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On f-biharmonic Submanifolds of Three Dimensional Trans-Sasakian Manifolds

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Abstract. The object of the present paper is to study f-biharmonic submanifolds of three dimensional trans-Sasakian manifolds. We find some necessary and sufficient conditions for such submanifolds to be f-biharmonic.

1. Introduction

Let M and N be two Riemannian manifolds, a harmonic map $\psi : M \to N$ is any critical point of the energy equation

$$E(\psi) = \frac{1}{2}\int_M |d\psi|^2 dv_g,$$

where dv_g denotes the volume element of g, and the Euler-Lagrange equation corresponding to $E(\psi)$ is $\tau(\psi) = \text{trace}\nabla d\psi = 0$.

In 1983, Eells and Lemaire [9] introduced the notion of biharmonic maps, which are a natural generalization of harmonic maps. A biharmonic map $\psi : M \to N$ is a critical point of the energy equation

$$E_2(\psi) = \frac{1}{2}\int_M |\tau\psi|^2 dv_g,$$

where dv_g denotes the volume element of g, and the Euler-Lagrange equation [15] corresponding to $E_2(\psi)$ is

$$\tau_2(\psi) = \Delta\tau(\psi) - \text{trace}(R^N(d\psi, \tau(\psi))d\psi) = 0. \tag{1.1}$$

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Here Δ is the Laplacian operator given by $\Delta V = \mathrm{tr}(\nabla^2 V)$, and R^N is the curvature tensor on the manifold N defined as $R^N(X,Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}$.

Let M be the submanifold of the manifold $\bar{M}$, if the biharmonic map $\psi : M \to \bar{M}$ is an isometric immersion then M is biharmonic submanifold of $\bar{M}$. In the paper [2], Baird studied conformal and semi-conformal biharmonic maps. Oniciuc studied biharmonic submanifolds of CP^n in [10]. He studied explicit formula for biharmonic submanifolds in Sasakian space forms and deduced some conditions in [11]. He proved a gap theorem for the mean curvature of certain complete proper biharmonic pmc submanifolds and classified proper biharmonic pmc surfaces in $S^n \times R$ in [12]. In [16], Oniciuc studied biharmonic constant mean curvature surface in the sphere. Recently, Oniciuc proved several unique continuation results for biharmonic maps between Riemannian manifolds in [19]. He studied biharmonic maps between Riemannian manifolds in [18]. Over the last few years many authors have studied biharmonic submanifolds, for example see [5, 10, 18]. Recently, Ou studied biharmonic maps form tori into a 2-sphere in [27]. In the paper [1], Ou studied biharmonic Riemannian submanifolds.

The notion of f-biharmonic maps was introduced by Lu [17]; it is a natural generalization of biharmonic maps. In the papers [21, 22], Ou studied f-biharmonic maps and f-biharmonic submanifolds. In these papers he proved that a f-biharmonic map from a compact Riemannian manifold into a non-positively curved manifold with constant f-bienergy density is a harmonic map. In [20], Ou characterized harmonic maps and minimal submanifolds using the concept of f-biharmonic maps and proved that the set of all f-biharmonic maps from a 2-dimensional domain is invariant under the conformal change of the metric on the domain. In [24], Roth studied f-biharonic submanifolds of generalized space forms. He deduced some necessary and sufficient conditions for f-biharmonicity in the general case and many particular cases. In [2] Baird and Fardon studied conformal and semi conformal biharmonic maps.

Let us consider the C^∞ differentiable function $f : M \to R$. Now, f-harmonic maps are the critical points of the f-energy functional $E_f(\psi)$ for the maps $\psi : M \to N$ between Riemannian manifolds, where

$$E_f(\psi) = \frac{1}{2}\int_M f|d\psi|^2 dv_g.$$

The Euler-Lagrange equation corresponding to $E_f(\psi)$ is given by

$$\tau_f(\psi) = f\tau(\psi) + d\psi(\mathrm{grad} f) = 0. \tag{1.2}$$

Analgously f-biharmonic maps are critical points of the f-bienergy functional $E_{2,f}(\psi)$ for maps $\psi : M \to N$ between Riemannian manifolds where

$$E_{2,f}(\psi) = \frac{1}{2}\int_M f|\tau\psi|^2 dv_g.$$

The Euler-Lagrange equation corresponding to $E_{2,f}(\psi)$ is given by

$$\tau_{2,f}(\psi) = f\tau_2(\psi) + (\Delta f)\tau(\psi) + 2\nabla^{\psi}_{(\mathrm{grad} f)}\tau(\psi) = 0. \tag{1.3}$$

Clearly, we have the following relationship among these different types of harmonic maps: Harmonic maps $\subset$ biharmonic maps $\subset$ f-biharmonic maps.

A f-biharmonic map is called a proper f-biharmonic map if it is neither a harmonic nor a biharmonic map. Also, we will call a f-biharmonic submanifold proper if it is neither minimal nor biharmonic.

The notion of trans-Sasakian Manifolds was introduced by Blair and Oubina [4, 23] as a generalization of Sasakian manifolds. Trans-Sasakian manifolds of type (α, β) are generalizations of α-Sasakian and β-Kenmotsu manifolds. It is known that a proper trans-Sasakian manifold exists only for dimension three and trans-Sasakian manifolds of type $(0,0), (0,\beta)$, and $(\alpha, 0)$ are known [14] as cosymplectic, $\beta-$Kenmotsu and α-Sasakian respectively. In higher dimension it is either α-Sasakian or β-Kenmotsu. In Differential Geometry of almost contact manifolds, submanifold theory has become an important topic of research. There are several works on invariant submanifolds. In [6], the authors studied invariant submanifolds of trans-Sasakian manifolds. Three dimensional trans-Sasakian Manifolds have been studied by the first author in the papers [8, 25, 26].

During last few years biharmonic maps on contact manifolds have become a popular area of research. So in the present paper we would like to study f-biharmonic maps on three dimensional trans-Sasakian manifolds. Precisely we study f-biharmonic submanifolds of three dimensional trans-Sasakian manifolds and find some conditions for the map f to be biharmonic or not.

The present paper is organized as follows: Section 1 is introductory. After the introduction we give some preliminaries in Section 2. In Section 3 we study f-biharmonic submanifolds of three-dimensional trans-Sasakian manifolds.

2. Preliminaries

Let $\bar{M}$ be an odd dimensional smooth differential manifold with an almost contact metric structure (ϕ, ξ, η, g), where ϕ is a (1,1)-tensor field, ξ is a vector field, η is a one form and g is a Riemannian metric on $\bar{M}$. For such manifolds, we know [3]

$$\phi^2 X = -X + \eta(X)\xi, \qquad \eta(\xi) = 1, \tag{2.1}$$

$$\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \tag{2.2}$$

$$\phi\xi = 0, \quad \eta o \phi = 0, \quad g(X, \phi Y) = -g(\phi X, Y) \tag{2.3}$$

for any $X, Y \in \chi(\bar{M})$, where $\chi(\bar{M})$ denotes the Lie algebra of all vector fields on $\bar{M}$.

For a contact metric manifold $(\bar{M}, \phi, \xi, \eta, g)$, we define a (1,1) tensor field h by $h = \frac{1}{2}\mathcal{L}_\xi \phi$ and $\mathcal{L}$ is the usual Lie derivative. Then h is symmetric and satisfies the following relations

$$h\xi = 0, \quad h\phi = -\phi h, \quad tr(h) = tr(\phi h) = 0, \quad \eta(hX) = 0 \tag{2.4}$$

for any $X, Y \in \chi(\bar{M})$.

Moreover, if $\bar{\nabla}$ denotes the Levi-Civita connection with respect to g, then the following relation holds

$$\bar{\nabla}_X \xi = -\phi X - \phi h X. \tag{2.5}$$

A connected manifold $\bar{M}$ with almost contact metric structure (ϕ, ξ, η, g) is called a trans-Sasakian manifold [23] if $(\bar{M} \times R, J, G)$ belongs to the class W_4 [13], where J is an almost complex structure on $\bar{M} \times R$ which is defined by

$$J(X, f\frac{d}{dt}) = (\phi X - f\xi, \eta(X)\frac{d}{dt})$$

for any vector field X on $\bar{M}$ and the smooth function f on $\bar{M} \times R$, and G is the usual product metric on $\bar{M} \times R$. According to [4], an almost contact metric manifold is a trans-Sasakian manifold if and only if

$$(\bar{\nabla}_X \phi)Y = \alpha(g(X, Y)\xi - \eta(Y)X) + \beta(g(\phi X, Y)\xi - \eta(Y)\phi X) \tag{2.6}$$

for smooth functions α, β on $\bar{M}$, where $\bar{\nabla}$ denote the covariant derivative with respect to g. Generally, $\bar{M}$, is said to be a trans-Sasakian manifold of type (α, β). In a three-dimensional trans-Sasakian manifold the curvature tensor with respect to the Levi-Civita connection $\bar{\nabla}$ is as follows [7]:

$$\begin{aligned} R(X, Y)Z = {} & (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y, Z)X - g(X, Z)Y) \\ & - g(Y, Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\ & - \eta(X)(\phi \mathrm{grad}\alpha - \phi \mathrm{grad}\beta) + (X\beta + (\phi X)\alpha)\xi] \\ & + g(X, Y)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\ & - \eta(Y)(\phi \mathrm{grad}\alpha - \phi \mathrm{grad}\beta) + (Y\beta + (\phi Y)\alpha)\xi] \\ & - [(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\ & + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\ & + [(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\ & + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y, \end{aligned} \tag{2.7}$$

where r is the scalar curvature of the manifold.

Let M^m $(m < n)$ be the submanifold of a contact metric manifold $\bar{M}^n$. Let ∇ and $\bar{\nabla}$ be the Levi-Civita connections of M and $\bar{M}$, respectively. Then for any vector fields $X, Y \in \chi(M)$, the second fundamental form σ is defined by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y). \tag{2.8}$$

For any section of the normal bundle $T^{\perp}M$, we have

$$\bar{\nabla}_X N = -A_N X + \nabla^{\perp} N, \tag{2.9}$$

where $\nabla^{\perp}$ denotes the normal bundle connection of M. The second fundamental form σ and the shape operator A_N are related by

$$g(A_N X, Y) = g(\sigma(X, Y), N). \tag{2.10}$$

For any vector field $X \in \chi(M)$, we can right

$$\phi X = TX + NX, \tag{2.11}$$

where TX is the tangential component of ϕX and NX is the normal component of ϕX. Similarly, for any vector field V in normal bundle we have

$$\phi V = tV + nV, \tag{2.12}$$

where tV and nV are the tangential and normal components of ϕV.

The submanifold M is said to be invariant if $\phi X \in TM$ for any vector field X. On other hand M is said to be an anti-invariant submanifold if $\phi X \in T^{\perp}M$ for any vector field X

3. f-biharmonic Submanifolds of Three-dimensional Trans-Sasakian Manifolds

We know for a isometric immersion ψ [24]

$$\tau(\psi) = \mathrm{tr}\nabla d\psi = \mathrm{tr}\sigma = mH, \tag{3.1}$$

where H is the mean curvature. Now using the equation (1.1) in the above equation we have

$$\tau_2(\psi) = m\Delta H - \mathrm{tr}(R(d\psi, mH)d\psi). \tag{3.2}$$

By some classical and straightforward computations, we have

$$\Delta H = \frac{m}{2}\mathrm{grad}|H|^2 + \mathrm{tr}(\sigma(., A_H.)) + 2\mathrm{tr}(A_{\nabla^{\perp}H}(.)) + \Delta^{\perp}H. \tag{3.3}$$

Using (3.3) in (3.2), we have

(3.4)

$$\tau_2(\psi) = \frac{m^2}{2}\mathrm{grad}|H|^2 + m\mathrm{tr}(\sigma(., A_H.)) + 2m\mathrm{tr}(A_{\nabla^{\perp}H}(.)) + m\Delta^{\perp}H - \mathrm{tr}(R(d\psi, mH)d\psi).$$

From the equation (1.3), we have the submanifold M is f-biharmonic if and only if

$$\tau_{2,f}(\psi) = f\tau_2(\psi) + (\Delta f)\tau(\psi) + 2\nabla^{\psi}_{(\mathrm{grad} f)}\tau(\psi) = 0. \tag{3.5}$$

By simple calculation we have the above equation is equivalent to

$$\tau_2(\psi) + m\frac{\Delta f}{f}H + 2m(-A_H\mathrm{grad}(\ln f) + \nabla^{\perp}_{\mathrm{grad}(\ln f)}H) = 0. \tag{3.6}$$

For a f-biharmonic submanifold of a three-dimensional trans-Sasakian manifold we have the following:

Theorem 3.1. *Let M be a submanifold of a three dimensional trans-Sasakian manifold $\bar{M}$. Then M is f-biharmonic if and only if the following equations hold*

$$\begin{aligned}
&\Delta^{\perp} H + \mathrm{tr}(\sigma(.,A_H.)) + \frac{\Delta f}{f} H + 2\nabla^{\perp}_{\mathrm{grad}(\ln f)} H \\
&= -2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))H + 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^{\perp} \\
&\quad - \eta(H)(N\mathrm{grad}\alpha - N\mathrm{grad}\beta) + \xi\beta H - \xi\alpha n(H)] \\
&\quad + [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))]H,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{grad}|H|^2 - 2\mathrm{tr}A_H \mathrm{grad}(\ln f) + 2\mathrm{tr}(A_{\nabla^{\perp} H}, .) \\
&= 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^T - \eta(H)(T\mathrm{grad}\alpha - T\mathrm{grad}\ \beta) \\
&\quad + t(H)\xi\alpha] - [(\mathrm{grad}\beta)^T\eta(H) + g(\mathrm{grad}\beta, H)\xi^T + g(\mathrm{grad}\alpha, \phi H)\xi^T \\
&\quad + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^T].
\end{aligned}$$

Proof. Form (2.7) we have

$$\begin{aligned}
R(X,Y)Z = {} & (\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))(g(Y,Z)X - g(X,Z)Y) \\
& - g(Y,Z)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\xi \\
& - \eta(X)(\phi\mathrm{grad}\alpha - \phi\mathrm{grad}\beta) + (X\beta + (\phi X)\alpha)\xi] \\
& + g(X,Y)[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\xi \\
& - \eta(Y)(\phi\mathrm{grad}\alpha - \phi\mathrm{grad}\beta) + (Y\beta + (\phi Y)\alpha)\xi] \\
& - [(Z\beta + (\phi Z)\alpha)\eta(Y) + (Y\beta + (\phi Y)\alpha)\eta(Z) \\
& + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(Y)\eta(Z)]X \\
& + [(Z\beta + (\phi Z)\alpha)\eta(X) + (X\beta + (\phi X)\alpha)\eta(Z) \\
& + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(X)\eta(Z)]Y. \qquad (3.7)
\end{aligned}$$

Let $\{e_1, e_2\}$ be an orthogonal basis of the tangent space at a point of M. Then we have from above

$$
\begin{aligned}
R(e_i,Y)e_i = {} & (\frac{r}{2}+2\xi\beta-2(\alpha^2-\beta^2))(g(H,e_i)e_i-g(e_i,e_i)H) \\
& - g(H,e_i)[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(e_i)\xi \\
& - \eta(e_i)(\phi\mathrm{grad}\alpha-\phi\mathrm{grad}\beta)+(e_i\beta+(\phi e_i)\alpha)\xi] \\
& + g(e_i,e_i)[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(H)\xi \\
& - \eta(H)(\phi\mathrm{grad}\alpha-\phi\mathrm{grad}\beta)+(H\beta+(\phi H)\alpha)\xi] \\
& - [(e_i\beta+(\phi e_i)\alpha)\eta(H)+(H\beta+(\phi H)\alpha)\eta(e_i) \\
& + (\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(H)\eta(e_i)]e_i \\
& + [(e_i\beta+(\phi e_i)\alpha)\eta(e_i)+(e_i\beta+(\phi e_i)\alpha)\eta(e_i) \\
& + (\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(e_i)\eta(e_i)]H.
\end{aligned}
\tag{3.8}
$$

Taking trace and using the equations (2.1), (2.11) and (2.12) we obtain

$$
\begin{aligned}
\mathrm{tr}(R(.,H).) = {} & -2(\frac{r}{2}+2\xi\beta-2(\alpha^2-\beta^2))H+2[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(H)\xi \\
& - \eta(H)(\phi\mathrm{grad}\alpha-\phi\mathrm{grad}\beta)+\xi\beta H-\xi\alpha\phi(H)]-[(\mathrm{grad}\beta)^T\eta(H) \\
& + g(\mathrm{grad}\beta,H)\xi^T+g(\mathrm{grad}\alpha,\phi H)\xi^T+(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(H)\xi^T] \\
& + [2\eta(\mathrm{grad}\beta)+(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))]H.
\end{aligned}
$$

Using the equations (3.4) and (3.6) we can obtain

$$
\begin{aligned}
\mathrm{tr}(R(.,H).) = {} & \mathrm{grad}|H|^2+\mathrm{tr}(\sigma(.,A_H.))+2\mathrm{tr}(A_{\nabla^\perp H}(.)) \\
& + \Delta^\perp H+\frac{\Delta f}{f}H-2(A_H\mathrm{grad}(\ln f))+2\nabla^\perp_{\mathrm{grad}(\ln f)}H.
\end{aligned}
$$

Therefore we have

$$
\begin{aligned}
& \mathrm{grad}|H|^2+\mathrm{tr}(\sigma(.,A_H.))+2\mathrm{tr}(A_{\nabla^\perp H}(.)) \\
& \quad + \Delta^\perp H+\frac{\Delta f}{f}H-2(A_H\mathrm{grad}(\ln f))+2\nabla^\perp_{\mathrm{grad}(\ln f)}H \\
& = -2(\frac{r}{2}+\xi\beta-2(\alpha^2-\beta^2))H \\
& \quad + 2[(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(H)\xi-\eta(H)(\phi\mathrm{grad}\alpha-\phi\mathrm{grad}\beta)+\xi\beta H-\xi\alpha\phi(H)] \\
& \quad - [(\mathrm{grad}\beta)^T\eta(H)+g(\mathrm{grad}\beta,H)\xi^T+g(\mathrm{grad}\alpha,\phi H)\xi^T \\
& \quad + (\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))\eta(H)\xi^T]+[2\eta(\mathrm{grad}\beta)+(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))]H.
\end{aligned}
$$

Comparing the tangent and normal components we have the result of the theorem.

Now we have the following as particular cases of the above theorem.

Corollary 3.1. *Let M be a submanifold of a three-dimensional trans-Sasakian manifold $\bar{M}$.*

(1) *If M is anti-invariant, M is f-biharmonic if and only if*

$$\begin{aligned}&\Delta^{\perp}H + tr(\sigma(.,A_H.)) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{grad(\ln f)}H\\ &= -2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2-\beta^2))H + 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))\eta(H)\xi^{\perp}\\ &\quad + \xi\beta H - \xi\alpha n(H)] + [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))]H,\end{aligned}$$

and

$$\begin{aligned}&grad|H|^2 - 2trA_H grad(\ln f) + 2tr(A_{\nabla^{\perp}H},.)\\ &= 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))\eta(H)\xi^T - \eta(H)(T grad\alpha - T grad\ \beta)\\ &\quad + t(H)\xi\alpha] - [(grad\beta)^T\eta(H) + g(grad\beta, H)\xi^T + g(grad\alpha, \phi H)\xi^T\\ &\quad + (\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))\eta(H)\xi^T].\end{aligned}$$

(2) *If M is invariant M is f-biharmonic if and only if*

$$\begin{aligned}&\Delta^{\perp}H + tr(\sigma(.,A_H.)) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{grad(\ln f)}H\\ &= -2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2-\beta^2))H + 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))\eta(H)\xi^{\perp}\\ &\quad - \eta(H)(N grad\alpha - N grad\beta) + \xi\beta H - \xi\alpha n(H)]\\ &\quad + [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))]H,\end{aligned}$$

and

$$\begin{aligned}&grad|H|^2 - 2trA_H grad(\ln f) + 2tr(A_{\nabla^{\perp}H},.)\\ &= 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))\eta(H)\xi^T + t(H)\xi\alpha] - [(grad\beta)^T\eta(H) +\\ &\quad g(grad\beta, H)\xi^T + g(grad\alpha, \phi H)\xi^T + (\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))\eta(H)\xi^T].\end{aligned}$$

(3) *If ξ is normal to M, M is f-biharmonic if and only if*

$$\begin{aligned}&\Delta^{\perp}H + tr(\sigma(.,A_H.)) + \frac{\Delta f}{f}H + 2\nabla^{\perp}_{grad(\ln f)}H\\ &= -2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2-\beta^2))H + 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))\eta(H)\xi^{\perp}\\ &\quad - \eta(H)(N grad\alpha - N grad\beta) + \xi\beta H - \xi\alpha n(H)]\\ &\quad + [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2-\beta^2))]H,\end{aligned}$$

and

$$\begin{aligned}&grad|H|^2 - 2trA_H grad(\ln f) + 2tr(A_{\nabla^\perp H}, .)\\&= 2[-\eta(H)(T grad\alpha - T grad\beta) + t(H)\xi\alpha] - [(grad\beta)^T\eta(H)].\end{aligned}$$

(4) *If ξ is tangent to M, M is f-biharmonic if and only if*

$$\begin{aligned}&\Delta^\perp H + tr(\sigma(., A_H .)) + \frac{\Delta f}{f}H + 2\nabla^\perp_{grad(\ln f)}H\\&= -2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))H + \xi\beta H - \xi\alpha n(H)]\\&\quad + [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))]H,\end{aligned}$$

and

$$\begin{aligned}&grad|H|^2 - 2trA_H grad(\ln f) + 2tr(A_{\nabla^\perp H}, .)\\&= 2t(H)\xi\alpha - [g(grad\beta, H)\xi^T + g(grad\alpha, \phi H)\xi^T],\end{aligned}$$

(5) *If M is a hypersurface, M is f-biharmonic if and only if*

$$\begin{aligned}&\Delta^\perp H + tr(\sigma(., A_H .)) + \frac{\Delta f}{f}H + 2\nabla^\perp_{grad(\ln f)}H\\&= -2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))H + 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^\perp\\&\quad - \eta(H)(N grad\alpha - N grad\beta) + \xi\beta H - \xi\alpha n(H)]\\&\quad + [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))]H,\end{aligned}$$

and

$$\begin{aligned}&grad|H|^2 - 2trA_H grad(\ln f) + 2tr(A_{\nabla^\perp H}, .)\\&= 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^T - \eta(H)(T grad\alpha - T grad\beta]\\&\quad - [(grad\beta)^T\eta(H) + g(grad\beta, H)\xi^T + g(grad\alpha, \phi H)\xi^T\\&\quad + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^T].\end{aligned}$$

Proof. Proof of the results is directly obtained from Theorem 3.1, using the following facts, respectively.

(1) If M is invariant then $N = 0$.

(2) If M is anti-invariant then $T = 0$.

(3) If ξ is normal to M then $\xi^T = 0$.

(4) If ξ is tangent to M then $\eta(H) = 0$ and $\xi^{\perp} = 0$.

(5) If M is a hypersurface then $tH = 0$. □

Theorem 3.2. *Let M be a submanifold of a three dimensional trans-Sasakian manifold $\bar{M}$ with non zero constant mean curvature H and ξ is tangent to M, then M proper f-biharmonic if and only if*

$$|\sigma|^2 = -\frac{3r}{2} - 7\xi\beta + 7(\alpha^2 - \beta^2) - \frac{\Delta f}{f},$$

and $A_H \mathrm{grad}(\ln f) = 0$, or equivalent if and only if

$$\mathrm{Scal}_M = \frac{3r}{2} + 9\xi\beta - 8(\alpha^2 - \beta^2) + \frac{\Delta f}{f} - 3|H|^2.$$

Proof. Let M be a f biharmonic submanifold of $\bar{M}$ with constant mean curvature and ξ tangent to M then from the previous corollary we have

$$\begin{aligned}
&\Delta^{\perp} H + \mathrm{tr}(\sigma(., A_H.)) + \frac{\Delta f}{f} H + 2\nabla^{\perp}_{\mathrm{grad}(\ln f)} H \\
&= -2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))H + 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^{\perp} \\
&\quad - \eta(H)(N\mathrm{grad}\alpha - N\mathrm{grad}\beta) + \xi\beta H - \xi\alpha n(H)] \\
&\quad + [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))]H,
\end{aligned}$$

and

$$\begin{aligned}
&\mathrm{grad}|H|^2 - 2\mathrm{tr}A_H \mathrm{grad}(\ln f) + 2\mathrm{tr}(A_{\nabla^{\perp} H}, .) \\
&= 2[-\eta(H)(T\mathrm{grad}\alpha - T\mathrm{grad}\beta) + t(H)\xi\alpha] - [(\mathrm{grad}\beta)^T \eta(H)].
\end{aligned}$$

Since ξ is tangent to M then the equations are of the form

$$\begin{aligned}
\mathrm{tr}(\sigma(., A_H.)) = &-2(\frac{r}{2} + 2\xi\beta - 2(\alpha^2 - \beta^2))H + 2[(\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))\eta(H)\xi^{\perp} \\
&- \eta(H)(N\mathrm{grad}\alpha - N\mathrm{grad}\beta) + \xi\beta H - \xi\alpha n(H)] \\
&+ [2\xi\beta + (\frac{r}{2} + \xi\beta - 3(\alpha^2 - \beta^2))]H - \frac{\Delta f}{f} H,
\end{aligned}$$

and $A_H \mathrm{grad}(\ln f) = 0$. Thus, the second equation is trivial and the first equation becomes

$$\mathrm{tr}\sigma(., A_H.) = [-\frac{3r}{2} - 7\xi\beta + 7(\alpha^2 - \beta^2) - \frac{\Delta f}{f}]H. \tag{3.9}$$

Now since $\mathrm{tr}\sigma(., A_H.) = |\sigma|^2 H$ and H is non zero, so we have form above equation

$$|\sigma|^2 = -\frac{3r}{2} - 7\xi\beta + 7(\alpha^2 - \beta^2) - \frac{\Delta f}{f}.$$

Now from the Gauss formula we have

$$\text{Scal}_M = \sum_{i,j} g(R(e_i, e_j)e_j, e_i) - |\sigma|^2 - 2H^2. \tag{3.10}$$

Using (2.7) in the above equation we have

$$\text{Scal}_M = \frac{3r}{2} 9\xi\beta - 8(\alpha^2 - \beta^2) + \frac{\Delta f}{f} - 3|H|^2.$$ □

Corollary 3.2. *Let M be a submanifold of a three dimensional trans-Sasakian manifold $\bar{M}$ with non zero constant mean curvature H and ξ is tangent to M. If the functions α, β satisfy the inequality*

$$-\frac{3r}{2} - 7\xi\beta + 7(\alpha^2 - \beta^2) \leq \frac{\Delta f}{f}$$

then M is not f-biharmonic.

Proof. Form the Theorem 3.2 we know that M is f-biharmonic if and only if its second fundamental form σ satisfies the inequality

$$|\sigma|^2 = -\frac{3r}{2} - 7\xi\beta + 7(\alpha^2 - \beta^2) - \frac{\Delta f}{f},$$

Since $|\sigma|^2 \geq 0$, this is not possible if

$$-\frac{3r}{2} - 7\xi\beta + 7(\alpha^2 - \beta^2) \leq \frac{\Delta f}{f}. \tag{3.11}$$ □

Theorem 3.3. *Let M be a submanifold of a three dimensional trans-Sasakian manifold $\bar{M}$ with non zero constant mean curvature H such that ξ and ϕH are tangent to M. Define $F(f, \alpha, \beta)$ on M by*

$$F(f, \alpha, \beta) = -2r - 9\xi\beta + 9(\alpha^2 - \beta^2) - \frac{\Delta f}{f}.$$

Then

(1) *if inf $F(f, \alpha, \beta)$ is non-positive, M is not f-biharmonic.*

(2) *if $F(f, \alpha, \beta)$ is positive and M is proper f-biharmonic then*

$$0 < |H|^2 \leq \frac{1}{2} F(f, \alpha, \beta).$$

Proof. M is proper f-biharmonic submanifold with constant mean curvature H and ξ is tangent to M, so we have form Corollary 3.1

$$\begin{aligned}&\Delta^{\perp}H+\mathrm{tr}(\sigma(.,A_H.))+\frac{\Delta f}{f}H+2\nabla^{\perp}_{\mathrm{grad}(\ln f)}H\\&=-2(\frac{r}{2}+2\xi\beta-2(\alpha^2-\beta^2))H+\xi\beta H-\xi\alpha n(H)]\\&\quad+[2\xi\beta+(\frac{r}{2}+\xi\beta-3(\alpha^2-\beta^2))]H\end{aligned}$$

and

$$\begin{aligned}&\mathrm{grad}|H|^2-2\mathrm{tr}A_H\mathrm{grad}(\ln f)+2\mathrm{tr}(A_{\nabla^{\perp}H},.)\\&=2t(H)\xi\alpha-[g(\mathrm{grad}\beta,H)\xi^T+g(\mathrm{grad}\alpha,\phi H)\xi^T].\end{aligned}$$

Given that ϕH is tangent to M, so $tH=0$. Therefore form the above equation we have

$$\begin{aligned}\Delta^{\perp}H+\mathrm{tr}(\sigma(.,A_H.))&=[-2r-9\xi\beta+9(\alpha^2-\beta^2)-\frac{\Delta f}{f}]\\&=F(f,\alpha,\beta)H,\end{aligned}$$

where

$$F(f,\alpha,\beta)=-2r-9\xi\beta+9(\alpha^2-\beta^2)-\frac{\Delta f}{f}.$$

Taking inner product by H of the equation (**??**), we have

$$<\Delta^{\perp}H,H>+<\mathrm{tr}(\sigma(.,A_H.)),H>=F(f,\alpha,\beta)|H|^2.$$

Now using the results $<\mathrm{tr}(\sigma(.,A_H.)),H>=|A_H|^2$, and $\Delta|H|^2=2(<\Delta^{\perp}H,H>-|\nabla^{\perp}H|^2)$, in the above equation we have

$$|A_H|^2+|\Delta^{\perp}H|^2=F(f,\alpha,\beta)|H|^2. \tag{3.12}$$

By using the Cauchy-Schwarz inequality $|A_H|^2\geq\frac{1}{2}\mathrm{tr}(A_H)=2|H|^4$, the equation reduces to

$$F(f,\alpha,\beta)|H|^2=|A_H|^2+|\nabla^{\perp}H|^2\geq 2|H|^4+|\nabla^{\perp}H|^2\geq 2|H|^4.$$

Therefore $F(f,\alpha,\beta)\geq 2|H|^2$, since $|H|$ is positive. This proves the theorem. □

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