Nonlinear Functional Analysis and Applications
Vol. 26, No. 2 (2021), pp. 289-301
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.02.04
http://nfaa.kyungnam.ac.kr/journal-nfaa

COMMON FIXED POINT RESULTS FOR MAPPINGS UNDER NONLINEAR CONTRACTION OF CYCLIC FORM IN b-METRIC SPACES

Ayat Rabaiah[1], Abdallah Tallafha[2] and Wasfi Shatanawi[3]

[1]Department of Mathematics, Faculty of Science
The University of Jordan, Amman, Jordan
e-mail: aya9160322@ju.edu.jo; ayatrabaiah@yahoo.com

[2]Department of Mathematics, Faculty of Science
The University of Jordan, Amman, Jordan
e-mail: a.tallafha@ju.edu.jo

[3]Department of Mathematics, Faculty of General Science
Prince Sultan University, Riyadh, Saudi Arabia
Department of Mathematics, Faculty of Science
Hashemite University, Zarqa, Jordan
e-mail: wshatanawi@psu.edu.sa; swasfi@hu.edu.jo

Abstract. In this research, we interpret the notion of a b-cyclic $(\Phi, C, D)-$contraction for the pair (g, S) of self-mappings on the set Y. We employ our definition to introduce some common fixed point theorems for the two mappings g and S under a set of conditions. Also we introduce an example to support our results.

1. Introduction

Many years ago, different results were obtained in fixed point theory in b-metric spaces. A main topic in the fixed point theory is the cyclic contraction. Kirk et al. [15] established the first result in this interesting field.

[0]Received September 6, 2020. Revised December 9, 2020. Accepted February 5, 2021.
[0]2010 Mathematics Subject Classification: 54H25, 47H10, 34B14.
[0]Keywords: Metric spaces, common fixed point, altering distance function, almost contraction, b-metric spaces.
[0]Corresponding author: A. Rabaiah(ayatrabaiah@yahoo.com).

Now a days, others attained important outcomes in this dominant field see [20, 21, 29, 30]

We start with the definition of a cyclic map.

Definition 1.1. ([29]) Let C and D be non-empty subsets of a metric space (Y, d) and S: $C \cup D \to C \cup D$. Then S is called a cyclic map if $S(C) \subseteq D$ *and* $S(D) \subseteq C$.

In 2003, Kirk et al. [15] gave the following interesting theorem in fixed point theory for a cyclic map.

Theorem 1.2. ([15]) *Let C and D be nonempty closed subsets of a complete metric space (Y, d). Suppose that $S : C \cup D \to C \cup D$ is a cyclic map such that*

$$d(Sx, Sy) \leq kd(x, y), \quad \forall x, y \in D.$$

If $k \in [0, 1)$, then S has a unique fixed point in $C \cap D$.

Some of contractive conditions are based on functions called control function which alter the distance between two points in a metric space. Such functions were inaugurated by Khan et al. [17]

Definition 1.3. ([17]) The function $\Phi : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied:

(1) Φ is continuous and nondecreasing,
(2) $\Phi(\zeta) = 0$ if and only if $\zeta = 0$.

Definition 1.4. ([6, 11]) Let Y be a nonempty set and $b \geq 1$ be a given real number. A function d: $Y \times Y \to [0, \infty)$ is called b-metric. If it satisfies the following properties for each $y_1, y_2, y_3 \in Y$,

(1) $d(y_1, y_2) = 0$ if and only if $y_1 = y_2$,
(2) $d(y_1, y_2) = d(y_2, y_1)$,
(3) $d(y_1, y_3) \leq b[d(y_1, y_2) + d(y_2, y_3)]$.

The pair (Y, d) is called a b-metric space.

Example 1.5. Let $Y = l_P(R)$ with $0 < p < 1$, where $l_p(R) = \{y_n \subset R : \Sigma_{n=1}^{\infty} |y_n|^p < \infty\}$.

Define $d : Y \times Y \to R^+$ by:

$$d(y, z) = (\Sigma_{n=1}^{\infty} |y_n - z_n|^p)^{\frac{1}{p}},$$

where $y = \{y_n\}, z = \{z_n\}$. Then d is a b-metric space (see [12]) with coefficient $b = \frac{1}{p}$.

Example 1.6. Let $Y = L_p[0,1]$ be the space of all real function $x(t), t \in [0,1]$ such that for $0 < p < 1$,

$$\int_0^1 |y(t)|^p < \infty.$$

Define $d : Y \times Y \to R^+$ by:

$$d(x,y) = \left(\int_0^1 |y(t) - z(t)|^p \, dt \right)^{\frac{1}{p}}.$$

Then d is a b-metric space (see [12]) with coefficient $b = 2^{\frac{1}{p}}$.

The above examples show that class of b-metric space is larger than the class of metric spaces. When $b = 1$, the concept of b-metric coincides with the concept of metric spaces. Many authors introduce many fixed point theorems in the notion of metric spaces, for more details see [1, 2, 3, 5, 7, 8, 9, 16, 22, 24, 25, 34, 35, 36, 37, 38, 39, 40, 41, 42]. Also, for some work on b-metric, we refer the reader to [4, 10, 13, 18, 19, 23, 26, 27, 28, 31, 32, 33].

Definition 1.7. ([13]) Let (Y,d) be a b- metric space.

(1) A sequence $\{y_n\}$ in Y is said to be Cauchy, if $d(y_n, y_m) \to 0$ as $n, m \to \infty$.
(2) A sequence $\{y_n\}$ in Y is said to be convergent, if there exists $y \in Y$ such that $d(y_n, y) \to 0$ as $n \to \infty$ and we write $\lim_{n\to\infty} y_n = y$.
(3) The b-metric space (Y,d) is said to be complete if every Cauchy sequence in Y is convergent.

Theorem 1.8. ([14]) *Let (Y,d) be a complete b-metric space with constant $b \geq 1$, such that b-metric is a continuous functional. Let $S : Y \to Y$ be a contraction with constant $k \in [0,1)$ such that $kb < 1$. Then S has a unique fixed point.*

The justification of this paper is to acquire common fixed point results for mapping satisfying nonlinear contractive conditions of a cyclic form based on the notion of an altering distance function.

2. The main results

We begin with the following definition.

Definition 2.1. Let (Y,d) be a b-metric space and C, D be nonempty closed subsets of Y. Let $g, S : Y \to Y$ be two mappings. The pair (g,S) is called a b-cyclic (Φ, C, D)-contraction, if the following conditions are satisfied:

(1) Φ is an altering distance function,
(2) $C \cup D$ has a cyclic representation *w.r.t.* the pair (g, S); that is $g(C) \subseteq D$, $S(D) \subseteq C$ and $Y = C \cup D$,
(3) there exists $\delta > 0$ with $b^2\delta < 1$ such that for all $x, y \in Y$ with $x \in C$ and $y \in D$, we have

$$\Phi\left(bd\left(gx, Sy\right)\right) \leqslant \Phi\left(\delta \max\left\{d\left(x,y\right), d\left(x,gx\right), d\left(y,Sy\right), \frac{1}{2b}d\left(x,Sy\right), \frac{1}{2b}d\left(gx,y\right)\right\}\right). \quad (2.1)$$

From this point till the end of the paper, by Φ we mean altering distance function unless otherwise stated and Y stands for a complete b-metric space. In the rest of this paper, we also mean by N set of non negative integer numbers.

Theorem 2.2. *Let (Y, d) be a b-complete metric space and C, D be nonempty closed subsets of Y. Let $g, S : Y \to Y$ be two mapping. Assume the following:*

(1) *the pair $(g, S)$is a b-cyclic (Φ, C, D) contraction,*
(2) *g or S is continuous.*

Then g and S have a common fixed point.

Proof. Choose $y_0 \in C$, let $y_1 = gy_0$. Since $gC \subseteq D$, we have $y_1 \in D$. Also, let $y_2 = Sy_1$. Since $SD \subseteq C$, we have $y_2 \in C$. Continuing this process, we can construct a sequence $\{y_n\}$ in Y such that $y_{2n+1} = gy_{2n}$, $y_{2n+2} = Sy_{2n+1}$, $y_{2n} \in C$ and $y_{2n+1} \in D$.

We divide our proof into the following steps:

Step 1. We will show that $\{y_n\}$ is a Cauchy sequence in (Y, d).
Subcase 1: Suppose that $y_{2n} = y_{2n+1}$ for some $n \in N$. Since y_{2n} and y_{2n+1} are elements in Y with $y_{2n} \in C$ and $y_{2n+1} \in D$, we have

$$\begin{aligned}
&\Phi\left(bd\left(y_{2n+1}, y_{2n+2}\right)\right) \\
&= \Phi\left(d\left(gy_{2n}, Sy_{2n+1}\right)\right) \\
&\leqslant \Phi\left(\delta \max\left\{d\left(y_{2n}, y_{2n+1}\right), d\left(y_{2n}, gy_{2n}\right), d\left(y_{2n+1}, Sy_{2n+1}\right),\right.\right. \\
&\qquad \left.\left.\frac{1}{2b}d\left(y_{2n}, Sy_{2n+1}\right), \frac{1}{2b}d\left(gy_{2n}, y_{2n+1}\right)\right\}\right) \\
&= \Phi\left(\delta \max\left\{d\left(y_{2n}, y_{2n+1}\right), d\left(y_{2n}, y_{2n+1}\right), d\left(y_{2n+1}, y_{2n+2}\right),\right.\right. \\
&\qquad \left.\left.\frac{1}{2b}d\left(y_{2n}, y_{2n+2}\right), \frac{1}{2b}d\left(y_{2n+1}. y_{2n+1}\right)\right\}\right)
\end{aligned}$$

$$\leq \Phi\left(\delta d\left(y_{2n+1}, y_{2n+2}\right)\right)$$
$$\leq \Phi(\delta b d\left(y_{2n+1}, y_{2n+2}\right)).$$

By properties of ϕ, we have $bd(y_{2n+1}, y_{2n+2}) \leq \delta bd(y_{2n+1}, y_{2n+2})$. Since $\delta b < 1$, we have $bd\left(y_{2n+1}, y_{2n+2}\right) = 0$ and hence $y_{2n+2} = y_{2n+1}$.

Similarly, we may show that $y_{2n+3} = y_{2n+2}$. Hence $\{y_n\}$ is a constant sequence in Y, so it is a Cauchy sequence in (Y, d).

Subcase 2: $y_{2n} \neq y_{2n+1}$ for all $n \in N$. Given $n \in N$. If n is even, then $n = 2q$ for some $q \in N$.

Since $y_{2q} \in C$, $y_{2q+1} \in D$ and y_{2q}, y_{2q+1} are elements in Y, we have

$$\begin{aligned}
\Phi\left(bd\left(y_{n+1}, y_{n+2}\right)\right) &= \Phi\left(bd\left(y_{2q+1}, y_{2q+2}\right)\right) \\
&= \Phi\left(bd\left(gy_{2q}, Sy_{2q+1}\right)\right) \\
&\leq \Phi\Bigg(\delta \max\Bigg\{d\left(y_{2q}, y_{2q+1}\right), d\left(y_{2q}, gy_{2q}\right), d\left(y_{2q+1}, Sy_{2q+1}\right), \\
&\qquad \frac{1}{2b} d\left(y_{2q}, Sy_{2q+1}\right), \frac{1}{2b} d\left(gy_{2q}, y_{2q+1}\right)\Bigg\}\Bigg) \\
&= \Phi\Bigg(\delta \max\Bigg\{d\left(y_{2q}, y_{2q+1}\right), d\left(y_{2q+1}, y_{2q+2}\right), \\
&\qquad \frac{1}{2b} d\left(y_{2q}, y_{2q+2}\right), \frac{1}{2b} d\left(y_{2q+1}, y_{2q+2}\right)\Bigg\}\Bigg) \\
&\leq \Phi\Bigg(\delta \max\Bigg\{d\left(y_{2q}, y_{2q+1}\right), d\left(y_{2q}, y_{2q+2}\right)\Bigg\}\Bigg) \\
&\leq \Phi\Bigg(\delta b \max\Bigg\{d\left(y_{2q}, y_{2q+1}\right), d\left(y_{2q}, y_{2q+2}\right)\Bigg\}\Bigg).
\end{aligned}$$

If

$$\max\{d\left(y_{2q}, y_{2q+1}\right), d\left(y_{2q+1}, y_{2q+2}\right)\} = d\left(y_{2q+1}, y_{2q+2}\right),$$

then

$$\begin{aligned}
\Phi\left(bd\left(y_{2q+1}, y_{2q+2}\right)\right) &\leq \Phi\left(\delta d\left(y_{2q+1}, y_{2q+2}\right)\right) \\
&\leq \Phi(\delta bd\left(y_{2q+1}, y_{2q+2}\right)) \\
&< \Phi(d\left(y_{2q+1}, y_{2q+2}\right)) \\
&\leq \Phi(bd\left(y_{2q+1}, y_{2q+2}\right)),
\end{aligned}$$

which is a contradiction. Thus

$$\max\{d\left(y_{2q}, y_{2q+1}\right), d\left(y_{2q+1}, y_{2q+2}\right)\} = d\left(y_{2q}, y_{2q+1}\right). \tag{2.2}$$

Therefore

$$\begin{aligned}\Phi\left(bd\left(y_{2q+1},y_{2q+2}\right)\right) &\leq \Phi\left(\delta d\left(y_{2q},y_{2q+1}\right)\right)\\ &\leq \Phi\left(\delta bd\left(y_{2q},y_{2q+1}\right)\right). \end{aligned} \tag{2.3}$$

If n is odd, then $n=2q+1$ for some $q\in N$. Since y_{2q+2} and y_{2q+1} are elements in Y with $y_{2q+2}\in C$ and $y_{2q+1}\in D,$ we have

$$\begin{aligned}&\Phi\left(bd\left(y_{n+2},y_{n+1}\right)\right)\\ &=\Phi\left(bd\left(y_{2q+3},y_{2q+2}\right)\right)\\ &=\Phi(bd(gy_{2q+2},Sy_{2q+1}))\\ &\leq\Phi(\max\delta\{d\left(y_{2q+2},y_{2q+1}\right),d\left(y_{2q+2}.gy_{2q+2}\right),d\left(y_{2q+2},Sy_{2q+1}\right),\\ &\quad\frac{1}{2b}d\left(y_{2q+2,Sy_{2q+1}}\right),\frac{1}{2b}d\left(gy_{2q+2},y_{2q+1}\right)\})\\ &\leq\Phi\left(\delta\max\left\{d\left(y_{2q+2},y_{2q+1}\right),d\left(y_{2q+2},y_{2q+3}\right),\right.\right.\\ &\quad\left.\left.\frac{1}{2b}d\left(y_{2q+2},y_{2q+2}\right),\frac{1}{2b}d\left(y_{2q+3},y_{2q+1}\right)\right\}\right)\\ &\leq\Phi\left(\delta\max\left\{d\left(y_{2q+2},y_{2q+1}\right),d\left(y_{2q+2},y_{2q+3}\right)\right\}\right)\\ &\leq\Phi\left(\delta b\max\left\{d\left(y_{2q+2},y_{2q+1}\right),d\left(y_{2q+2},y_{2q+3}\right)\right\}\right).\end{aligned}$$

If

$$\max\{d\left(y_{2q+2},y_{2q+1}\right),d\left(y_{2q+2},y_{2q+3}\right)=d\left(y_{2q+2},y_{2q+3}\right),$$

then

$$\Phi\left(bd\left(y_{2q+2},y_{2q+3}\right)\right) \leq \Phi\left(\delta bd\left(y_{2q+2},y_{2q+3}\right)\right).$$

Properties of ϕ implies that

$$bd\left(y_{2q+2},y_{2q+3}\right) \leq \delta bd\left(y_{2q+2},y_{2q+3}\right)<bd\left(y_{2q+2},y_{2q+3}\right),$$

which is a contradiction. Therefore

$$\max\{d\left(y_{2q+2},y_{2q+1}\right),d\left(y_{2q+2},y_{2q+3}\right)\}=d\left(y_{2q+2},y_{2q+1}\right), \tag{2.4}$$

and hence

$$\Phi\left(bd\left(y_{2q+3},y_{2q+2}\right)\right)\leq\Phi\left(\delta bd\left(y_{2q+2},y_{2q+1}\right)\right). \tag{2.5}$$

From (2.3) and (2.5), we have

$$\Phi\left(bd\left(y_{n+1},y_{n+2}\right)\right)\leq\Phi\left(\delta bd\left(y_{n},y_{n+1}\right)\right)\leq\Phi\left(bd\left(y_{n},y_{n+1}\right)\right). \tag{2.6}$$

Since Φ is an altering distance function, we have $\{d\left(y_{n+1},y_{n+2}\right):n\in\mathbb{N}\cup\{0\}\}$ is a bounded nonincreasing sequence. Thus there exists $\zeta\geqslant 0$ such that

$$\lim_{n\to\infty} d\left(y_n, y_{n+1}\right) = \zeta.$$

On letting $n \to \infty$ in (2.6), we have

$$\Phi\left(b\zeta\right) \le \Phi\left(\delta b\zeta\right).$$

Claim: $\zeta = 0$. Suppose to the contrary, that is, $\zeta \neq 0$. By properties of ϕ, we have

$$b\zeta \le \delta b\zeta < \zeta,$$

which is a contradiction. Therefore $\zeta = 0$. Thus

$$\lim_{n\to\infty} d(y_n, y_{n+1}) = 0. \tag{2.7}$$

Next, we show that $\{y_n\}$ is a Cauchy sequence in b-metric space (Y, d). It is sufficient to show that $\{y_{2n}\}$ is a Cauchy sequence in (Y, d). Suppose to the contrary, that is, $\{y_{2n}\}$ is not a Cauchy sequence in (Y, d). Then there exists $\epsilon > 0$ for which we can find two subsequences $\{y_{2m(i)}\}$ and $\{y_{2n(i)}\}$ of $\{y_{2n}\}$ such that $n(i)$ is the smallest index for which

$$n(i) > m(i) > i, \quad d\left(y_{2m(i)}, y_{2n(i)}\right) \ \ge \ \epsilon. \tag{2.8}$$

This means that

$$d\left(y_{2m(i)}, y_{2n(i)-2}\right) < \epsilon. \tag{2.9}$$

From (2.8), (2.9) and the definition of the b-metric space, we get

$$\begin{aligned}
\epsilon &\le d\left(y_{2m(i)}, y_{2n(i)}\right)\\
&\le bd\left(y_{2m(i)}, y_{2n(i)-2}\right) + bd\left(y_{2n(i)-2}, y_{2n(i)}\right)\\
&\le bd\left(y_{2m(i)}, y_{2n(i)-2}\right) + b^2 d\left(y_{2n(i)-2}, y_{2n(i)-1}\right) + b^2 d\left(y_{2n(i)-1}, y_{2n(i)}\right)\\
&\le \epsilon b + b^2 d\left(y_{2n(i)-2}, y_{2n(i)-1}\right) + b^2 d\left(y_{2n(i)-1}, y_{2n(i)}\right).
\end{aligned}$$

By taking the sup limit of above inequalities using (2.7), we have

$$\epsilon \le \limsup_{i\to+\infty} d\left(y_{2m(i)}, y_{2n(i)}\right) \le \epsilon b. \tag{2.10}$$

Again, from (2.8) and the definition of the b-metric space, we get

$$\begin{aligned}
\epsilon &\le d\left(y_{2m(i)}, y_{2n(i)}\right)\\
&\le b(\left(d\left(y_{2m(i)}, y_{2m(i)+1}\right) + d\left(y_{2m(i)+}, y_{2n(i)}\right)\right).
\end{aligned}$$

On taking the limsup in above inequalities and using (2.7), we get

$$\epsilon \le \limsup_{i\to+\infty} bd\left(y_{2m(i)+1}, y_{2n(i)}\right). \tag{2.11}$$

Again, from the definition of the b-metric space, we get

$$d\left(y_{2m(i)}, y_{2n(i)-1}\right) \le b(\left(d\left(y_{2m(i)}, y_{2n(i)}\right) + d\left(y_{2n(i)+}, y_{2n(i)-1}\right)\right).$$

On taking the limsup in above inequalities and using (2.7) and (2.10), we get

$$\limsup_{i\to+\infty} bd\left(y_{2m(i)}, y_{2n(i)-1}\right) \leq \epsilon b^2. \tag{2.12}$$

Again, from the definition of the b-metric space, we get that

$$d\left(y_{2n(i)+1}, y_{2n(i)-1}\right) \leq \underline{\mathrm{d}}\left(y_{2n(i)+1}, y_{2n(i)}\right) + d\left(y_{2n(i)}, y_{2n(i)-1}\right).$$

On taking the limsup in above inequalities and using the properties of Φ , we get

$$\limsup_{i\to+\infty} bd\left(y_{2n(i)+1}, y_{2n(i)-1}\right) = 0. \tag{2.13}$$

Since $y_{2m(i)} \in C$ and $y_{2n(i)-1} \in D$, we have

$$\begin{aligned}
\Phi\left(bd\left(y_{2m(i)+1}, y_{2n(i)}\right)\right) &= \Phi\left(bd\left(gy_{2m(i)}, Sy_{2n(i)-1}\right)\right) \\
&\leq \Phi\Bigg(\max\delta\Big\{ d\left(y_{2m(i)}, y_{2n(i)-1}\right), d\left(y_{2m(i)}, y_{2m(i)}\right), \\
&\qquad d\left(y_{2n(i)-1}, Sy_{2n(i)-1}\right), \\
&\qquad \frac{1}{2b} d\left(y_{2m(i)}, gy_{2n(i)-1}\right), \frac{1}{2b} d\left(gy_{2m(i)}, y_{2n(i)-1}\right) \Big\}\Bigg) \\
&= \Phi\Bigg(\delta\max\Big\{ d\left(y_{2m(i)}, y_{2n(i)-1}\right), d\left(y_{2m(i)}, y_{2m(i)+1}\right), \\
&\qquad d\left(y_{2n(i)-1}, y_{2n(i)}\right), \\
&\qquad \frac{1}{2b} d\left(y_{2m(i)}, y_{2n(i)}\right), \frac{1}{2b} d\left(y_{2n(i)+1}, y_{2n(i)-1}\right) \Big\}\Bigg).
\end{aligned}$$

Taking the limsup in above inequalities, and using the properties of Φ and (2.7), (2.10), (2.11), (2.12) and (2.13), we get

$$\Phi\left(\epsilon\right) \leq \Phi\left(\epsilon\delta b^2\right).$$

Again, properties of Φ implies that $\epsilon \leq \epsilon\delta b^2$. Since $b^2\delta < 1$, we have $\epsilon = 0$, a contradiction. Thus $\{y_n\}$ is a Cauchy sequence in (Y, d).

Step 2: Existence of a common fixed point.

Since (Y, d) is a complete b-metric space and $\{y_n\}$ is a Cauchy sequence in Y we have $\{y_n\}$ converges to some $v \in Y$, that is, $\lim_{n\to+\infty} d\left(y_n, v\right) = 0$. Therefore,

$$\lim_{n\to\infty} y_n = \lim_{n\to\infty} y_{2n-1} = \lim_{n\to+\infty} y_{2n} = v. \tag{2.14}$$

Since $\{y_{2n}\}$ is a sequence in C. C is closed and $y_{2n} \to v$, we have $v \in C$. Also, since $\{y_{2n+1}\}$ is a sequence in D, D is closed and $y_{2n+1} \to v$, we have $v \in D$.

Now, we show that v is a fixed point of g and S. Without loss of generality, we may assume that g is continuous, since $y_{2n} \to v$, we get $y_{2n+1} = gy_{2n} \to gv$. By the uniqueness of limit, we have $v = gv$.

Now, we show that $v = Sv$. Since $v \in C$ and $v \in D$, we have

$$\begin{aligned}\Phi\left(bd\left(v, Sv\right)\right) &= \Phi\left(bd\left(gv, Sv\right)\right)\\ &\leq \Phi(\delta \max\{d\left(gv, Sv\right), d\left(v, gv\right), d\left(v, Sv\right),\\ &\qquad \frac{1}{2b}d\left(v, Sv\right), \frac{1}{2b}d\left(gv, v\right)\})\\ &= \Phi\left(\delta d\left(v, Sv\right)\right).\end{aligned}$$

Properties of Φ implies that

$$bd(v, Sv) \leq \delta d(v, Sv),$$

the last inequality only if $d(v, Sv) = 0$, and hence $v = Sv$. □

If we take $\Phi = I\left[0, +\infty\right]$ is the identity function in Theorem 2.2 we have the following result.

Corollary 2.3. *Let (Y, d) be a b-metric space and C, D be nonempty closed subsets of Y. Let $g, S : Y \to Y$ be two mappings and $C \cup D$ has a b-cyclic representation with respect to the pair (g, S). Suppose there exists $\delta > 0$ with $b^2\delta < 1$ such that for all $x, y \in Y$ with $x \in C$ and $y \in Y$, we have*

$$bd\left(gx, Sy\right) \leq \delta \max\left\{d\left(x, y\right), d\left(x, gx\right), d\left(y, Sy\right), \frac{1}{2b}d\left(x, Sy\right), \frac{1}{2b}d\left(gx, y\right)\right\}.$$

If g or S is continuous, then g and S have a common fixed point.

By taking $g = S$ in Theorem 2.2, we have the following result.

Corollary 2.4. *Let (Y, d) be a b- metric space and C, D be nonempty closed subsets of Y with $Y = C \cup D$. Let $g, S : Y \to Y$ be two mappings. Suppose there exists $\delta > 0$ with $b^2\delta < 1$ such that for all $x, y \in Y$ with $x \in C$ and $y \in Y$, we have*

$$\begin{aligned}&\Phi\left(bd\left(gx, gy\right)\right)\\ &\leq \Phi\left(\delta \max\left\{d\left(x, y\right), d\left(x, gx\right), d\left(y, gy\right), \frac{1}{2b}d\left(x, gy\right), \frac{1}{2b}d\left(gx, y\right)\right\}\right).\end{aligned}$$

Assume that g is a continuous and cyclic map, Then g has a fixed point.

By taking $C = D = Y$ in Theorem 2.2, we have the following result.

Corollary 2.5. *Let (Y,d) be a b- metric space. Let $g,S:Y\to Y$ be two mappings. Suppose there exists $\delta>0$ with $b^2\delta<1$ such that for all $x,y\in Y$, we have*

$$\begin{aligned}&\Phi\left(bd\left(gx,Sy\right)\right)\\&\leq\Phi\left(\delta\max\left\{d\left(x,y\right),d\left(x,gx\right),d\left(y,Sy\right),\frac{1}{2b}d\left(x,Sy\right),\frac{1}{2b}d\left(gx,y\right)\right\}\right).\end{aligned}$$

If g or S is continuous, then g and S have a common fixed point.

Example 2.6. Let $Y=\{1,2,3,4,5\}$. Define $d:Y\times Y\to[0,+\infty)$ by
$d\left(x,x\right)=0$ if $x\in\{1,2,3,4,5\}$;
$d\left(x,y\right)=1$ if $x,y\in\{1,2,3,4\}$ and $x\neq y$;
$d\left(x,y\right)=20$ if $x\in\{1,2,3\}$ and $y=5$;
$d\left(x,y\right)=20$ if $x=5$ and $y\in\{1,2,3\}$;
$d\left(x,y\right)=12$ if $x,y\in\{4,5\}$ and $x\neq y$.

Define $g:Y\to Y$ by $g\left(x\right)=1$ if $x\in\{1,2,3,4\}$ and $g\left(5\right)=4$. Also, define $S:Y\to Y$ by $S\left(x\right)=1$ if $x\in\{1,2,3,4\}$ and $S\left(5\right)=3$. Also, define $\Phi:[0,+\infty)\to[0,+\infty)$ via $\Phi\left(t\right)=\frac{t}{4}$. Let $C=\{1,3,5\}$ and $D=\{1,2,4\}$. Then

(1) (Y,d) is a complete b-metric space,
(2) $C\cup D$ has cyclic representation with respect to the pair (g,S),
(3) for every two elements $x,y\in Y$ with $x\in C$ and $y\in D$, we have

$$\begin{aligned}&\Phi\left(2d\left(gx,Sy\right)\right)\\&\leq\Phi\left(\frac{1}{8}max\left\{d\left(x,y\right),d\left(x,gx\right),d\left(y,Sy\right),\frac{1}{4}d\left(x,Sy\right),\frac{1}{4}d\left(gx,y\right)\right\}\right).\end{aligned}$$

The proof of (1) is obvious with $b=2$. To prove part (2), since $gC=\{1,4\}\subseteq D$ and $SD=\{1\}\subseteq C$, we can say that $C\cup D$ has b-cyclic representation with respect to the pair (g,S). To prove part (3), we have the following two cases:

Case I: Let $x=1,3$ and $y\in D$. Then $g\left(x\right)=1$ and $S\left(y\right)=1$ and hence $\Phi\left(d\left(gx,Sy\right)\right)=0$. Thus we have

$$\begin{aligned}&\Phi\left(2d\left(gx,Sy\right)\right)\\&\leq\Phi\left(\frac{1}{8}max\left\{d\left(x,y\right),d\left(x,gx\right),d\left(y,Sy\right),\frac{1}{4}d\left(x,Sy\right),\frac{1}{4}d\left(gx,y\right)\right\}\right).\end{aligned}$$

Case II: Let $x=5$ and $y\in D\{1,2\}$. Then $g\left(x\right)=4$ and $S\left(y\right)=1$. Hence $\Phi\left(2d\left(gx,Sy\right)\right)=\Phi\left(2d\left(4,1\right)\right)=\Phi\left(2\right)=\frac{1}{2}$ and $d\left(x,y\right)=10$. Thus,

$$\Phi\left(2d\left(gx,Sy\right)\right)=\frac{1}{2}\leq\frac{5}{8}=\Phi\left(\frac{1}{8}d(x,y)\right)$$
$$\leq\Phi\left(\frac{1}{8}max\left\{d\left(x,y\right),d\left(x,gx\right),d\left(y,Sy\right)\frac{1}{4}d\left(x,Sy\right),\frac{1}{4}d\left(gx,y\right)\right\}\right)$$
$$=\Phi\left(\frac{5}{2}\right).$$

Similarly, we can deal with the case $x=5$ and $y=4$. Thus g and S satisfy all the hypothesis of Theorem 2.2. Hence g and S have a common fixed point. Here 1 is the common fixed point of g and S.

References

[1] M. Abbas, W. Shatanawi, S. Farooq and Z.D. Mitrovic, *On a JH-operators pair of type (A) with applications to integral equations*, J. Fixed Point Theory Appl., **22** (2020), Article number 72.

[2] K. Abodayeh, T. Qawasmeh, W. Shatanawi and A. Tallafha, *$E_\phi-$contraction and some fixed point results via modified -distance mappings in the frame of complete quasi metric spaces and applications,* Inter. J. Elect. Comput. Eng., **10** (2020), 3839–3853.

[3] E. Ameer, H. Aydi, H.A. Hammad, W. Shatanawi and N. Mlaiki, *On $(\phi,\psi)-$metric spaces with applications,* Symmetry, **12** (2020), Article number 1459.

[4] H.Aydi, E. Karapinar and W. Shatanawi, *Coupled fixed point results for (ψ,ϕ)-weakly contractive contractive condition in ordered partial metric space,* Comput. Math. Appl., **62** (2011), 4449–4460.

[5] H. Aydi, W. Shatanawi, M, Postolache, Z. Mustafa and N. Tahat, *Theorems for Boyd-Wong-type contractions in ordered metric spaces,* Abstr. Appl. Anal., **2012** (2012), Article number 359054.

[6] A. Bakhtin, *The contraction mapping principle in almost metric spaces*, Funct. Anal., **30** (1989), 26–37.

[7] A. Bataihah, W. Shatanawi, T. Qawasmeh and R. Hatamleh, *On H-Simulation functions and fixed point results in the setting of wt-distance mappings with application on matrix equations,* Mathematics, **8** (2020), Article number 837.

[8] A. Bataihah, W. Shatanawi and A. Tallafha, *Fixed point results with simulation functions,* Nonlinear Funct. Anal. Appl., **25** (2020), 13–23.

[9] A. Bataihah, A. Tallafha and W. Shatanawi, *Fixed point results with $\Omega-$distance by utilizing simulation functions,* Italian J. Pure and Appl. Math., **43** (2020), 185–196.

[10] Y.J. Cho, B.E. Rhoades, R. Saadait, B. Samet and W. Shatanawi *Nonlinear coupled fixed point theorems in ordered generalized metric space with integral type,* Fixed Point Theory Appl., **2012**. Article ID (2012)

[11] S. Czerwik, *Contraction mappings in b-metric spaces,* Acta Math. Inform. Univa. Ostra., **1** (1993), 5–11.

[12] S. Czerwik, *Nonlinear set-valued contraction mappings in b-metric space,* Atti Sem. Mat. Uniu. Modena, **46** (1998), 263–276.

[13] U. Kadak, *On the classical sets of sequences with fuzzy b-metric,* Gen. Math. Notes. **23**(1) (2014), 89–108.

[14] N. Kir and H. Kiziltun, *On some well Know fixed point theorems in b-metric space,* Turk. J. Anal. Number Theory, **1** (2013), 13–16.
[15] W.A. Kirk, S.P. Srinavasan and P. Veeramanyi, *Fixed points for mapping satisfying cyclical conditions,* Fixed Point Theory Appl., **4** (2003), 79–89.
[16] A. Khan, T. Abdeljawad, W. Shatanawi and H. Khan, *Fixed point theorems for Quadruple self-mappings satisfying integral type inequalities,* Filoma, **34**(3) (2020), 905–917, https://doi.org/10.2298/FIL2003905K
[17] M.S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distances between the points,* Bull. Aust. Math. Soc., **30** (1984), 1–9.
[18] K. Kukic, W. Shatanawi and M.G. Filipovic, *Khan and Ciric contraction princples in almost b-metric space,* U.P.B. Sci. Bull., Series A, **82**, Iss. 1, (2020).
[19] A. Mukheimer, N. Mlaiki, K. Abodayeh and W.Shatanawi, *New theorems on extended b-metric spaces under new contractions,* Nonlinear Anal, Model. Con., **24**(6) (2019), 870-883.
[20] M. Pcurar and I.A. Rus, *Fixed point theory for cyclic ϕ-contractions,* Nonlinear Anal., **72** (2010), 1181–1187.
[21] G. Petruel, *Cyclic representations and periodic points,* Stud. Univ. Babe-Bolyai, Math., **50** (2005), 107–112.
[22] H. Qawaqneh, M.S.M. Noorani and W. Shatanawi, *Fixed point theorems for $(\alpha, k, \theta)-$contractive multi-valued mapping in b-metric space and applications,* Int. J. Math. Comput. Sci., **14**(1) (2019), 263-283.
[23] H. Qawaqneh, M.S.M. Noorani, S. Shatanawi, H. Aydi and H. Alsamir, *Fixed point results for multi-valued contractions in b-metric spaces and an application,* Mathematics, **7** (2019), Article number 132.
[24] T. Qawasmeh, A. Tallafha and W. Shatanawi, *Fixed and common fixed point theorems through modified $\omega-$distance mappings,* Nonlinear Funct. Anal. Appl., **24** (2019), 221–239.
[25] T. Qawasmeh, A. Tallafha and W. Shatanawi, *Fixed point theorems through modified w-distance and application to nontrivial equations,* Axioms, **8** (2019), Article Number 57.
[26] K.P.R. Rao, W. Shatanawi, G.N.V. Kishore, K. Abodayeh and D.R. Prasad, *Existeness and uniqeness of Suzuki type results in $S_b$metric spaces with applications to integral equations,* Nonlinear Funct. Anal. Appl., **23** (2018), 225–245.
[27] J.R. Roshan, V. Parvaneh, S. Sedghi, N. Shobkolaei and W. Shatanawi, *Common fixed points of almost generalized $(\psi, \phi)_s$-contractive mappings in ordered bmetric spaces,* Fixed Point Theory Appl., **2013**:159 (2013).
[28] S. Sedghi, N. Shobkolaei, J. Rezaei Roshan and W. Shatanawi, *Coupled fixed point theorems in G_b-metric spaces,* Matematicki Vesnik, **66**(2) (2014), 190–201.
[29] W. Shatnawi and Postolache Mihai, *Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces,* A springer open journal, (2013).
[30] W. Shatanawi, and S. Manro, *Fixed point results for cyclic (ψ, ϕ, A, B)-contraction in partial metric space,* Fixed point Theory Appl., **2012** (2012). Article ID. 165.
[31] W. Shatanawi and B. Samet, *On (ψ, ϕ)-weakly contractive condition in partially ordered metric space,* Comput. Math. Appl., **62** (2011), 3204–3214.
[32] W. Shatanawi, *Some coincidence point result in cone metric space*, Math. Comput .Modle., **55** (2012), 2023–2028.
[33] W. Shatanawi, *Some fixed point theorems in ordered G-metric space and applications,* Abstr. Appl. Anal., **2011** (2011), Article ID 126205.

[34] W. Shatanawi, *On w-compatible mappings and common coupled coincidence point in cone metric spaces,* Appl. Math. Lett., **25** (2012), 925-931.

[35] W. Shatanawi, *Some fixed point results for a generalized ψ-weak contraction mappings in orbitally metric spaces,* Chaos, Solitons and Fractals, **45** (2012), 520-526.

[36] W. Shatanawi, A. Bataihah and A. Tallafha, *Four-step iteration scheme to approximate fixed point for weak contractions,* Comput. Materials Continua CMC, **64** (2020), 1491–1504.

[37] W. Shatanawi, E. Karapnar and H. Aydi, *Coupled coincidence points in partially ordered cone metric spaces with a c-distance,* J. Appl. Math., **2012** (2012), Article number 312078.

[38] W. Shatanawi, Z. Mustafa and N. Tahat, *Some coincidence point theorems for nonlinear contraction in ordered metric spaces,* Fixed Point Theory Appl., **2011** (2011), Article number 68.

[39] W. Shatanawi and A. Pitea, *Fixed and coupled fixed point theorems of Ω-distance for nonlinear contraction,* Fixed Point Theory Appl., **2013**(11) (2013), DOI: 10.1186/1687-1812-2013-275.

[40] W. Shatanawi, A. Pitea and R. Lazovi, *Contraction conditions using comparison functions on b-metric spaces,* Fixed Point Theory Appl., **2013**:120 (2013).

[41] W. Shatanawi and M. Postolache, *Common fixed point theorems for dominating and weak annihilator mappings in ordered metric spaces,* Fixed Point Theory Appl., **2013** (2013), Article number 271.

[42] W. Shatanawi, V.C. Rajic, S.C. Radenovic and A. Al-Rawashdeh, *Mizoguchi-Takahashi-type theorems in tvs-cone metric spaces,* Fixed Point Theory Appl., **2012** (2012), Article number 106.