{n(k)}) =& \max\Bigg\{\rho(Tx_{m(k)}, Tx_{n(k)}), \\ & \frac{\rho(Sx_{m(k)}, Tx_{m(k)}) + \rho(Sx_{n(k)}, Tx_{n(k)})}{2}, \\ & \frac{\rho(Sx_{m(k)}, Tx_{n(k)}) + \rho(Sx_{n(k)}, Tx_{m(k)})}{2s}\Bigg\} \\ =& \max\Bigg\{\rho(y_{m(k)-1}, y_{n(k)-1}), \frac{\rho(y_{m(k)}, y_{m(k)-1}) + \rho(y_{n(k)}, y_{n(k)-1})}{2}, \\ & \frac{\rho(y_{m(k)}, y_{n(k)-1}) + \rho(y_{n(k)}, y_{m(k)-1})}{2s}\Bigg\}. \end{aligned}$$

Now, taking upper limit as $k \to \infty$ and using Proposition 2.3, we have

$$\lim_{n\to\infty} N(x_{m(k)}, x_{n(k)}) \leq \max\left\{s\varepsilon, 0, \frac{\varepsilon + \varepsilon s^2}{2s}\right\} = \varepsilon s. \tag{4.5}$$

Taking upper limit as $k \to \infty$ in (4.4) and using (4.5), we get

$$\begin{aligned} \psi(s\varepsilon) \leq \psi(s^2\varepsilon) &\leq \limsup_{k\to\infty} \psi[s^2 \rho(y_{m(k)}, y_{n(k)})] \\ &\leq \limsup_{k\to\infty} \psi[s^2 \rho(Sx_{m(k)}, Sx_{n(k)})] \\ &\leq \limsup_{k\to\infty} \beta[\psi[N(x_{m(k)}, x_{n(k)})]] \limsup_{k\to\infty} \psi[N(x_{m(k)}, x_{n(k)})] \\ &\leq \beta[\psi(s\varepsilon)]\psi(s\varepsilon) < \frac{1}{s}\psi(s\varepsilon) \leq \psi(s\varepsilon), \end{aligned}$$

which is a contradiction. Thus, we conclude that (y_n) is a b-Cauchy sequence in M. □

Now, we prove that the Jungck iterative process converges to the POC of a ψ-Geraghty-Zamfirescu contraction pair.

Proposition 4.2. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ be two selfmappings with $SM \subset TM$. Assume that S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings. If $TM \subset M$ is a complete subspace of M. Then, for any $x_0 \in M$, the Jungck sequence defined in* (4.1) *converges to $z \in M$, $C(S,T) \neq \{\emptyset\}$ and $PC(S,T) = \{z\}$.*

Proof. By Proposition 4.1, the Jungck sequence (y_n) is a b-Cauchy sequence in M. Then, $(Tx_{n+1}) \subset TM$ is a b-Cauchy sequence and since TM is complete, we have

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_{n+1} = Tu, \quad \text{for some} \quad u \in M.$$

We have been prove that the Jungck sequence (y_n) converges to some $z \in M$. The rest of the proof runs as the proof of Proposition 3.2.. □

4.2. Convergence of the Jungck-Mann and Jungck-Ishikawa Iterations

We recall that in 1970, W. Takahashi [22] introduce the notion of convex structure on metric spaces, which is a natural generalization of convexity in normed linear spaces as A. A. Abdelhakim establishes in his 2016 paper [2]. In 2012, M. O. Olatinwo and M. Postolache [15] proved some stability results in metric space endowed with this structure for Jungck–Mann and Jungck–Ishikawa iterations for nonselfmappings satisfying certain general contractivity condition. Later on, in 2013, A. Razani and M. Bagherboum [18] reintroduce this concept in the framework of b-metric spaces and proved the convergence and stability of Jungck-type interative procedures in this setting.

Definition 4.1. Let (M, ρ) be a b-metric space. A mapping $\mathcal{W} : M \times M \times [0,1] \longrightarrow M$ is said to be a *convex structure* on M if for each $(x, y, \lambda) \in M \times M \times [0,1]$ and $z \in M$,

$$\rho(z, \mathcal{W}(x, y, \lambda)) \leq \lambda\rho(z, x) + (1-\lambda)\rho(z, y).$$

A b-metric space (M, ρ) equipped with the convex structure $\mathcal{W}$ is called a convex b-metric space, and it is denoted by $(M, \rho, \mathcal{W})$.

The b-metric space given in Example 2.1(1) is a convex b-metric space with convex structure $\mathcal{W}(x, y, \lambda) = \lambda x + (1-\lambda)y$.

From now on, it is assumed that $(M, \rho, \mathcal{W})$ is a convex b-metric space with parameter s and that $S, T : N \longrightarrow M$ are two nonself mappings on a subset N of M such that $S(N) \subset T(N)$, where $T(N)$ is a complete subspace of M.

For any $x_0 \in M$, let (x_n) be the sequence generated by the so-called Jungck-Mann iterative procedure:

$$Tx_{n+1} = \mathcal{W}(Tx_n, Sx_n, \alpha_n), \qquad n = 0, 1, \ldots$$

where (α_n) is a sequence of real numbers such that $0 \leq \alpha_n \leq 1$.

Theorem 4.3. *Let* $(M, \rho, \mathcal{W})$ *be a convex b-metric space with* $s \geq 1$*, and let* $S, T : N \longrightarrow M$ *be generalized* ψ*-Geraghty-Zamfirescu contraction mappings having a point of coincidence and* $\psi \in \Psi$ *be a convex function. Let* (α_n) *be a real sequence in* $[0, 1]$ *such that* $\sum_{k=0}^{\infty} \frac{s+1-(s-1)\alpha_k}{s} = \infty$*. Then, for any* $x_0 \in N$*, the sequence defined by the Jungck-Mann iterative process converges to the point of coincidence of* S *and* T*.*

Proof. Let z be the unique coincidence point of (S, T), i.e., $Sz = Tz = p$. We are going to prove that the Jungck-Mann iterative process converges to p. Notice that

$$\rho(Tx_{n+1}, p) =\rho(\mathcal{W}(Tx_n, Sx_n, \alpha_n), p) \leq \alpha_n \rho(Tx_n, p) + (1 - \alpha_n)\rho(Sx_n, p).$$

Since ψ is increasing and convex, using (3.2), we obtain

$$\begin{aligned} \psi(s\rho(Tx_{n+1}, p)) &\leq \alpha_n \psi(s\rho(Tx_n, p)) + (1 - \alpha_n)\psi(s\rho(Sx_n, p)) \\ &\leq \alpha_n \psi(s\rho(Tx_n, p)) + (1 - \alpha_n)\psi(s^2\rho(Sx_n, Sz)) \\ &< \alpha_n \psi(s\rho(Tx_n, p)) + \frac{(1 - \alpha_n)}{s}\psi(N(x_n, z)) \end{aligned} \tag{4.6}$$

where,

$$\begin{aligned} N(x_n, z) =&\max\left\{\rho(Tx_n, Tz), \frac{\rho(Sx_n, Tx_n) + \rho(Sz, Tz)}{2}, \frac{\rho(Sx_n, Tz) + \rho(Sz, Tx_n)}{2s}\right\} \\ =&\max\left\{\rho(Tx_n, p), \frac{\rho(Sx_n, Tx_n)}{2}, \frac{\rho(Sx_n, p) + \rho(Tx_n, p)}{2s}\right\}. \end{aligned}$$

Note that

$$\frac{\rho(Sx_n, Tx_n)}{2} \leq s\frac{\rho(Sx_n, p) + \rho(Tx_n, p)}{2} \leq s \max\{\rho(Sx_n, p), \rho(Tx_n, p)\}$$

and

$$\frac{\rho(Sx_n, p) + \rho(Tx_n, p)}{2s} \leq s\frac{\rho(Sx_n, p) + \rho(Tx_n, p)}{2} \leq s \max\{\rho(Sx_n, p), \rho(Tx_n, p))\}.$$

Therefore,

$$N(x_n, z) \leq s \max\{\rho(Sx_n, p), \rho(Tx_n, p)\}.$$

Now, if $\rho(Sx_n, p) \geq \rho(Tx_n, p)$, from condition (3.2), we get

$$\begin{aligned} \psi(s\rho(Sx_n, Sz)) \leq&\psi(s^2\rho(Sx_n, Sz)) \\ \leq&\beta(\psi(N(x_n, z)))\psi(N(x_n, z)) \\ <&\frac{1}{s}\psi(s\rho(Sx_n, Sz)), \end{aligned}$$

a contradiction. Hence, $\psi(N(x_n,z)) \leq \psi(s\rho(Tx_n,p))$. In this way, estimate (4.6) is now given by

$$\begin{aligned}\psi(s\rho(Tx_{n+1},p)) <&\alpha_n\psi(s\rho(Tx_n,p)) + \frac{(1-\alpha_n)}{s}\psi(s\rho(Tx_n,p))\\ =&\left(\alpha_n + \frac{1-\alpha_n}{s}\right)\psi(s\rho(Tx_n,p)).\end{aligned}\tag{4.7}$$

On the other hand, note that

$$\alpha_n + \frac{1-\alpha_n}{s} = 1 - \left(1 - \frac{(s-1)\alpha_n + 1}{s}\right).$$

Moreover, since $0 \leq \alpha_n \leq 1$, then $(s-1)\alpha_n \leq s-1$, or equivalently, $\frac{s-1}{s}\alpha_n \leq 1 - \frac{1}{s}$. I.e.,

$$1 - \frac{(s-1)\alpha_n + 1}{s} \geq 0.$$

Now, recursively, from (4.7), we obtain

$$\begin{aligned}\psi(s\rho(Tx_{n+1},p)) &< \left(1 - \left(1 - \frac{(s-1)\alpha_n + 1}{s}\right)\right)\psi(s\rho(Tx_n,p))\\ &\vdots\\ &< \prod_{k=0}^{n}\left(1 - \left(1 - \frac{(s-1)\alpha_k + 1}{s}\right)\right)\psi(s\rho(Tx_0,p)).\end{aligned}$$

Taking $\limsup$ as $n \to \infty$, using the fact that $\sum_{k=0}^{\infty} \frac{s+1-(s-1)\alpha_n}{s} = \infty$, we conclude that

$$\lim_{n\to\infty} \psi(s\rho(Tx_{n+1},p)) = 0.$$

The condition $\psi \in \Psi$ implies that $\lim_{n\to\infty} \rho(Tx_{n+1},p) = 0$. Thus, we obtain the conclusion. □

Now, for any $x_0 \in M$, the sequence (x_n) generated by the following iterative two-step process:

$$\begin{aligned}Tx_{n+1} &= \mathcal{W}(Tx_n, Sy_n, \alpha_n),\\ Ty_n &= \mathcal{W}(Tx_n, Sx_n, \beta_n), \qquad n = 0, 1, \ldots\end{aligned}$$

where (α_n) and (β_n) are sequences of real numbers such that $0 \leq \alpha_n, \beta_n \leq 1$, is called the Jungck-Ishikawa iterative procedure.

In the next result we prove the convergence of the Jungck-Ishikawa iterative scheme for generalized ψ-Geraghty-Zamfirescu contraction mappings.

Theorem 4.4. *Let $(M, \rho, \mathcal{W})$ be a convex b-metric space with $s \geq 1$, and let $S, T : N \longrightarrow M$ be generalized ψ-Geraghty-Zamfirescu contraction mappings having* *a point of coincidence and $\psi \in \Psi$ be a convex function. Let (α_n) and (β_n) be a real sequences in $[0,1]$ such that $\sum_{k=0}^{\infty} \frac{s+1-(s-1)\alpha_k}{s} = \infty$. Then, for any $x_0 \in N$, the sequence defined by the Jungck-Ishikawa iterative process converges to the point of coincidence of S and T.*

Proof. Let z be the unique coincidence point of (S,T), i.e., $Sz = Tz = p$. Now,

$$\begin{aligned}
\psi(s\rho(Tx_{n+1},p)) &= \psi(s\rho(\mathcal{W}(Tx_n, Sy_n, \alpha_n), p)) \\
&\leq \alpha_n \psi(s\rho(Tx_n,p)) + (1-\alpha_n)\psi(s\rho(Sy_n,p)) \\
&\leq \alpha_n \psi(s\rho(Tx_n,p)) + (1-\alpha_n)\psi(s^2\rho(Sy_n,Sz)) \\
&< \alpha_n \psi(s\rho(Tx_n,p)) + \frac{1-\alpha_n}{s}\psi(N(y_n,z)).
\end{aligned}$$

As in the proof of Theorem 4.3, we conclude that

$$\psi(s\rho(Tx_{n+1},p)) < \alpha_n \psi(s\rho(Tx_n,p)) + \frac{1-\alpha_n}{s}\psi(s\rho(Ty_n,p)).$$

Now, as above, we have

$$\begin{aligned}
\psi(s\rho(Ty_n,p)) &= \psi(s\rho(\mathcal{W}(Tx_n, Sx_n, \beta_n), p)) \\
&\leq \beta_n \psi(s\rho(Tx_n,p)) + (1-\beta_n)\psi(s\rho(Sx_n,p)) \\
&\leq \beta_n \psi(s\rho(Tx_n,p)) + (1-\beta_n)\psi(s^2\rho(Sx_n,Sz)) \\
&< \beta_n \psi(s\rho(Tx_n,p)) + \frac{1-\beta_n}{s}\psi(N(x_n,z)) \\
&\leq \beta_n \psi(s\rho(Tx_n,p)) + \frac{1-\beta_n}{s}\psi(s\rho(Tx_n,p)).
\end{aligned}$$

Thus, from the previous estimates we obtain

$$\begin{aligned}
\psi(s\rho(Tx_{n+1},p)) &< \left[\alpha_n + \frac{1-\alpha_n}{s}\beta_n + \frac{(1-\alpha_n)(1-\beta_n)}{s^2}\right]\psi(s\rho(Tx_n,p)) \\
&\leq \left[\alpha_n + \frac{1-\alpha_n}{s}\beta_n + \frac{(1-\alpha_n)(1-\beta_n)}{s}\right]\psi(s\rho(Tx_n,p)) \\
&= \left[\alpha_n + \frac{1-\alpha_n}{s}\right]\psi(s\rho(Tx_n,p)),
\end{aligned}$$

which is the estimate (4.7). The conclusion then is obtained following the proof of Theorem 4.3. □

4.3. Stability Results

A convergent sequence (x_n) generated by an iterative process is said numerically stable iff a sequence (y_n) approximately close to (x_n) converges to the same limit. A. S. Ostrowski [16] appears to be the first to discuss the stability of iterative procedures on metric spaces, since then, the stability theory has extensively been studied. In 2005, S. L. Singh, C. Bhatnagar and S. N. Mishra [20] introduce the notion of the stability of iterative procedures for a pair of self-maps of a metric space (M, d) and develop the theory for this kind of procedures.

Definition 4.2. Let $(M, \rho, \mathcal{W})$ be a convex b-metric space, let N be a subset of M, and let $S, T : N \longrightarrow M$ be such that $S(N) \subset T(N)$. For any $x_0 \in N$, let the sequence (Tx_n) generated by the iterative procedure

$$Tx_{n+1} = f(Tx_n, Sx_n, \alpha_n), \quad 0 \leq \alpha_n \leq 1, \quad n = 0, 1, 2, \ldots \tag{4.8}$$

converging to p. Also, let $(Ty_n) \subset M$ be an arbitrary sequence and let

$$\varepsilon_n = \rho(Ty_{n+1}, f(Ty_n, Sy_n, \alpha_n)).$$

The iterative process (4.8) will be called (S,T)-*stable* if $\lim_{n\to\infty} \varepsilon_n = 0$ implies that $\lim_{n\to\infty} Ty_n = p$.

Notice that $f(Tx_n, Sx_n, \alpha_n) \equiv Sx_n$ corresponds to the Jungck iteration, $f(Tx_n, Sx_n, \alpha_n) \equiv \mathcal{W}(Tx_n, Sx_n, \alpha_n)$ to the Jungck-Mann process and if we consider $f(Tx_n, Sx_n, \alpha_n) \equiv \mathcal{W}(Tx_n, Sy_n, \alpha_n)$, with $Ty_n = \mathcal{W}(Tx_n, Sx_n, \beta_n)$ then, it the corresponds to the Jungck-Ishikawa iterative scheme.

In order to prove the (S,T)-stability of the Jungck, Jungck-Mann and Jungck-Ishikawa iterative schemes we assume that $s > 2$, and we will use the following fact concerning to recurrent inequalities. See, e.g., Lemma 1.6 of [5].

Lemma 4.5. *Let (a_n), (b_n) be sequences of nonnegative numbers and $0 \leq q < 1$, so that*

$$a_{n+1} \leq q a_n + b_n, \quad \textit{for all } n \geq 0.$$

If $\lim_{n\to\infty} b_n = 0$, then $\lim_{n\to\infty} a_n = 0$.

Theorem 4.6. *Let (M, ρ) be a metric space with $s > 2$. Let $S, T : M \longrightarrow M$ be generalized ψ-Geraghty-Zamfirescu mappings and let $\psi \in \Psi$ be a convex subadditive function. Let z be a coincidence point of S and T, that is, $Sz = Tz = p$. Let $x_0 \in M$ and suppose the sequence (Tx_n) generated by the Jungck iteration $Tx_{n+1} = Sx_n$, $n = 0, 1, \ldots$ converges to p. Then, for any arbitrary sequence $(Ty_n) \subset M$ and ε_n as in Definition 4.2,*

$$\lim_{n\to\infty} Ty_n = p, \ \textit{if and only if} \ \lim_{n\to\infty} \varepsilon_n = 0.$$

Proof. Let (Ty_n) be an arbitrary sequence and define

$$\varepsilon_n = \rho(Ty_{n+1}, Sy_n), \quad n = 0, 1, \ldots$$

Using the s-triangle inequality we get,

$$\rho(Ty_{n+1}, p) \leq s[\rho(Ty_{n+1}, Sy_n) + \rho(Sy_n, p)]$$

$$= s\varepsilon_n + s\rho(Sy_n, Sz).$$

Since $\psi \in \Psi$ is subadditive, we have

$$\begin{aligned} \psi(\rho(Ty_{n+1}, p)) &\leq \psi(s\varepsilon_n) + \psi(s\rho(Sy_n, Sz)) \\ &\leq \psi(s\varepsilon_n) + \psi(s^2\rho(Sy_n, Sz)). \end{aligned} \tag{4.9}$$

Now, from condition (3.2) we have

$$\begin{aligned} \psi(s^2\rho(Sy_n, Sz)) &\leq \beta(\psi(N(y_n, z)))\psi(N(y_n, z)) \\ &< \frac{1}{s}\psi(N(y_n, z)), \end{aligned} \tag{4.10}$$

where,

$$\begin{aligned} N(y_n, z) &= \max\left\{\rho(Ty_n, Tz), \frac{\rho(Sy_n, Ty_n) + \rho(Sz, Tz)}{2}, \frac{\rho(Sy_n, Tz) + \rho(Sz, Ty_n)}{2s}\right\} \\ &= \max\left\{\rho(Ty_n, Tz), \frac{1}{2}\rho(Sy_n, Ty_n), \frac{\rho(Sy_n, Tz) + \rho(Sz, Ty_n)}{2s}\right\}. \end{aligned}$$

Note that

$$\frac{1}{2}\rho(Sy_n, Ty_n) \leq \frac{s}{2}[\rho(Ty_n, Tz) + \rho(Tz, Sy_n)].$$

Also,

$$\frac{\rho(Sy_n, Tz) + \rho(Sz, Ty_n)}{2s} \leq \frac{s}{2}[\rho(Ty_n, Tz) + \rho(Tz, Sy_n)].$$

Moreover, the condition $s > 2$ implies that

$$\rho(Ty_n, Tz) \leq \frac{s}{2}[\rho(Ty_n, Tz) + \rho(Tz, Sy_n)].$$

Thus, we conclude that

$$N(y_n, z) \leq \frac{s}{2}[\rho(Ty_n, Tz) + \rho(Tz, Sy_n)].$$

With this upper bound, estimate (4.10) takes the form

$$\begin{aligned} \psi(s^2\rho(Sy_n, Sz)) &< \frac{1}{2s}[\psi(s\rho(Ty_n, Tz)) + \psi(s\rho(Tz, Sy_n))] \\ &\leq \frac{1}{2s}[\psi(s\rho(Ty_n, Tz)) + \psi(s^2\rho(Tz, Sy_n))]. \end{aligned}$$

Since $Tz = Sz = p$, we get

$$\psi(s^2\rho(Sy_n, Sz)) < \frac{1}{2s-1}\psi(s\rho(Ty_n, Tz)).$$

Thus, inequality (4.9) is now given by

$$\psi(\rho(Ty_{n+1},p)) < \psi(s\varepsilon_n) + \frac{1}{2s-1}\psi(s\rho(Ty_n,p)).$$

Notice that a subadditive increasing function ψ satisfies that $\psi(sa) \le \lfloor s \rfloor \psi(a)$, where $\lfloor s \rfloor$ is the smallest integer greater or equal to s. Therefore,

$$\psi(\rho(Ty_{n+1},p)) < \psi(s\varepsilon_n) + \frac{\lfloor s \rfloor}{2s-1}\psi(\rho(Ty_n,p)).$$

Taking into account that $s > 2$ and

$$\lfloor s \rfloor \le \begin{cases} s, & \text{if } s \in \mathbb{Z}_+ \\ s+1, & \text{if } s \in \mathbb{R}_+ \setminus \mathbb{Z}_+ \end{cases}$$

we conclude that $\lfloor s \rfloor/(2s-1) < 1$. Therefore, from Lemma 4.5 we conclude that

$$\lim_{n\to\infty} \psi(\rho(Ty_n,p)) = 0, \quad \text{if } \lim_{n\to\infty} \psi(s\varepsilon_n) = 0.$$

Since $\psi \in \Psi$, this is equivalent to

$$\lim_{n\to\infty} \rho(Ty_n,p) = 0, \quad \text{if } \lim_{n\to\infty} \varepsilon_n = 0.$$

On the other hand,

$$\begin{aligned} 0 \le \varepsilon_n &= \rho(Ty_{n+1}, Sy_n) \\ &\le s\rho(Ty_{n+1},p) + s\rho(Sy_n,p) \\ &< s\rho(Ty_{n+1},p) + \frac{\lfloor s \rfloor}{2s-1}\rho(Ty_n,p). \end{aligned}$$

Using that $\psi \in \Psi$ is a subadditive function, we get

$$0 \le \psi(\varepsilon_n) < \psi(s\rho(Ty_{n+1},p)) + \lfloor s \rfloor \psi\left(\frac{1}{2s-1}\rho(Ty_n,p)\right).$$

Taking limits as $n \to \infty$, we have

$$0 \le \lim_{n\to\infty} \psi(\varepsilon_n) < \lim_{n\to\infty} \psi(s\rho(Ty_{n+1},p)) + \lfloor s \rfloor \lim_{n\to\infty} \psi\left(\frac{1}{2s-1}\rho(Ty_n,p)\right).$$

Therefore, we conclude that

$$\lim_{n\to\infty} \psi(\varepsilon_n) = 0, \quad \text{equivalently}, \quad \lim_{n\to\infty} \varepsilon_n = 0$$

whenever,

$$\lim_{n\to\infty} \psi(s\rho(Ty_n,p)) = 0, \quad \text{equivalently}, \quad \lim_{n\to\infty} Ty_n = p,$$

which completes the proof. □

To prove the (S,T)-stability of the Jungck-Mann and Jungck-Ishikawa iterations we will assume, in addition to $s>2$, that there exists $q\in[0,1)$ such that the sequence (α_n) satisfy that

$$0\le \alpha_n\le \frac{(2s-1)q-\lfloor s\rfloor}{2\lfloor s\rfloor(s-1)},\quad \text{for all } n=0,1,2,\dots \tag{4.11}$$

where $\lfloor s\rfloor$ is the smallest integer greater or equal to s and

$$\frac{\lfloor s\rfloor}{2s-1}<q<1. \tag{4.12}$$

Notice that if (4.12) holds, then

$$0<\frac{(2s-1)q-\lfloor s\rfloor}{2\lfloor s\rfloor(s-1)}<1.$$

In fact,

$$\begin{aligned}\frac{\lfloor s\rfloor}{2s-1}<q &\Longleftrightarrow 0<(2s-1)q-\lfloor s\rfloor\\ &\Longleftrightarrow 0<\frac{(2s-1)q-\lfloor s\rfloor}{2\lfloor s\rfloor(s-1)}.\end{aligned}$$

On the other hand,

$$\begin{aligned}q<1 &\Longleftrightarrow (2s-1)q<2s-1\le\lfloor s\rfloor(2s-2)+\lfloor s\rfloor\\ &\Longleftrightarrow \frac{(2s-1)q}{2\lfloor s\rfloor(s-1)}<1+\frac{\lfloor s\rfloor}{2\lfloor s\rfloor(s-1)}\\ &\Longleftrightarrow \frac{(2s-1)q-\lfloor s\rfloor}{2\lfloor s\rfloor(s-1)}<1.\end{aligned}$$

Thus, under this condition, $(\alpha_n)\subset[0,1]$ for all $n=0,1,\dots$

Theorem 4.7. *Let $(M,\ \rho,\ \mathcal{W})$ be a convex metric space with $s>2$. Let $S,T: M\longrightarrow M$ be generalized ψ-Geraghty-Zamfirescu mappings and let $\psi\in\Psi$ be a convex subadditive function. Let z be a coincidence point of S and T, that is, $Sz=Tz=p$. Let $x_0\in M$ and suppose that the sequence (Tx_n) generated by the Jungck-Mann iteration $Tx_{n+1}=\mathcal{W}(Tx_n,Sx_n,\alpha_n)$, $(n=0,1,\dots)$ with (α_n) satisfying* (4.11) *converges to p. Then, for any arbitrary sequence $(Ty_n)\subset M$ and ε_n as in Definition 4.2,*

$$\lim_{n\to\infty}Ty_n=p,\ \textit{if and only if}\ \lim_{n\to\infty}\varepsilon_n=0.$$

Proof. Let $(Ty_n)\subset M$ be an arbitrary sequence. From the s-triangle inequality and the convex structure we have

$$\begin{aligned}\rho(Ty_{n+1},p) &\le s[\rho(Ty_{n+1},\mathcal{W}(Ty_n,Sy_n,\alpha_n))+\rho(\mathcal{W}(Ty_n,Sy_n,\alpha_n),p)]\\ &\le s\varepsilon_n+s[\alpha_n\rho(Ty_n,p)+(1-\alpha_n)\rho(Sy_n,p)].\end{aligned}$$

Since $\psi\in\Psi$ is convex and subadditive, we get

$$\psi(\rho(Ty_{n+1},p))\le\psi(s\varepsilon_n)+\alpha_n\psi(s\rho(Ty_n,p))+(1-\alpha_n)\psi(s\rho(Sy_n,p)).$$

Now, due to the fact that $p=Tz=Sz$, for some $z\in M$, in Theorem 4.6 we prove that

$$\psi(s\rho(Sy_n,p))<\frac{1}{2s-1}\psi(s\rho(Ty_n,p)).$$

Therefore,

$$\begin{aligned}\psi(\rho(Ty_{n+1},p)) &< \psi(s\varepsilon_n)+\alpha_n\psi(s\rho(Ty_n,p))+\frac{1-\alpha_n}{2s-1}\psi(s\rho(Ty_n,p))\\ &\le\psi(s\varepsilon_n)+\left(\alpha_n+\frac{1-\alpha_n}{2s-1}\right)\lfloor s\rfloor\psi(\rho(Ty_n,p)),\end{aligned}\tag{4.13}$$

where $\lfloor s\rfloor$ is the smallest integer greater or equal to s. Now, since

$$\alpha_n\le\frac{(2s-1)q-\lfloor s\rfloor}{2\lfloor s\rfloor(s-1)}=\frac{2s-1}{2(s-1)}\frac{q}{\lfloor s\rfloor}-\frac{1}{2(s-1)}$$

then,

$$\frac{1}{2(s-1)}+\alpha_n\le\frac{2s-1}{2(s-1)}\frac{q}{\lfloor s\rfloor}.$$

Equivalently,

$$\frac{1}{2s-1}+\frac{2(s-1)}{2s-1}\alpha_n=\alpha_n+\frac{1-\alpha_n}{2s-1}\le\frac{q}{\lfloor s\rfloor}.$$

Hence, we have

$$\psi(\rho(Ty_{n+1},p))<\psi(s\varepsilon_n)+q\psi(\rho(Ty_n,p)).$$

Thus, from Lemma 4.5, we conclude that

$$\lim_{n\to\infty}\psi(\rho(Ty_n,p))=0,\quad\text{equivalently,}\quad\lim_{n\to\infty}\rho(Ty_n,p)=0$$

if

$$\lim_{n\to\infty}\psi(s\varepsilon_n)=0,\quad\text{equivalently,}\quad\lim_{n\to\infty}\varepsilon_n=0.$$

On the other hand,

$$\begin{aligned}0\le\varepsilon_n &= \rho(Ty_{n+1},\mathcal{W}(Ty_n,Sy_n,\alpha_n))\\ &\le s\rho(Ty_{n+1},p)+s\rho(\mathcal{W}(Ty_n,Sy_n,\alpha_n),p)\\ &\le s\rho(Ty_{n+1},p)+s[\alpha_n\rho(Ty_n,p)+(1-\alpha_n)\rho(Sy_n,p)].\end{aligned}$$

Now,

$$0 \le \psi(\varepsilon_n) \le \psi(s\rho(Ty_{n+1}, p)) + \alpha_n \psi(s\rho(Ty_n, p)) + (1 - \alpha_n)\psi(s\rho(Sy_n, Sz))$$

with

$$\psi(s\rho(Sy_n, Sz)) < \frac{1}{2s-1}\psi(s\rho(Ty_n, p)).$$

Thus,

$$0 \le \psi(\varepsilon_n) < \psi(s\rho(Ty_{n+1}, p)) + \alpha_n \psi(s\rho(Ty_n, p)) + \frac{1-\alpha_n}{2s-1}\psi(s\rho(Ty_n, p)).$$

In this way, using the fact that $\psi \in \Psi$, we conclude that

$$\text{if } \lim_{n\to\infty} Ty_n = p, \quad \text{then } \lim_{n\to\infty} \varepsilon_n = 0.$$

This completes the proof. □

Theorem 4.8. *Let $(M, \rho, \mathcal{W})$ be a convex metric space with $s > 2$. Let $S, T : M \longrightarrow M$ be generalized ψ-Geraghty-Zamfirescu mappings and let $\psi \in \Psi$ be a convex subadditive function. Let z be a coincidence point of S and T, that is, $Sz = Tz = p$. Let $x_0 \in M$ and suppose that the sequence (Tx_n) generated by the Jungck-Ishkawa iteration*

$$\begin{aligned} Tx_{n+1} &= \mathcal{W}(Tx_n, Sy_n, \alpha_n) \\ Ty_n &= \mathcal{W}(Tx_n, Sx_n, \beta_n), \qquad n = 0, 1, \dots \end{aligned}$$

with $(\alpha_n), (\beta_n) \subset [0,1]$ and (α_n) satisfying (4.11), converges to p. Then, for any arbitrary sequence $(Tz_n) \subset M$ and ε_n as in Definition 4.2,

$$\lim_{n\to\infty} Tz_n = p, \text{ if and only if } \lim_{n\to\infty} \varepsilon_n = 0.$$

Proof. Let $(Tz_n) \subset M$ be an arbitrary sequence. From the s-triangle inequality we have

$$\rho(Tz_{n+1}, p) \le s[\rho(Tz_{n+1}, \mathcal{W}(Tz_n, Sw_n, \alpha_n)) + \rho(\mathcal{W}(Tz_n, Sw_n, \alpha_n), p)],$$

where $Tw_n = \mathcal{W}(Tz_n, Sz_n, \beta_n)$. Then, from the convex structure we obtain

$$\rho(Tz_{n+1}, p) \le s\varepsilon + \alpha_n s\rho(Tz_n, p) + (1 - \alpha_n)s\rho(Sw_n, p).$$

Since $\psi \in \Psi$ is convex and subadditive, we get

$$\psi(\rho(Tz_{n+1}, p)) \le \psi(s\varepsilon) + \alpha_n \psi(s\rho(Tz_n, p)) + (1 - \alpha_n)\psi(s\rho(Sw_n, p)).$$

From the proof of Theorem 4.7 we known that

$$\psi(s\rho(Sw_n, p)) < \frac{1}{2s-1}\psi(s\rho(Tw_n, p)),$$

then, we have the following estimate

$$\begin{aligned}\psi(s\rho(Tw_n,p)) &= \psi(s\rho(\mathcal{W}(Tz_n,Sz_n,\beta_n),p))\\ &\leq \beta_n\psi(s\rho(Tz_n,p)) + (1-\beta_n)\psi(s\rho(Sz_n,p))\\ &< \beta_n\psi(s\rho(Tz_n,p)) + \frac{1-\beta_n}{2s-1}\psi(s\rho(Tz_n,p))\\ &\leq \beta_n\lfloor s\rfloor\psi(\rho(Tz_n,p)) + \frac{1-\beta_n}{2s-1}\lfloor s\rfloor\psi(\rho(Tz_n,p)).\end{aligned}$$

Therefore, we obtain

$$\begin{aligned}&\psi(\rho(Tz_{n+1},p))\\ &< \psi(s\varepsilon) + \alpha_n\lfloor s\rfloor\psi(\rho(Tz_n,p)) + \frac{1-\alpha_n}{2s-1}\left[\beta_n\lfloor s\rfloor + \frac{1-\beta_n}{2s-1}\lfloor s\rfloor\right]\psi(\rho(Tz_n,p))\\ &\leq \psi(s\varepsilon) + \alpha_n\lfloor s\rfloor\psi(\rho(Tz_n,p)) + \frac{1-\alpha_n}{2s-1}\left[\beta_n\lfloor s\rfloor + (1-\beta_n)\lfloor s\rfloor\right]\psi(\rho(Tz_n,p))\\ &= \psi(s\varepsilon) + \alpha_n\lfloor s\rfloor\psi(\rho(Tz_n,p)) + \frac{1-\alpha_n}{2s-1}\lfloor s\rfloor\psi(\rho(Tz_n,p))\end{aligned}$$

which is the same estimate (4.13). The conclusion is obtained by repeating the arguments of the proof of Theorem 4.7. □

5. Common Fixed Point Theorems

In this section we provide conditions to guarantee the existence of common fixed points for ψ-Geraghty-Zamfirescu pairs.

First, we would like to point out that weakly compatibility is a minimal requirement for the existence of common fixed points for contractive pair of mappings. For a discussion in the subject see, e.g., [13, 14] and references therein. In this direction, in 2006, G. Jungck and B. E. Rhoades [10] proved that any pair of weakly compatible mappings with a unique point of coincidence u satisfy that u is its unique common fixed point.

Lemma 5.1.(Jungck-Rhoades, [10]) *Let S and T be weakly compatible selfmaps of a set $M \neq \emptyset$. If S and T have a unique POC $z = Su = Tu$, then z is the unique common fixed point of S and T.*

Theorem 5.2. *Let $(M,\, \rho)$ be a b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ be two selfmappings that are generalized ψ-Geraghty-Zamfirescu contraction mappings. If $SM \subset TM$ and $TM \subset M$ is complete, then*

(i) *S and T have a unique POC and*

(ii) *If S and T are weakly compatible, then S and T have a unique common fixed point.*

Proof. Since $SM \subset TM$ and $TM \subset M$ is complete, then from Proposition 4.2, the pair (T, S) have a unique point of coincidence. Now, under the hypothesis (T, S) being weakly compatible, Lemma 5.1, implies that (S, T) has a unique common fixed point. □

In the following result, by using the b-Property (EA), we drop the condition "$TM \subset M$ is complete" in Theorem 5.2.

Theorem 5.3. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ be two selfmappings satisfying the b-property (EA). Assume that S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings. If $TM \subset M$ is closed, then*

(i) *S and T have a unique POC and*

(ii) *If S and T are weakly compatible, then S and T have a unique common fixed point in M.*

Proof. Since S and T satisfy the b-property (EA), there exists a sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = z, \quad \text{for some} \quad z \in M.$$

Since $TM \subset M$ is closed, we have

$$\lim_{n\to\infty} Tx_n = z = Tu \quad \text{for some} \quad u \in M.$$

In the proof of Proposition 3.2 we show that the limit above implies that $z = Tu = Su$. Finally, Lemma 5.1, implies that (S, T) has a unique common fixed point. □

Since two non-compatible selfmappings of a b-metric space (M, ρ) with $s \geq 1$ satisfy the b-property (EA), we get the following result.

Corollary 5.4. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ be two non-compatible selfmappings. Assume that S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings. If $TM \subset M$ is closed, then*

(i) *S and T have a unique POC and*

(ii) *If S and T are weakly compatible, then S and T have a unique common fixed point in M.*

In the next result we use the b-CLR_T-property and we drop the conditions $SM \subset TM$ and closedness of the range of any mapping in Theorem 5.2.

Theorem 5.5. *Let (M, ρ) be a b-metric space with $s \geq 1$ and let $S, T : M \longrightarrow M$ be two selfmappings satisfying the b-CLR_T-property. Assume that S and T are generalized ψ-Geraghty-Zamfirescu contraction mappings. Then,*

(i) *S and T have a unique POC and*

(ii) *If S and T are weakly compatible, then S and T have a unique common fixed point in M.*

Proof. Since S and T satisfy the b-CLR_T-property, then there exists a sequence (x_n) in M such that

$$\lim_{n\to\infty} Sx_n = \lim_{n\to\infty} Tx_n = Tz, \quad \text{for some} \quad z \in M.$$

Therefore, there exists $u \in M$ such that $Tz = u$. The rest of the proof follows as in Theorem 5.3. □

6. Conclusions

The class of generalized ψ-Geragthy-Zamfirescu mappings includes several classical contractive-type mappings as particular cases, by considering T or ψ the identity map. Also, we would like to point out that condition (3.1) for $\beta \in \mathcal{B}_s$ is not explicitly used to prove our results, so we can consider β as a constant function without altering the conclusions. We can see the inclusion of the extra functions ψ and β as a pointwise control for the contractive inequality, which is very convenient for the construction of examples, specially when we are dealing with discontinuous mappings.

On the other hand, since weakly compatibility is a minimal non-commutative requirement for the existence of common fixed points for pair of contractive-type mappings, and in virtue of the nice result of Jungck-Rhoades (Lemma 5.1), the investigation of the existence of a unique common fixed point is reduced to prove the existence of a unique POC. To show the existence of such POC, we analyze the limits of some sequences, hence it is expected to impose some convergence conditions to the mappings under study. The b-property (EA) and the b-CLR_T-property are natural substitutes to the completeness of the b-metric space, so the conditions imposed in our results are sharp in this sense.

This convergence approach also allows us to study the convergence and stability of iterative schemes, and Takahashi's convex structure allows to pose these results in the framework of linear normed spaces, or other metric spaces enhanced with some geometric structure as, for instance, CAT(0) and hyperbolic metric spaces. However, it is worth to mention that our stability results run only for b-metric spaces with $s > 2$, thus it cannot be used in the setting of metric spaces.

Acknowledgements. The authors are thankful to the referee for the very constructive comments and suggestions that led to an improvement of the paper, particularly, in regards to the consideration of the Jungck-Ishikawa iteration.

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