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WEIGHTED FRACTIONAL INEQUALITIES USING MARICHEV-SAIGO-MAEDA FRACTIONAL INTEGRAL OPERATOR

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ABSTRACT. In this paper, we investigate several new weighted fractional integral inequalities by considering Marichev-Saigo-Maeda (MSM) fractional integral operator.

1. INTRODUCTION

The concept of fractional calculus is the generalization of traditional calculus into non-integer differential and integral order. Fractional calculus has found significant important due to its application in various fields of science and engineering such as life sciences, chemical science and physical sciences. Fractional integral inequalities plays a very important role in a different fields of mathematics, especially for continuous dependence of solutions, uniqueness of solution in fractional differential equation. During last two decades, many mathematicians have worked on the different type of fractional integral inequalities and applications by using the Riemann-Liouville, Erdelyi-Kober, Saigo, Hadamard, generalized fractional integral, k-fractional integral operator and generalized k-fractional integral operator, see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15]. Recently, Baleanu et al. [2], Chinchane V. L., et al. [4], established fractional integral inequalities using the generalized k-fractional integral operator in terms of the Gauss hypergeometric functions. In [16], Houas M. investigated

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certain weighted integral inequalities involving the fractional hypergeometric operators. Recently, Tassaddiq A. and et al. [17] obtained Minkowski-type fractional integral inequalities using Marichev-Saigo-Maeda fractional integral operator. In [18, 19], Joshi S. and et al., investigated the Grüss-type inequality and Chebyshev type inequalities by employing Marichev-Saigo-Maeda fractional integral operator. Marichev [20] introduced generalization of the hypergeometric fractional integral including Saigo operator (also see [15]). In [21], Saigo and Maeda have worked on the hypergeometric fractional integral in terms of any complex order with Appell function in the kernel. S. D. Purohit and et al.[13] introduced generalized operators of fractional integration involving Appell's function $F_3(.)$ due to Marichev-Saigo-Maeda. Motivated from [13, 15, 17, 18, 19, 20, 21, 22], our aim is to establish some new weighted fractional integral inequalities by using Marichev-Saigo-Maeda fractional integral operators. The paper has been organized as follows. In Section 2, we define basic definitions and a lemma related to Marichev-Saigo-Maeda fractional derivatives and integrals. In Section 3, we give weighted fractional integral inequalities by employing Marichev-Saigo-Maeda fractional integral operator. In Section 4, we give concluding remarks.

2. Preliminaries

Here, we present some basic notations, definitions and lemma of Marichev-Saigo-Maeda fractional integral operator which are useful later.

Definition 1. *A real valued function $g(\tau), \tau \geq 0$ is said be in $C_\mu([a,b]), \mu \in \mathbb{R}$ if there exist $\sigma \in \mathbb{R}$ such that $\sigma > \mu$ and $\Phi(\tau) \in C([a,b])$.*

Definition 2. [13, 17, 19, 21] *Let $v, v^{'}, \xi, \xi^{'}, \vartheta \in \mathbb{C}$, $x > 0$ and $\Re(\vartheta) > 0$, then Marichev-Saigo-Maeda (MSM) fractional integral is defined by*

$$(\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta} f)(x) = \frac{x^{-v}}{\Gamma(\vartheta)} \int_0^x (x-t)^{\vartheta-1} t^{-v^{'}} F_3(v, v^{'}, \xi, \xi^{'}; \vartheta; 1 - \frac{t}{x}, 1 - \frac{x}{t}) f(t) dt. \quad (2.1)$$

Where $F_3(.)$ is the Appel function defined by [23] as

$$F_3(v, v^{'}, \xi, \xi^{'}; \gamma; x; y) = \sum_{m,n=0}^{\infty} \frac{(v)_m (v^{'})_m (\xi)_m (\xi)^{'}_n}{(\gamma)_{m+n}} \frac{x^m y^n}{m! n!}, max(x, y) < 1,$$

and $(v)_m = v(v+1)...(v+m-1)$ is the pochhammer symbol.

Lemma 2.1. *Let $v, v^{'}, \xi, \xi^{'}, \vartheta, \rho \in \mathbb{C}, x > 0$ be such that $\Re(\vartheta) > 0$ and $\Re(\tau) > max\{0, \Re(v - v^{'} - \xi - \vartheta), \Re(v^{'} - \xi^{'})\}$. Then there exists the relation*

$$\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta} x^{\rho-1}(x) = \frac{\Gamma(\rho)\Gamma(\rho+\vartheta-v-v^{'}-\xi)\Gamma(\rho+\xi^{'}-v^{'})}{\Gamma(\rho+\xi^{'})\Gamma(\rho+\vartheta-v-v^{'})\Gamma(\rho+\vartheta-v^{'}-\xi)} x^{\rho-v-v^{'}+\vartheta-1}.$$

If we consider $\rho = 1$ in Lemma 2.1, then we get following relation

$$(\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[1])(x) = \frac{\Gamma(1+\vartheta-v-v^{'}-\xi)\Gamma(1+\xi^{'}-v^{'})}{\Gamma(1+\xi^{'})\Gamma(1+\vartheta-v-v^{'})\Gamma(1+\vartheta-v^{'}-\xi)} x^{-v-v^{'}+\vartheta}. \quad (2.2)$$

Consider a function

$$\begin{aligned}\mathfrak{J}(x,t) &= \frac{x^{-v}}{\Gamma(\vartheta)}(x-t)^{\vartheta-1}t^{-v'}F_3(v,v',\xi,\xi';\vartheta;1-\frac{t}{x},1-\frac{x}{t}) \\ &= \frac{x^{-v}}{\Gamma(\vartheta)}(x-t)^{\vartheta-1}t^{-v'}\left[(1+\frac{v'(\xi)}{\vartheta})\frac{1-x}{t}+\frac{v(\xi)}{\vartheta}\frac{1-t}{x}+...\right].\end{aligned} \tag{2.3}$$

Clearly, the function $\mathfrak{J}(x,t)$ remains positive because all terms of Eq. (2.3) are positive.

3. Weighted Fractional Integral Inequalities

In this section, we establish some weighted fractional integral inequalities using Marichev-Saigo-Maeda fractional integral operator.

Theorem 3.1. *Let* f *be a positive and continuous functions on* $[0,\infty)$*, such that*

$$(\sigma^{\varrho}\mathrm{f}^{\varrho}(\tau)-\tau^{\varrho}\mathrm{f}^{\varrho}(\sigma))(\mathrm{f}^{\varpi-\lambda}(\tau)-\mathrm{f}^{\varpi-\lambda}(\sigma))\geq 0, \tag{3.1}$$

and $w:[0,\infty)\to\mathbb{R}^+$ *be a positive continuous function. Then for all* $x,\varrho>0$*,* $\varpi\geq\lambda>0$ $v,v',\xi,\xi',\vartheta\in\mathbb{C}$*,* $\Re(\vartheta)>0$*, we have*

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)] \\ &\leq \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)]\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)],\end{aligned}$$

where $v'>-1$*,* $1>max\left\{0,\Re(v+v'+\xi-\vartheta),\Re(v'-\xi')\right\}$
$(\vartheta-v')>max\,(1-\xi,1-v)$*.*

Proof:- Since f is a positive and continuous function on $[0,\infty)$, then for all $\varrho>0,\varpi\geq 0,\lambda>0$, $\tau,\sigma\in(0,x),x>0$. From (3.1), we have

$$\begin{aligned}&\sigma^{\varrho}\mathrm{f}^{\varpi-\lambda}(\sigma)\mathrm{f}^{\varrho}(\tau)+\tau^{\varrho}\mathrm{f}^{\varpi-\lambda}(\tau)\mathrm{f}^{\varrho}(\sigma) \\ &\leq \sigma^{\varrho}\mathrm{f}^{\varpi+\varrho-\lambda}(\tau)+\tau^{\varrho}\mathrm{f}^{\varpi+\varrho-\lambda}(\sigma).\end{aligned} \tag{3.2}$$

Again, multiplying both sides of (3.2) by $\mathfrak{J}(x,\tau)w(\tau)\mathrm{f}^{\lambda}(\tau)$, $\tau\in(0,x)$, $x>0$, then integrating resulting identity with respect to τ from 0 to x we get

$$\begin{aligned}&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})\sigma^{\varrho}\mathrm{f}^{\varpi-\lambda}(\sigma)w(\tau)\mathrm{f}^{\varrho+\lambda}(\tau)d\tau \\ &+\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})\tau^{\varrho}\mathrm{f}^{\varpi}(\tau)\mathrm{f}^{\varrho}(\sigma)w(\tau)d\tau \\ &\leq\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})\sigma^{\varrho}\mathrm{f}^{\varpi+\varrho}(\tau)w(\tau)d\tau \\ &+\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})\tau^{\varrho}\mathrm{f}^{\varpi+\varrho-\lambda}(\sigma)w(\tau)\mathrm{f}^{\lambda}(\tau)d\tau,\end{aligned}$$

consequently,

$$\begin{aligned}&\sigma^{\varrho}\mathrm{f}^{\varpi-\lambda}(\sigma)\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]+\mathrm{f}^{\varrho}(\sigma)\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)]\\&\leq\sigma^{\varrho}\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)]+\mathrm{f}^{\varrho+\varpi-\lambda}(\sigma)\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)].\end{aligned}\tag{3.3}$$

Multiplying both sides of Eq. (3.3) by $\mathfrak{J}(x,\sigma)w(\sigma)\mathrm{f}^{\lambda}(\sigma),\sigma\in(0,x),x>0$ which is positive, and integrating the obtained result with respective to σ from 0 to x, we get

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)]\\&+\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\\&\leq\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)]\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)]\\&+\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)]\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)],\end{aligned}$$

which completes the proof.
Now, we give our main result.

Theorem 3.2. *Let* f *be a positive and continuous functions on* $[0,\infty)$ *and satisfying (3.1). Let* $w:[0,\infty)\to\mathbb{R}^{+}$ *be a positive continuous function. Then for all* $x>0$*,* $v,v^{'},\xi,\xi^{'},\vartheta,\alpha,\alpha^{'},\beta,\beta^{'},\theta\in\mathbb{C},\Re(\vartheta),\Re(\theta)>0$*, we have*

$$\begin{aligned}&\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)]\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\\&+\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)]\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\\&\leq\ \mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)]\\&+\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)]\,\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)],\end{aligned}$$

where $v^{'},\alpha^{'}>-1,1>\max\left\{0,\Re(v+v^{'}+\xi-\vartheta),\Re(v^{'}-\xi^{'})\right\},(\vartheta-v^{'})>\max(1-\xi,1-v)$*,* $1>\max\left\{0,\Re(\alpha+\alpha^{'}+\beta-\theta),\Re(\alpha^{'}-\beta^{'})\right\}(\theta-\alpha^{'})>\max(1-\beta,1-\alpha)$.

Proof:- Multiplying both sides of (3.2) by $\mathfrak{J}(x,\sigma)w(\sigma)\mathrm{f}^{\lambda}(\sigma)=\frac{x^{-\alpha}}{\Gamma(\theta)}(x-\sigma)^{\theta-1}\sigma^{-\alpha^{'}}F_3(\alpha,\alpha^{'},\beta,\beta^{'};\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}^{\lambda}(\sigma)$ $(\sigma\in(0,x),\ x>0)$, this function remains positive under the conditions stated with the theorem.

Integrating the obtained result with respective to σ from 0 to x, we get

$$\begin{aligned}
&\frac{\mathrm{f}^{\varrho}(\tau)\, x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\sigma^{\varrho}\mathrm{f}^{\varpi}(\sigma)d\sigma\\
&+\frac{\tau^{\varrho}\mathrm{f}^{\varpi-\lambda}(\tau)\, x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}^{\varrho+\lambda}(\sigma)d\sigma\\
&\le \frac{\mathrm{f}^{\varpi+\varrho-\lambda}(\tau)\, x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\sigma^{\varrho}\mathrm{f}^{\lambda}(\sigma)d\sigma\\
&+\frac{\tau^{\varrho}\, x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}^{\varpi+\varrho}(\sigma)d\sigma,
\end{aligned}$$

consequently

$$\begin{aligned}
&\mathrm{f}^{\varrho}(\tau)\ \mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)]+\tau^{\varrho}\mathrm{f}^{\varpi-\lambda}(\tau)\ \mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\\
&\le \mathrm{f}^{\varrho+\varpi-\lambda}(\tau)\ \ \mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)]+\tau^{\varrho}\ \mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)].
\end{aligned}\tag{3.4}$$

Multiplying both sides of (3.4) by $\mathfrak{J}(x,\tau)w(\tau)\mathrm{f}^{\lambda}(\tau), \tau\in(0,x), x>0$ which is positive, and integrating the obtained result with respective to σ from 0 to x, we get

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)x^{\varrho}\mathrm{f}^{\varpi}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})w(\tau)\mathrm{f}^{\varrho+\lambda}(\tau)d\tau\\
&+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})w(\tau)\tau^{\varrho}\mathrm{f}^{\varpi}(\tau)d\tau\\
&\le \mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)x^{\varrho}\mathrm{f}^{\lambda}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})w(\tau)\mathrm{f}^{\varpi+\varrho}(\tau)d\tau\\
&+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[wx\mathrm{f}^{\varrho+\varpi}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x (x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})w(\tau)\tau^{\varrho}\mathrm{f}^{\lambda}(\tau)d\tau
\end{aligned}$$

This completes the proof of Theorem 3.2.

Theorem 3.3. *Let* f *and* h *be positive and continuous functions on* $[0,\infty)$*, such that*

$$(\mathrm{h}^{\varrho}(\sigma)\mathrm{f}^{\varrho}(\tau)-\mathrm{h}^{\varrho}(\tau)\mathrm{f}^{\varrho}(\sigma))(\mathrm{f}^{\varpi-\lambda}(\tau)-\mathrm{f}^{\varpi-\lambda}(\sigma))\ge 0,\tag{3.5}$$

and let $w:[0,\infty)\to\mathbb{R}^{+}$ *be a positive continuous function. Then for all* $x,\varrho>0,\varpi\geq\lambda>0$ $v,v^{'},\xi,\xi^{'},\vartheta,\in\mathbb{C},\Re(\vartheta)>0$, *we have*

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\\&\leq\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\lambda}(x)],\end{aligned}$$

where $v^{'}>-1$, $1>max\left\{0,\Re(v,+v^{'}+\xi-\vartheta),\Re(v^{'}-\xi^{'})\right\}$
$(\vartheta-v^{'})>max\,(1-\xi,1-v)$.

Proof:- Let $(\tau,\sigma)\in(0,\sigma)$, $x>0$, for any $\varpi>\lambda>0$, $\varrho>0$. From (3.5), we have

$$\mathrm{h}^{\varrho}(\sigma)\mathrm{f}^{\varpi-\lambda}(\sigma)\mathrm{f}^{\varrho}(\tau)+\mathrm{h}^{\varrho}(\tau)\mathrm{f}^{\varrho}(\sigma)\mathrm{f}^{\varpi-\lambda}(\tau)\leq\mathrm{h}^{\varrho}(\sigma)\mathrm{f}^{\varpi+\varrho-\lambda}(\tau)+\mathrm{h}^{\varrho}(\tau)\mathrm{f}^{\varpi+\varrho-\lambda}(\sigma).\quad(3.6)$$

Multiplying both sides of (3.6) by $\mathfrak{J}(x,\tau)w(\tau)\mathrm{f}^{\lambda}(\tau)$, $\tau\in(0,x)$, $x>0$, then integrating resulting identity with respect to τ from 0 to x, we obtain

$$\begin{aligned}&\frac{\mathrm{h}^{\varrho}(\sigma)\mathrm{f}^{\varpi-\lambda}(\sigma)\,x^{-v}}{\Gamma(\eta)}\int_{0}^{x}(x-\tau)^{\eta-1}\tau^{-v^{'}}F_{3}(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{f}^{\varrho+\lambda}(\tau)]d\tau\\&+\frac{\mathrm{f}^{\varrho}(\sigma)x^{-v}}{\Gamma(\eta)}\int_{0}^{x}(x-\tau)^{\eta-1}\tau^{-v^{'}}F_{3}(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{h}^{\varrho}(\tau)\mathrm{f}^{\varpi}(\tau)]d\tau\\&\leq\ \frac{\mathrm{h}^{\varrho}(\sigma)x^{-v}}{\Gamma(\eta)}\int_{0}^{x}(x-\tau)^{\eta-1}\tau^{-v^{'}}F_{3}(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{f}^{\varpi+\varrho}(\tau)]d\tau\\&+\frac{\mathrm{f}^{\varrho+\varpi-\lambda}(\sigma)x^{-v}}{\Gamma(\eta)}\int_{0}^{x}(x-\tau)^{\eta-1}\tau^{-v^{'}}F_{3}(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{h}^{\varrho}(\tau)\mathrm{f}^{\lambda}(\tau)]d\tau.\end{aligned}$$

Thus, we obtain

$$\begin{aligned}&\mathrm{h}^{\varrho}(\sigma)\mathrm{f}^{\varpi-\lambda}(\sigma)\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]+\mathrm{f}^{\varrho}(\sigma)\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\\&\leq\mathrm{h}^{\varrho}(\sigma)\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varpi+\varrho}(x)]+\mathrm{f}^{\varrho+\varpi-\lambda}(\sigma)\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\lambda}(x)]\end{aligned}\quad(3.7)$$

Multiplying both sides of (3.7) by $\mathfrak{J}(x,\sigma)w(\sigma)\mathrm{f}^{\lambda}(\sigma)$, then integrating the resulting inequality with respect to σ over $(0,x)$, we obtain

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{h}^{\varrho}(\sigma)\mathrm{f}^{\varpi}(\sigma)d\sigma\\
&+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})\mathrm{f}^{\varrho+\lambda}(\sigma)w(\sigma)d\sigma\\
&\leq\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varpi+\varrho}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})\mathrm{f}^{\lambda}(\sigma)w(\sigma)\mathrm{h}^{\varrho}(\sigma)d\sigma\\
&+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\lambda}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}^{\varpi+\varrho}(\sigma)d\sigma,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)\\
&+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[\mathrm{f}^{\varrho+\lambda}(x)w(x)]\\
&\leq\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varpi+\varrho}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[\mathrm{f}^{\lambda}(x)w(x)\mathrm{h}^{\varrho}(x)]\\
&+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\lambda}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varpi+\varrho}(x)],
\end{aligned}$$

which completes the proof.

Theorem 3.4. *Let* f *and* h *be two positive and continuous functions on* $[0,\infty)$ *and satisfying* (3.5). *Let* $w:[0,\infty)\to\mathbb{R}^+$ *be a positive continuous function. Then for all* $x>0$, $v,v',\xi,\xi',\vartheta,\alpha,\alpha',\beta,\beta',\theta\in\mathbb{C}$, $\mathfrak{R}(\vartheta),\mathfrak{R}(\theta)>0$, *we have*

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\\
&+\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\\
&\leq\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{h}^{\varrho}\mathrm{f}^{\lambda}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}^{\varpi+\varrho}(x)]\\
&+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}^{\varrho+\varpi-\lambda}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\lambda}(x)],
\end{aligned}$$

where $v^{'}, \alpha^{'} > -1$, $1 > \max\left\{0, \Re(v+v^{'}+\xi-\vartheta), \Re(v^{'}-\xi^{'})\right\}$,$(\vartheta - v^{'}) > \max(1-\xi, 1-v)$, $1 > \max\left\{0, \Re(\alpha+\alpha^{'}+\beta-\theta), \Re(\alpha^{'}-\beta^{'})\right\}$ $(\theta-\alpha^{'}) > \max(1-\beta, 1-\alpha)$.

Proof:- Multiplying the inequality (3.7) by $\mathfrak{J}(x,\sigma)w(\sigma)\mathrm{f}^{\lambda}(\sigma)$, $\sigma \in (0,x), x > 0$, this function remains positive under the conditions stated with the theorem. Integrating the obtained result with respective to σ from 0 to x, we get

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\times\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha^{'}}F_3(\alpha,\alpha^{'},\beta,\beta^{'};\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{h}^{\varrho}(\sigma)\mathrm{f}^{\varpi}(\sigma)d\sigma\\
&+\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\times\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha^{'}}F_3(\alpha,\alpha^{'},\beta,\beta^{'};\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})\mathrm{f}^{\varrho+\lambda}(\sigma)w(\sigma)d\sigma\\
&\le \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varpi+\varrho}(x)]\times\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha^{'}}F_3(\alpha,\alpha^{'},\beta,\beta^{'};\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})\mathrm{f}^{\lambda}(\sigma)w(\sigma)\mathrm{h}^{\varrho}(\sigma)d\sigma\\
&+\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\lambda}(x)]\times\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha^{'}}F_3(\alpha,\alpha^{'},\beta,\beta^{'};\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}^{\varpi+\varrho}(\sigma)d\sigma,
\end{aligned}$$

which implies that,

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\\
&+\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}^{\varrho+\lambda}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\varpi}(x)]\\
&\le \mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{h}^{\varrho}\mathrm{f}^{\lambda}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}^{\varpi+\varrho}(x)]\\
&+\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}^{\varrho+\varpi}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}^{\lambda}(x)],
\end{aligned}$$

Thus, proof is completed.
Next, we shall propose a new generalization of weighted fractional integral inequalities using a family of n positive functions defined on $[0,\infty)$.

Theorem 3.5. *Let* $\mathrm{f_i}, \mathrm{i} = 1,...,\mathrm{n}$ *be* n *positive and continuous functions on* $[0,\infty)$ *such that*

$$(\sigma^{\varrho}\mathrm{f}_r^{\varrho}(\tau) - \tau^{\varrho}\mathrm{f}_r^{\varrho}(\sigma))(\mathrm{f}_r^{\varpi-\lambda_r}(\tau) - \mathrm{f}^{\varpi-\lambda_r}(\sigma)) \ge 0. \tag{3.8}$$

Let $w : [0,\infty) \to \mathbb{R}^{+}$ *be a positive and continuous functions. Then for all* $v, v^{'}, \xi, \xi^{'}, \vartheta \in \mathbb{C}$, $\Re(\vartheta) > 0$, $\varpi \ge \lambda_r > 0, r \in \{1,...,n\}$ *and* $v^{'} > -1$, $1 > max\left\{0, \Re(v, +v^{'}+\xi-\vartheta), \Re(v^{'}-\xi^{'})\right\}$

$x, \varrho > 0\ (\vartheta - v^{'}) > max\,(1-\xi, 1-v)$, *the following fractional inequality*

$$\begin{aligned}&\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &\leq\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)],\end{aligned} \tag{3.9}$$

is valid.

proof:- Suppose $\mathrm{f_i}, \mathrm{i} = 1, ..., \mathrm{n}$ are n positive and continuous functions on $[0, \infty)$, then for any fixed $r \in \{1, ..., n\}$ and for any $\varrho > 0, \varpi \geq \lambda_r > 0, \tau, \sigma \in (0, x), x > 0$. From (3.8), we have

$$\begin{aligned}&\sigma^{\varrho}\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\mathrm{f}_r^{\varrho}(\tau)+\tau^{\varrho}\mathrm{f}_r^{\varrho}(\sigma)\mathrm{f}_r^{\varpi-\lambda_r}(\tau)\\ &\leq\sigma^{\varrho}\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\tau)+\tau^{\varrho}\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\sigma),\end{aligned} \tag{3.10}$$

multiplying both sides of (3.10) by $\mathfrak{J}(x,\tau)w(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(0, x)$, we obtain

$$\begin{aligned}&\sigma^{\varrho}\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v^{'}}F_3(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{f}_r^{\varrho}(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau\\ &+\mathrm{f}^{\varrho}(\sigma)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v^{'}}F_3(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\tau^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau\\ &\leq\sigma^{\varrho}\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v^{'}}F_3(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{f}_r^{\varpi+\varrho}(\tau)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau\\ &+\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\sigma)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v^{'}}F_3(v,v^{'},\xi,\xi^{'};\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\tau^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau,\end{aligned}$$

consequently

$$\begin{aligned}&\sigma^{\varrho}\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &+\mathrm{f}^{\varrho}(\sigma)\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &\leq\sigma^{\varrho}\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\ &+\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\sigma)\,\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)].\end{aligned} \tag{3.11}$$

Again, multiplying the inequality (3.11) by $\mathfrak{J}(x,\sigma)w(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma), \sigma \in (0, x), x > 0$, this function remains positive under the conditions stated with the theorem. Integrating the obtained result with respective to σ from 0 to x, we have

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\sigma^{\varrho}\mathrm{f}_r^{\varpi}(\sigma)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&+\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}_r^{\varrho}(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&\leq\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\times\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\sigma^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&+\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\sigma)^{\eta-1}\sigma^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}_r^{\varpi+\varrho}(\sigma)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma,
\end{aligned}$$

therefore,

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[x^{\varrho}\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&+\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&+\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)].
\end{aligned}$$

This completes the inequality (3.9).

Theorem 3.6. *Let* $\mathrm{f_i}, \mathrm{i}=1,...,\mathrm{n}$ *be* n *positive and continuous functions on* $[0,\infty)$*and satisfying* (3.8). *Let* $w:[0,\infty)\to\mathbb{R}^+$ *be a positive and continuous functions. Then for all* $x>0$, $v,v',\xi,\xi',\vartheta,\alpha,\alpha',\beta,\beta',\theta\in\mathbb{C}$, $\Re(\vartheta),\Re(\theta)>0$, $\varrho>0$, $\varpi\geq\lambda_r>0, r\in\{1,...,n\}$, *then the inequality*

$$\begin{aligned}
&\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)x^{\varrho}\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&+\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq\ \mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&+\ \mathfrak{I}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{I}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)],
\end{aligned}\tag{3.12}$$

is valid, where $v',\alpha'>-1$, $1>\max\left\{0,\Re(v+v'+\xi-\vartheta),\Re(v'-\xi')\right\}$, $(\vartheta-v')>\max(1-\xi,1-v)$, $1>\max\left\{0,\Re(\alpha+\alpha'+\beta-\theta),\Re(\alpha'-\beta')\right\}(\theta-\alpha')>\max(1-\beta,1-\alpha)$.

Proof:- Multiplying the inequality (3.11) by $\mathfrak{J}(x,\sigma)w(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma), \sigma \in (0,x), x > 0$, this function remains positive under the conditions stated with the theorem. Integrating the obtained result with respective to σ from 0 to x, we have

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\sigma^{\varrho}\mathrm{f}_r^{\varpi}(\sigma)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&+\ \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}_r^{\varrho}(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&\leq\ \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\sigma^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma\\
&+\ \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)x^{\varrho}\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\frac{x^{-\alpha}}{\Gamma(\theta)}\int_0^x (x-\sigma)^{\theta-1}\sigma^{-\alpha'}F_3(\alpha,\alpha',\beta,\beta';\theta;1-\frac{\sigma}{x},1-\frac{x}{\sigma})w(\sigma)\mathrm{f}_r^{\varpi+\varrho}(\sigma)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)d\sigma,
\end{aligned}$$

which gives the inequality (3.12).

Theorem 3.7. *Let* $\mathrm{f_i}, \mathrm{i}=1,...,\mathrm{n}$ *be* h *positive and continuous functions on* $[0,\infty)$. *such that*

$$(\mathrm{h}^{\varrho}(\sigma)\mathrm{f}_r^{\varrho}(\tau)-\mathrm{h}^{\varrho}(\tau)\mathrm{f}_r^{\varrho}(\sigma))(\mathrm{f}_r^{\varpi-\lambda_r}(\tau)-\mathrm{f}_r^{\varpi-\lambda_r}(\sigma))\geq 0. \tag{3.13}$$

and let $w:[0,\infty)\to\mathbb{R}^+$ *be a positive and continuous function. Then for all* $x,\varrho>0$, $v,v',\xi,\xi',\vartheta,\in\mathbb{C}$, $\Re(\vartheta)>0$, $\varrho>0$, $\varpi\geq\lambda_r>0, r\in\{1,...,n\}$, *the following fractional inequality*

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq\ \mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]
\end{aligned}$$

is valid, where $v'>-1, 1>\max\left\{0,\Re(v+v'+\xi-\vartheta),\Re(v'-\xi')\right\}, (\vartheta-v')>\max(1-\xi,1-v)$.

Proof:- Let $\tau,\sigma\in(0,x), x>0$, for any $\varrho>0, \varpi\geq\lambda_i>0, r\in\{1,2,...n\}$. From (3.13), we have

$$\begin{aligned}
&\mathrm{h}^{\varrho}(\sigma)\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\mathrm{f}_r^{\varrho}(\tau)+\mathrm{f}_r^{\varrho}(\sigma)\mathrm{h}_r^{\varrho}(\tau)\mathrm{f}_r^{\varpi-\lambda_r}(\tau)\\
&\leq \mathrm{h}^{\varrho}(\sigma)\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\tau)+\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\sigma)\mathrm{h}^{\varrho}(\tau),
\end{aligned} \tag{3.14}$$

multiplying both sides of (3.14) by $\mathfrak{J}(x,\tau)w(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(0,x)$, we obtain

$$\begin{aligned}
&\mathrm{h}^{\varrho}(\sigma)\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{f}_r^{\varrho}(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau\\
&+\mathrm{f}_r^{\varrho}(\sigma)\,\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{h}^{\varrho}(\tau)\mathrm{f}_r^{\varpi}(\tau)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau\\
&\leq \mathrm{h}^{\varrho}(\sigma)\,\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{h}^{\varrho}(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau\\
&+\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\sigma)\frac{x^{-v}}{\Gamma(\eta)}\int_0^x(x-\tau)^{\eta-1}\tau^{-v'}F_3(v,v',\xi,\xi';\eta;1-\frac{\tau}{x},1-\frac{x}{\tau})[w(\tau)\mathrm{f}_r^{\varpi+\varrho}(\tau)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(\tau)]d\tau,
\end{aligned}$$

so, we can write

$$\begin{aligned}
&\mathrm{h}^{\varrho}(\sigma)\mathrm{f}_r^{\varpi-\lambda_r}(\sigma)\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]+\\
&\mathrm{f}_r^{\varrho}(\sigma)\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq \mathrm{h}^{\varrho}(\sigma)\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\sigma)\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)].
\end{aligned}\tag{3.15}$$

Multiplying both sides of (3.15) by $\mathfrak{J}(x,\sigma)w(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)$, then integrating the resulting inequality with respect to σ over $(0,x)$, we have

$$\begin{aligned}
&2\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\;\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq 2\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\,\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)].
\end{aligned}$$

This completes the proof of Theorem 3.7.

Theorem 3.8. *Let* $\mathrm{f_i}, \mathrm{i}=1,...,\mathrm{n}$ *be* h *positive and continuous functions on* $[0,\infty)$. *such that*

$$(\mathrm{h}^{\varrho}(\sigma)\mathrm{f}_r^{\varrho}(\tau)-\mathrm{h}^{\varrho}(\tau)\mathrm{f}_r^{\varrho}(\sigma))(\mathrm{f}_r^{\varpi-\lambda_r}(\tau)-\mathrm{f}_r^{\varpi-\lambda_r}(\sigma))\geq 0,$$

and let $w:[0,\infty)\to\mathbb{R}^+$ *be a positive and continuous function. Then for all* $x>0$, $v,v',\xi,\xi',\vartheta,\alpha,\alpha',\beta,\beta',\theta\in\mathbb{C}$, $\Re(\vartheta),\Re(\theta)>0$, $\varrho>0$, $\varpi\geq\lambda_r>0, r\in\{1,...,n\}$, *then the inequality*

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\;\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]+\\
&\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\;\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&\leq\;\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)]\\
&+\mathfrak{J}_{0,x}^{\alpha,\alpha',\beta,\beta',\theta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathfrak{J}_{0,x}^{v,v',\xi,\xi',\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)],
\end{aligned}$$

is valid, where $v^{'}, \alpha^{'} > -1$, $1 > \max\left\{0, \Re(v + v^{'} + \xi - \vartheta), \Re(v^{'} - \xi^{'})\right\}$, $(\vartheta - v^{'}) > \max(1 - \xi, 1 - v)$, $1 > \max\left\{0, \Re(\alpha + \alpha^{'} + \beta - \theta), \Re(\alpha^{'} - \beta^{'})\right\}(\theta - \alpha^{'}) > \max(1 - \beta, 1 - \alpha)$.

Proof:- Multiplying both sides of (3.14) by $\mathfrak{J}(x,\sigma)w(\sigma)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\sigma)$, then integrating the resulting inequality with respect to σ over $(0,x)$, we have

$$\begin{aligned}
&\mathrm{f}_r^{\varrho}(\tau)\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)] + \\
&\mathrm{h}^{\varrho}(\tau)\mathrm{f}_r^{\varpi-\lambda_r}(\tau)\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)] \\
&\leq\ \mathrm{f}_r^{\varpi+\varrho-\lambda_r}(\tau)\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)] + \\
&\mathrm{h}^{\varrho}(\tau)\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)].
\end{aligned} \tag{3.16}$$

Multiplying both sides of (3.16) by $\mathfrak{J}(x,\tau)w(\tau)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(\tau)$, then integrating the resulting inequality with respect to τ over $(0,x)$, we have

$$\begin{aligned}
&\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)] + \\
&\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\mathrm{f}_r^{\varpi}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\ \mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}_r^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)] \\
&\leq\ \mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)] + \\
&\mathfrak{J}_{0,x}^{\alpha,\alpha^{'},\beta,\beta^{'},\theta}[w(x)\mathrm{f}_r^{\varpi+\varrho}(x)\Pi_{i\neq r}^{n}\mathrm{f}_i^{\lambda_i}(x)]\mathfrak{J}_{0,x}^{v,v^{'},\xi,\xi^{'},\vartheta}[w(x)\mathrm{h}^{\varrho}(x)\Pi_{i=1}^{n}\mathrm{f}_i^{\lambda_i}(x)].
\end{aligned}$$

Thus, proof is completed.

4. Concluding Remarks

Here, we studied of Marichev-Saigo-Maeda fractional integral operators and then we obtained some weighted fractional integral inequalities. In this paper, we briefly consider some implication of our main results. If we set $v^{'} = 0$ in the Eq. (2.1) would reduced immediately to Saigo type of fractional integral operators as in following relationship, see [13, 14, 18, 24],

$$\left(\mathfrak{J}_{0,x}^{v,0,\xi,\xi^{'},\vartheta}f\right)(x) = \left(\mathfrak{J}_{0,x}^{\vartheta,v-\vartheta,-\xi}f\right)(x),$$

where the hypergeometric operator that appear in the right hand side is defined as

$$\mathfrak{J}_{0,x}^{v,v^{'},\vartheta}f(x) = \frac{x^{-v-\xi}}{\Gamma(\vartheta)}\int_0^x (x-t)^{v-1}\,{}_2F_1(v+\xi; v; 1-\frac{t}{x})f(t)dt, (\vartheta > 0, v, \xi \in \mathbb{C}) \tag{4.1}$$

Further, operator (2.1) can be reduced to Erdelyi-Kober and Riemann-Liouville type of fractional integral operators which are special cases of Saigo fractional operator (4.1). The weighted fractional inequalities established in this paper give some contribution in the fields of fractional calculus and Marichev-Saigo-Maeda fractional integral operators. Moreover, they are expected to lead to some application for finding uniqueness of solutions in fractional differential equations.

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