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ASYMPTOTICS FOR AN EXTENDED INVERSE MARKOVIAN HAWKES PROCESS

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Abstract. Hawkes process is a self-exciting simple point process with clustering effect whose jump rate depends on its entire past history and has been widely applied in insurance, finance, queueing theory, statistics, and many other fields. Seol [27] proposed the inverse Markovian Hawkes processes and studied some asymptotic behaviors. In this paper, we consider an extended inverse Markovian Hawkes process which combines a Markovian Hawkes process and inverse Markovian Hawkes process with features of several existing models of self-exciting processes. We study the limit theorems for an extended inverse Markovian Hawkes process. In particular, we obtain a law of large number and central limit theorems.

1. Introduction and main results

We start with a general description of Hawkes process introduced by Brémaud and Massoulié [3]. Let N be a simple point process on $\mathbb{R}$ and let $\mathcal{F}_t^{-\infty} := \sigma(N(C), C \in \mathcal{B}(\mathbb{R}), C \subset (-\infty, t])$ be an increasing family of σ-algebras. Any nonnegative $\mathcal{F}_t^{-\infty}$-progressively measurable process λ_t with

$$\mathbb{E}\left[N(a,b]|\mathcal{F}_a^{-\infty}\right] = \mathbb{E}\left[\int_a^b \lambda_s ds|\mathcal{F}_a^{-\infty}\right]$$

a. s. for all interval $(a,b]$ is called an $\mathcal{F}_t^{-\infty}$-intensity of N. We use the notation $N_t := N(0,t]$ to denote the number of points in the interval $(0,t]$.

A general Hawkes process is a simple point process N admitting an $\mathcal{F}_t^{-\infty}$-intensity

$$\lambda_t := \lambda\left(\int_{-\infty}^t h(t-s)N(ds)\right),$$

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where $\lambda(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ is locally integrable and left continuous, $h(\cdot):\mathbb{R}^+\to\mathbb{R}^+$ and we always assume that $\|h\|_{L^1}=\int_0^\infty h(t)dt<\infty$. Here $\int_{-\infty}^t h(t-s)N(ds)$ stands for $\int_{(-\infty,t)} h(t-s)N(ds)$. We always assume that $N(-\infty,0]=0$, i.e., the Hawkes process has empty history. In the literatures, $h(\cdot)$ and $\lambda(\cdot)$ are usually referred to as exciting function and rate function respectively. The Hawkes process is linear if $\lambda(\cdot)$ is linear and it is nonlinear otherwise. In general, the model described above is non-Markovian since the future evolution of a self-exciting simple point process is controlled by the timing of past events but it is Markovian for a special case. Hawkes process has wide applications in neuroscience, seismology, DNA modeling, finance and many other fields. It has both self-exciting and clustering properties, which is very appealing to some financial applications. In particular, self-exciting and clustering properties of Hawkes process make it a viable candidate in modeling the correlated defaults and evaluating the credit derivatives in finance, for example, see Errais et al. [6] and Dassios and Zhao [5].

Hawkes [14] introduced the linear case, and the linear Hawkes process can be studied via immigration-birth representation, see e.g. Hawkes and Oakes [15]. The stability, law of large numbers, central limit theorem, large deviations, Bartlett spectrum etc. have all been studied and understood very well [1,2,14]. Almost all of the applications of Hawkes process in the literatures consider exclusively the linear case. Because of the lack of immigration-birth representation and computational tractability, nonlinear Hawkes process is much less studied. However, some efforts have already been made in this direction. Nonlinear case was first introduced by Brémaud and Massoulié [3]. Recently, Zhu [29–33] investigated several results for both linear and nonlinear model. The central limit theorem was investigated in Zhu [31] and the large deviation principles have been obtained in Zhu [30] and Zhu [30]. Limit theorems and rough fractional diffusions as scaling limits for nearly unstable Hawkes processes was obtained in Jaisson and Rosenbaum [18, 19]. Zhu [33] have also studied for applications to financial mathematics. In the recent paper of Seol [24], he considers the arrival time τ_n, i.e., the inverse process of Hawkes process, and studies the limit theorems (law of large numbers, central limit theorem and large deviations) for τ_n. Seol [23] studied the law of large numbers, central limit theorem and invariance principles for discrete Hawkes processes starting from empty history. Moderate deviation principle for marked Hawkes processes was investigated in Seol [25] and limit theorems for the compensator of Hawkes processes was studied by Seol [26]. There have been some progress made in the direction of asymptotic results other than the large time limits in Gao and Zhu [10], Gao and Zhu [11–13]. In the literature, there have been studies extending and modifying the classical Hawkes process. First, the baseline intensity can be chosen to be time-inhomogeneous (see Gao, Zhou, and Zhu [9]). Second, the immigrants can arrive according to a Cox process with shot noise intensity, in which case the model is known as the dynamic contagion model (see Dassios and Zhao [5]). Third, the immigrants can arrive according to a renewal process instead of a Poisson process, which generalizes the classical Hawkes process. This is known as the renewal Hawkes process (see Wheatley, Filimonov, and Sorrette [28]). Recently, Seol [27] introduced the inverse Markovian Hawkes processes which combines features of several existing models of self-exciting processes and studied the limit theorems for the inverse Markovian Hawkes processes. Some other variations and extensions of Hawkes process have been studied in the literature, see e.g. Dassios and Zhao [5], Zhu [34], Karabash and Zhu [20], Mehrdad and Zhu [22] and Ferro, Leiva and Møller [7].

In this paper, we consider an extended inverse Markovian Hawkes process which combines a Markovian Hawkes process and inverse Markovian Hawkes process with features of several existing models of self-exciting processes and has been widely applied in insurance, finance, queueing theory, statistic, and many other fields, and study the limit theorems for the extended inverse Markovian Hawkes process. There are some difference remarks on two processes between general Hawkes processes and inverse Markovian Hawkes process which is introduced in. In the Hawkes process, the more jumps you have in the past, the more jumps will be expected in the future. However, in the inverse Hawkes process, the more jumps you have in the past, the larger jumps will be expected in the future. It is worth mentioning that, for the Hawkes process, the self-excitation lies on the intensity for the Hawkes process, while for the inverse Hawkes process, the self-excitation is about the jump size. That is, in the Hawkes process, self-excitation represents the frequency, while, in the inverse Markovian Hawkes process, self-excitation represents the severity.

The structure of this paper is organized mainly as two parts. Some introductions and the main results are stated in Section 1. The proofs for the main theorems with some auxiliary results are contained in Section 2.

Here are some reviews for the results of Hawkes processes.

1.1. Limit theorems for Hawkes processes

The limit theorems for both linear and nonlinear Hawkes processes are well known and studied by many authors.

Linear model: When $\lambda(\cdot)$ is linear, say $\lambda(z) = \nu + z$ for some $\nu > 0$, and $\|h\|_{L^1} < 1$, we can use a very nice immigration-birth representation and the limit theorems are well understood and more explicitly represented. Limit theorems for linear marked Hawkes processes are obtained in Zhu [22]. There is the law of large numbers, see e.g. Daley and Vere-Jones [4]:

$$\frac{N_t}{t} \to \frac{\nu}{1 - \|h\|_{L^1}}, \text{ a.s. as } t \to \infty. \tag{1}$$

The functional central limit theorem for linear multivariate Hawkes process under certain assumptions have been obtained by Bacry et al. [1] and they proved under the additional condition $\int_0^\infty t^{\frac{1}{2}} h(t)dt < \infty$ that

$$\frac{N_{\cdot t} - \cdot \mu t}{\sqrt{t}} \to \sigma B(\cdot), \text{ as } t \to \infty, \tag{2}$$

where $B(\cdot)$ is a standard Brownian motion and

$$\mu = \frac{\nu}{1 - \|h\|_{L^1}} \text{ and } \sigma^2 = \frac{\nu}{(1 - \|h\|_{L^1})^3}.$$

The convergence used in (2) is weak convergence on $D[0,1]$, the space of càdlàg function on $[0,1]$, equipped with Skorokhod topology. Bordenave and Torrisi [2] proved that if $0 < \|h\|_{L^1} < 1$ and $\int_0^\infty th(t)dt < \infty$, then $\mathbb{P}(\frac{N_t}{t} \in \cdot)$ satisfies a large deviation principle with the good rate function $I(\cdot)$, which means that for any Borel set A,

$$\begin{aligned} -\inf_{x \in A^\circ} I(x) &\leq \liminf_{t \to \infty} \frac{1}{t} \log \mathbb{P}\Big(N_t/t \in A\Big) \\ &\leq \limsup_{t \to \infty} \frac{1}{t} \log \mathbb{P}\Big(N_t/t \in A\Big) \\ &\leq -\inf_{x \in \bar{A}} I(x), \end{aligned}$$

where A° denotes the interior of A and $\bar{A}$ is its closure and

$$I(x) = \begin{cases} x\theta_x + \nu - \frac{\nu x}{\nu + \|h\|_{L^1} x} & \text{if } x \in (0, \infty) \\ \nu & \text{if } x = 0 \\ +\infty & \text{if } x \in (-\infty, 0), \end{cases}$$

where $\theta = \theta_x$ is the unique solution in $(-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1}]$, of

$$\mathbb{E}(e^{\theta S}) = \frac{x}{\nu + x\|h\|_{L^1}}, \; x > 0, \tag{3}$$

where S in the above equation denotes the total number of descendants of an immigrant, including the immigrant himself.

Remark 1.1. The rate function described above $I(x)$ can be represented as more explicit form. Note that (see [17] for detail), for all $\theta \in (-\infty, \|h\|_{L^1} - 1 - \log \|h\|_{L^1}]$, $\mathbb{E}(e^{\theta S})$ satisfies

$$\mathbb{E}(e^{\theta S}) = e^\theta e^{\|h\|_{L^1}(\mathbb{E}(e^{\theta S}) - 1)}, \tag{4}$$

which implies that $\theta_x = \log\left(\frac{x}{\nu + x\|h\|_{L^1}}\right) - \|h\|_{L^1}\left(\frac{x}{\nu + x\|h\|_{L^1}} - 1\right)$. Substituting into the formula, we have

$$I(x) = \begin{cases} x \log\left(\frac{x}{\nu + x\|h\|_{L^1}}\right) - x + \|h\|_{L^1} x + \nu & \text{if } x \in (0, \infty) \\ \nu & \text{if } x = 0 \\ +\infty & \text{if } x \in (-\infty, 0). \end{cases}$$

Zhu [32] proved that if $\|h\|_{L^1} < 1$ and $\sup_{t>0} t^{3/2}h(t) \leq C < \infty$, then for any Borel set A and time sequence $\sqrt{n} \ll c(n) \ll n$, there exists a moderate deviation principle

$$-\inf_{x\in A^\circ} J(x) \leq \liminf_{t\to\infty} \frac{t}{c(t)^2} \log \mathbb{P}\left(\frac{1}{c(t)}(N_t - \mu t) \in A\right)$$
$$\leq \limsup_{t\to\infty} \frac{t}{c(t)^2} \log \mathbb{P}\left(\frac{1}{c(t)}(N_t - \mu t) \in A\right) \leq -\inf_{x\in \bar{A}} J(x),$$

where $J(x) = \frac{x^2(1-\|h\|_{L^1})^3}{2\nu}$.

Nonlinear model: When $\lambda(\cdot)$ is nonlinear, the usual immigration-birth representation no longer works and so nonlinear model is much harder to study. Brémaud and Massoulié [3] proved that there exists a unique stationary version of nonlinear Hawkes processes under certain conditions and the convergence to equilibrium of a non-stationary version. Massoulié [21] extended the stability results to nonlinear Hawkes processes with random marks and considered the Markovian case and also proved stability results without the Lipschitz condition for $\lambda(\cdot)$. In addition, Brémaud and Massoulié [3] considered the rate of extinction for nonlinear Hawkes process. Functional central limit theorem for nonlinear Hawkes process is obtained in Zhu [31] and Zhu [29] proved large deviation for a special case for nonlinear case when $h(\cdot)$ is exponential or sums of exponentials. Zhu [30] proved a process-level, i.e., level-3 large deviation principle for nonlinear Hawkes processes for general $h(\cdot)$ and hence by contradiction principle, the level-1 large deviation principle for $\mathbb{P}(\frac{N_t}{t} \in \cdot)$.

1.2. Statement of the main results

This section states the main results of this paper. We state asymptotic results for Hawkes process with an extended inverse Markovian. It mainly consists of the central limit theorems and law of large number.

A linear Hawkes process N_t has the intensity:

$$\lambda_t = \nu + \int_{-\infty}^{t-} h(t-s)dN_s, \tag{5}$$

where $\nu > 0$ is the baseline intensity, $h(\cdot)$ is the exciting function.

When $h(t) = \alpha e^{-\beta t}$ for $\alpha, \beta > 0$, the system of the Hawkes process is Markovian in the sense that $Z_t = \int_{-\infty}^{t-} \alpha e^{-\beta(t-s)} dN_s$ is Markovian satisfying the dynamics:

$$dZ_t = -\beta Z_t dt + \alpha dN_t, \tag{6}$$

where N_t has the intensity $\nu + Z_{t-}$ at time t, and Z_t process has the infinitesimal generator

$$\mathcal{L}f(z) = -\beta z f'(z) + (\nu + z)[f(z+\alpha) - f(z)]. \tag{7}$$

It is well known (See [13]) that

$$\frac{1}{t}\int_0^t Z_s ds \to \frac{\nu}{\beta-\alpha}, \tag{8}$$

a. s., and

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu}{\beta-\alpha}\cdot t\right] \to N\left(0, \frac{\alpha^2\nu\beta}{(\beta-\alpha)^3}\right), \tag{9}$$

in distribution as $t \to \infty$.

First, we assume that $N(-\infty, 0] = 0$, i.e., the Hawkes process has empty history. Seol [27] proposed an inverse Markovian Hawkes process and is defined as

$$dZ_t = -\beta Z_t dt + (\nu + \alpha Z_{t-})dN_t, \tag{10}$$

where N_t is Poisson with intensity 1 and $\nu > 0$, $\alpha > 0$, and $\beta > 0$. The Z_t process has the infinitesimal generator:

$$\mathcal{L}f(z) = -\beta z f'(z) + f(z + \alpha z + \nu) - f(z). \tag{11}$$

From the our assumption with the empty history $Z_0 = 0$, and for the Markovian case,

$$dZ_t = -\beta Z_t dt + (\alpha Z_{t-} + \nu)dN_t, \tag{12}$$

where N_t is Poisson with rate 1, and it follows that

$$d(e^{\beta t} Z_t) = (\alpha Z_{t-} + \nu)e^{\beta t} dN_t, \tag{13}$$

and since we assumed $Z_0 = 0$, we get

$$Z_t = \int_0^t (\alpha Z_{s-} + \nu)e^{-\beta(t-s)} dN_s. \tag{14}$$

It is known (See [27]) that

$$\lim_{t\to\infty} \frac{1}{t}\int_0^t Z_s ds = \frac{\nu}{\beta-\alpha}, \tag{15}$$

a. s., and

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu}{\beta-\alpha}\cdot t\right] \to N\left(0, \frac{\nu^2 + 2\nu\alpha\frac{\nu}{\beta-\alpha} + \alpha^2\frac{\nu^2(\alpha+\beta+2)}{(\beta-\alpha)(2\beta-2\alpha-\alpha^2)}}{(\beta-\alpha)^2}\right), \tag{16}$$

in distribution as $t \to \infty$.

The inverse Markovian Hawkes processes combines features of several existing models of self-exciting process and some remark on this model are as follows:

- when $\alpha = 0, Z_t = Z_0 e^{-\beta t} + \int_0^t \nu e^{-\beta(t-s)} dN_s$, which is a shot-noise process.

- when $\nu = 0, Z_t = Z_0 \exp(-\beta t + \log(1+\alpha) N_t)$, which is a jump-diffusion process with no diffusions.

In the Hawkes process, the jump intensity is linear in Z_t, while the jump size is constant α. In the inverse Hawkes process, the opposite is true, that is, the jump intensity is constant 1, while the jump size is linear in Z_t. In other words, in the Hawkes process, the more jumps you have in the past, the more jumps will be expected in the future, while in the inverse Hawkes process, the more jumps you have in the past, the larger jumps will be expected in the future. For the Hawkes process, the self-excitation lies on the intensity, that is the frequency, while for the inverse Hawkes process, the self-excitation is about the jump size, that is, the severity.

We introduce an extended inverse Markovian Hawkes process, a new model combining the Hawkes process and the inverse Hawkes process:

$$dZ_t = -\beta Z_t dt + \alpha_1 dN_t^{(1)} + (\nu_2 + \alpha_2 Z_{t-}) dN_t^{(2)}, \tag{17}$$

where $N_t^{(1)}$ is a simple point process with intensity $\nu_1 + Z_{t-}$ at time t and $N_t^{(2)}$ is a Poisson process with intensity 1, where β, α_1, α_2, ν_1 and ν_2 are all positive constants. We state the assumptions which we will use throughout the paper.

Assumption 1.2.

1. $N(-\infty, 0] = 0$, i.e., Hawkes process has empty history,
2. $M_1 = \beta - \alpha_1 + \alpha_2 > 0$ for some $\alpha_1 > 0, \alpha_2 > 0$ and $\beta > 0$,
3. $M_2 = 2\beta - 2\alpha_1 - 2\alpha_2 - \alpha_2^2 > 0$.

The infinitesimal generator of Z_t process is given by

$$\mathcal{L}f(z) = -\beta z f'(z) + (\nu_1 + z)[f(z+\alpha_1) - f(z)] + f(z + \nu_2 + \alpha_2 z) - f(z). \tag{18}$$

The followings are our main results.

Theorem 1.3 (Law of Large Number)**.** *Assume that Assumption 1.2 is satisfied and the process Z_t is defined in* (17)*. Then, we have*

$$\lim_{t\to\infty} \frac{1}{t} \int_0^t Z_s ds = \frac{M_3}{M_1}, \tag{19}$$

a.s. as $t \to \infty$, where the constants are

$$M_1 = \beta - \alpha_1 + \alpha_2,$$
$$M_3 = \nu_1 \alpha_1 + \nu_2.$$

Theorem 1.4 (Central Limit Theorem)**.** *Assume that Assumption 1.2 is satisfied and the process Z_t is defined in* (17)*. Then, we have*

$$\frac{1}{\sqrt{t}} \left[\int_0^t Z_s ds - \frac{M_3}{M_1} \cdot t \right] \to N\left(0, \sigma^2\right), \tag{20}$$

in distribution as $t \to \infty$, where

$$\sigma^2 := \frac{1}{(\beta - \alpha_1 - \alpha_2)^2} \left[\alpha_1^2 (\nu_1 + \mathbb{E}[Z_\infty]) + \mathbb{E}[(\nu_1 + \alpha_2 Z_\infty)^2] \right]$$

$$= \frac{(\alpha_1^2\nu_1 + \nu_1^2)M_1M_2 + (\alpha_1^2\nu_1 + 2\nu_1\alpha_2)M_2M_3 + \alpha_2^2(M_3M_5 + M_1M_4)}{M_1^3M_2},$$

and $M_i, i \in 1,2,3,4,5$ *are constants and*

$$\begin{aligned} M_1 &= \beta - \alpha_1 + \alpha_2, \\ M_2 &= 2\beta - 2\alpha_1 - 2\alpha_2 - \alpha_2^2, \\ M_3 &= \nu_1\alpha_1 + \nu_2, \\ M_4 &= \nu_1\alpha_1^2 + \nu_2^2, \\ M_5 &= \alpha_1^2 + 2\nu_1\alpha_1 + 2\nu_2\alpha_2 + 2\nu_2. \end{aligned}$$

2. Proofs of the main results

In this section, we give the proofs of the main theorems and related auxiliary results. The following are key results to solve the main results. The first and second moments for Z_t can be obtained in Section 2.1 and the main theorems are completed in Section 2.2.

2.1. Some auxiliary results

Proposition 2.1. *Suppose that Z_t is a stochastic process which is defined in* (17). *Then under the Assumption 1.2, we have*

(i) *The first moment for Z_t is*

$$\mathbb{E}[Z_t] = (1 - e^{-M_1t})\frac{M_3}{M_1},$$

(ii) *The second moment for Z_t is*

$$\mathbb{E}[Z_t^2] = \left(1 - e^{-M_3t}\right)\frac{M_3M_5}{M_1M_2} + \left(1 - e^{-M_2t}\right)\frac{M_4}{M_2},$$

where $M_i, i \in 1,2,3,4,5$ *are constants and*

$$\begin{aligned} M_1 &= \beta - \alpha_1 + \alpha_2, \\ M_2 &= 2\beta - 2\alpha_1 - 2\alpha_2 - \alpha_2^2, \\ M_3 &= \nu_1\alpha_1 + \nu_2, \\ M_4 &= \nu_1\alpha_1^2 + \nu_2^2, \\ M_5 &= \alpha_1^2 + 2\nu_1\alpha_1 + 2\nu_2\alpha_2 + 2\nu_2. \end{aligned}$$

Proof. To prove this, we note that we have the following form for a extended inverse Markovian Hawkes process defined in (17),

$$\mathbb{E}f(Z_t) = f(Z_0) + \int_0^t \mathbb{E}\mathcal{L}f(Z_s)ds. \tag{21}$$

Taking $f(z) = z$ and $f(z) = z^2$ give us two explicit forms

$$\mathbb{E}[Z_t] = z_0 + \mathbb{E}[\int_0^t \mathcal{L}Z_sds]$$

and

$$\mathbb{E}[Z_t^2] = z_0^2 + \mathbb{E}[\int_0^t \mathcal{L}Z_s^2 ds].$$

(i) Since

$$\mathcal{L}z = -\beta z + (\nu_1 + z)\alpha_1 + \nu_2 + \alpha_2 z, \tag{22}$$

we have

$$\mathbb{E}[Z_t] = z_0 + \mathbb{E}[\int_0^t \mathcal{L}Z_s ds] = z_0 + \int_0^t (\nu_1\alpha_1 + \nu_2 + (\alpha_2 + \alpha_1 - \beta)\mathbb{E}[Z_s])ds.$$

Taking derivative with respect to t to the both sides provide

$$\frac{d}{dt}\mathbb{E}[Z_t] = \nu_1\alpha_1 + \nu_2 + (\alpha_2 + \alpha_1 - \beta)\mathbb{E}[Z_t] \text{ and } \mathbb{E}[Z_0] = 0.$$

Solving differential equation yields

$$\mathbb{E}[Z_t] = -\frac{\nu_1\alpha_1 + \nu_2}{\alpha_2 + \alpha_1 - \beta} + \frac{\nu_1\alpha_1 + \nu_2}{\alpha_2 + \alpha_1 - \beta} e^{-(\beta - \alpha_2 - \alpha_1)t}.$$

Thus we have

$$\mathbb{E}[Z_t] = (1 - e^{-M_1 t})\frac{M_3}{M_1},$$

where

$$M_1 = \beta - \alpha_1 + \alpha_2,\ M_3 = \nu_1\alpha_1 + \nu_2. \tag{23}$$

(ii) Since

$$\mathcal{L}z^2 = -2\beta z^2 + (\nu_1 + z)(\alpha_1^2 + 2\alpha_1 z) + (\nu_2 + \alpha_2 z)^2 + 2z(\nu_2 + \alpha_2 z), \tag{24}$$

we have

$$\begin{aligned} \mathbb{E}[Z_t^2] &= z_0^2 + \mathbb{E}[\int_0^t \mathcal{L}Z_s^2 ds] \\ &= z_0^2 + \int_0^t (\alpha_2^2 + 2\alpha_2 + 2\alpha_1 - 2\beta)\mathbb{E}[Z_s^2] \\ &\quad + (\alpha_1^2 + 2\nu_2\alpha_2 + 2\nu_1\alpha_1 + 2\nu_2)\mathbb{E}[Z_s] + \nu_1\alpha_1^2 + \nu_2^2 ds. \end{aligned} \tag{25}$$

Taking derivative with respect to t to the both sides provide

$$\begin{aligned} \frac{d}{dt}\mathbb{E}[Z_t^2] = &(\alpha_2^2 + 2\alpha_2 + 2\alpha_1 - 2\beta)\mathbb{E}[Z_t^2] \\ &+ (\alpha_1^2 + 2\nu_2\alpha_2 + 2\nu_1\alpha_1 + 2\nu_2)\mathbb{E}[Z_t] + \nu_1\alpha_1^2 + \nu_2^2, \end{aligned} \tag{26}$$

and so

$$\begin{aligned} &\frac{d}{dt}\mathbb{E}[Z_t^2] + (2\beta - \alpha_2^2 - 2\alpha_2 - 2\alpha_1)\mathbb{E}[Z_t^2] \\ &= (\alpha_1^2 + 2\nu_2\alpha_2 + 2\nu_1\alpha_1 + 2\nu_2)\Big(-\frac{\nu_1\alpha_1 + \nu_2}{\alpha_2 + \alpha_1 - \beta} \end{aligned}$$

$$
\begin{aligned}
&+\frac{\nu_1\alpha_1+\nu_2}{\alpha_2+\alpha_1-\beta}e^{-(\beta-\alpha_2-\alpha_1)t}+z_0e^{-(\beta-\alpha_2-\alpha_1)t}\Big)+\nu_1\alpha_1^2+\nu_2^2\\
=\;&-\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{\alpha_2+\alpha_1-\beta}\\
&+\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{\alpha_2+\alpha_1-\beta}e^{-(\beta-\alpha_2-\alpha_1)t}\\
&+(\nu_1\alpha_1^2+\nu_2^2)
\end{aligned}
$$

and $\mathbb{E}[Z_0^2]=0$. Solving differential equation using integrating factor $\mu(t)=e^{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}$ yields

$$
\begin{aligned}
&(e^{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\mathbb{E}[Z_t^2])'\\
=\;&-\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{\alpha_2+\alpha_1-\beta}e^{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\\
&+\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{\alpha_2+\alpha_1-\beta}e^{(\beta-\alpha_2^2-\alpha_2-\alpha_1)t}\\
&+(\nu_1\alpha_1^2+\nu_2^2)e^{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t},
\end{aligned}
\tag{27}
$$

and so

$$
\begin{aligned}
&e^{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\mathbb{E}[Z_t^2]\\
=\;&\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{\beta-\alpha_2-\alpha_1}\int_0^t e^{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)s}ds\\
&+\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{\alpha_2+\alpha_1-\beta}\int_0^t e^{(\beta-\alpha_2^2-\alpha_2-\alpha_1)s}ds\\
&+(\nu_1\alpha_1^2+\nu_2^2)\int_0^t e^{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)s}ds.
\end{aligned}
\tag{28}
$$

Thus

$$
\begin{aligned}
&\mathbb{E}[Z_t^2]\\
=\;&\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{(\beta-\alpha_2-\alpha_1)(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)}\Big(1-e^{-(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\Big)\\
&+\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{(\alpha_2+\alpha_1-\beta)(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)}\Big(e^{(\alpha_2+\alpha_1-\beta)t}-e^{-(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\Big)\\
&+\frac{(\nu_1\alpha_1^2+\nu_2^2)}{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)}\Big(1-e^{-(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\Big)+Ce^{-(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}.
\end{aligned}
\tag{29}
$$

Thus, by the boundary condition

(30)

$$\begin{aligned}
&\mathbb{E}[Z_t^2]\\
&= \frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{(\beta-\alpha_2-\alpha_1)(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)}\Big(1-e^{-(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\Big)\\
&\quad+\frac{(\nu_1\alpha_1+\nu_2)(\alpha_1^2+2\nu_2\alpha_2+2\nu_1\alpha_1+2\nu_2)}{(\alpha_2+\alpha_1-\beta)(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)}\Big(e^{(\alpha_2+\alpha_1-\beta)t}-e^{-(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\Big)\\
&\quad+\frac{(\nu_1\alpha_1^2+\nu_2^2)}{(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)}\Big(1-e^{-(2\beta-\alpha_2^2-2\alpha_2-2\alpha_1)t}\Big).
\end{aligned}$$

Therefore, we have

$$\mathbb{E}[Z_t^2]=\Big(1-e^{-M_3t}\Big)\frac{M_3M_5}{M_1M_2}+\Big(1-e^{-M_2t}\Big)\frac{M_4}{M_2},$$

where $M_i, i\in 1,2,3,4,5$ are constants and

$$\begin{aligned}
M_1&=\beta-\alpha_1+\alpha_2,\\
M_2&=2\beta-2\alpha_1-2\alpha_2-\alpha_2^2,\\
M_3&=\nu_1\alpha_1+\nu_2,\\
M_4&=\nu_1\alpha_1^2+\nu_2^2,\\
M_5&=\alpha_1^2+2\nu_1\alpha_1+2\nu_2\alpha_2+2\nu_2.
\end{aligned}$$

□

Corollary 2.2. *Suppose that Z_t is a stochastic process which is defined in* (17). *Then under the Assumption 1.2, we have*

(i)

$$\mathbb{E}[Z_\infty]=\frac{M_3}{M_1}, \tag{31}$$

(ii)

$$\mathbb{E}[Z_\infty^2]=\frac{M_3M_5+M_1M_4}{M_1M_2}, \tag{32}$$

where $M_i, i\in 1,2,3,4,5$ are constants and

$$\begin{aligned}
M_1&=\beta-\alpha_1+\alpha_2,\\
M_2&=2\beta-2\alpha_1-2\alpha_2-\alpha_2^2,\\
M_3&=\nu_1\alpha_1+\nu_2,\\
M_4&=\nu_1\alpha_1^2+\nu_2^2,\\
M_5&=\alpha_1^2+2\nu_1\alpha_1+2\nu_2\alpha_2+2\nu_2.
\end{aligned}$$

Proof. It follows that Z_t is uniformly integrable. If we take the limit to Proposition 2.1 as $t\to\infty$, the proofs are completed. □

2.2. Proofs of the main theorems

The followings are the proof of the main theorems in the paper.

Proof of Theorems 1.3. By the definition of Z_t process and by using Foster-Lyapunov criterion (See [8] for details) together with the condition 2 of the Assumption 1.2, we can show that Z_t is ergodic. Therefore, By ergodic theorem and Corollary 2.2(i), we have

$$\frac{1}{t}\int_0^t Z_s ds \to \mathbb{E}[Z_\infty] = \frac{\nu_1\alpha_1 + \nu_2}{\beta - \alpha_1 - \alpha_2}, \tag{33}$$

a.s. as $t \to \infty$. This completes the proof of Theorem 1.3. □

Completion of the Proof of Theorem 1.4. Note that

$$f(Z_t) - f(Z_0) - \int_0^t \mathcal{L}f(Z_s)ds, \tag{34}$$

is a martingale. Let $f(z) = z$, then

$$Z_t - Z_0 - \int_0^t [-\beta Z_s + (\nu_1 + Z_s)\alpha_1 + \nu_2 + \alpha_2 Z_s]ds, \tag{35}$$

is a martingale, which implies that

$$M_t := \int_0^t Z_s ds - \frac{\nu_1\alpha_1 + \nu_2}{\beta - \alpha_1 - \alpha_2}t + \frac{Z_t}{\beta - \alpha_1 - \alpha_2} - \frac{Z_0}{\beta - \alpha_1 - \alpha_2} \tag{36}$$

is a martingale. Note that

$$\begin{aligned} dM_t = Z_t dt - \frac{\nu_1\alpha_1 + \nu_2}{\beta - \alpha_1 - \alpha_2}dt \\ + \frac{1}{\beta - \alpha_1 - \alpha_2}(-\beta Z_t dt + \alpha_1 dN_t^{(1)} + (\nu_2 + \alpha_2 Z_{t-})dN_t^{(2)}), \end{aligned} \tag{37}$$

and thus the quadratic variation of M_t is given by

$$\frac{1}{(\beta - \alpha_1 - \alpha_2)^2}\alpha_1^2 N_t^{(1)} + \frac{1}{(\beta - \alpha_1 - \alpha_2)^2}\int_0^t (\nu_1 + \alpha_2 Z_{s-})^2 dN_s^{(2)}. \tag{38}$$

And we have

$$\begin{aligned} &\frac{1}{t(\beta - \alpha_1 - \alpha_2)^2}\alpha_1^2 N_t^{(1)} + \frac{1}{t(\beta - \alpha_1 - \alpha_2)^2}\int_0^t (\nu_1 + \alpha_2 Z_{s-})^2 dN_s^{(2)} \\ &\to \sigma^2 := \frac{1}{(\beta - \alpha_1 - \alpha_2)^2}\left[\alpha_1^2(\nu_1 + \mathbb{E}[Z_\infty]) + \mathbb{E}[(\nu_1 + \alpha_2 Z_\infty)^2]\right], \end{aligned} \tag{39}$$

a.s. as $t \to \infty$. By the central limit theorem for the martingales (See Theorem VIII-3.11 of [16] for details),

$$\frac{M_t}{\sqrt{t}} \to N\left(0, \sigma^2\right), \tag{40}$$

in distribution as $t \to \infty$, where σ^2 is defined in (39). Finally, $\frac{Z_t}{\sqrt{t}} \to 0$ and $\frac{Z_0}{\sqrt{t}} \to 0$ in probability as $t \to \infty$. Hence, with the formulas for $\mathbb{E}[Z_\infty]$ and $\mathbb{E}[Z_\infty^2]$ in Corollary 2.2, we get

$$\frac{1}{\sqrt{t}}\left[\int_0^t Z_s ds - \frac{\nu_1\alpha_1+\nu_2}{\beta-\alpha_1-\alpha_2}\cdot t\right] \to N\left(0,\sigma^2\right), \tag{41}$$

in distribution as $t \to \infty$, where

(42)

$$\sigma^2 := \frac{(\alpha_1^2\nu_1+\nu_1^2)M_1M_2 + (\alpha_1^2\nu_1+2\nu_1\alpha_2)M_2M_3 + \alpha_2^2(M_3M_5+M_1M_4)}{M_1^3M_2}$$

and $M_i, i \in 1,2,3,4,5$ are constants and

$$\begin{aligned}
M_1 &= \beta - \alpha_1 + \alpha_2,\\
M_2 &= 2\beta - 2\alpha_1 - 2\alpha_2 - \alpha_2^2,\\
M_3 &= \nu_1\alpha_1 + \nu_2,\\
M_4 &= \nu_1\alpha_1^2 + \nu_2^2,\\
M_5 &= \alpha_1^2 + 2\nu_1\alpha_1 + 2\nu_2\alpha_2 + 2\nu_2.
\end{aligned}$$

This completes the proof of Theorem 1.4 □

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