Nonlinear Functional Analysis and Applications
Vol. 26, No. 2 (2021), pp. 225-236
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.02.01
http://nfaa.kyungnam.ac.kr/journal-nfaa

PARAMETRIC EQUATIONS OF SPECIAL CURVES LYING ON A REGULAR SURFACE IN EUCLIDEAN 3-SPACE

Abderrazzak El Haimi[1] and Amina Ouazzani Chahdi[2]

[1]Laboratory of Analysis, Algebra and Applications (L3A), Faculty of Sciences Ben M'sik
Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco
e-mail: `abderrazzakelhaimi@gmail.com`

[2]Laboratory of Analysis, Algebra and Applications (L3A), Faculty of Sciences Ben M'sik
Hassan II University of Casablanca, P.O Box 7955 Sidi Othman, Casablanca, Morocco
e-mail: `ouazzaniamina@hotmail.com`

Abstract. In this paper, we determine position vector of a line of curvature of a regular surface which is relatively normal-slant helix, with respect to Darboux frame. Then, a vector differential equation is established by means Darboux formulas, in the case of the geodesic torsion is vanishes. In terms of solution, we determine the parametric representation of a line of curvature which is relatively normal-slant helix, with respect to standard frame in Euclidean 3-space. Thereafter, we apply this result to find the position vector of a line of curvature which is isophote curve.

1. Introduction

Curves theory is an important branch in the differential geometry studies. We have a lot of special curves such as circular helices, general helices, slant helices, k-slant helices etc. Characterizations of these special curves are heavily studied for a long time and are still studies. We can see the applications of helical structures in nature and mechanic tools. In the field of computer aided design and computer graphics, helices can be used for the tool path

[0]Received May 2, 2020. Revised September 14, 2020. Accepted October 13, 2020.
[0]2010 Mathematics Subject Classification: 53A04.
[0]Keywords: Darboux frame, line of curvature, relatively normal-slant helix, isophote curve.
[0]Corresponding author: A. E. Haimi(`abderrazzakelhaimi@gmail.com`).

description, the simulation of kinematic motion or design of highways. Also we can see the helix curve or helical structure in fractal geometry.

In a recent paper, Doğan and Yayli [3] study isophote curves and their characterizations in Euclidean 3-space. An isophote curve is defined as a curve on a surface whose unit normal field restricted to the curve makes a constant angle with a fixed direction. They also obtain the axis of an isophote curve. In 2017, Macit et al. [7] have defined a relatively normal-slant helix on a surface by using the Darboux frame (T, V, U) along the curve whose vector field V makes a constant angle with a fixed direction.

The determining of the position vector of some different curves according to the intrinsic equations $\kappa = \kappa(s)$ and $\tau = \tau(s)$ (where κ and τ are the curvature and torsion of the curve) is considered as a one of important subjects. Recently, the parametric representation of genral helices [6, 10] and slant helices and slant slant helices as an important special curves in euclidean space E^3 are deduced in [1, 2, 4].

In this work, first, we establish position vector of a line of curvature which is relatively normal-slant helix with respect to Darboux frame. Second, we use vector differential equations established by means Darboux frame in Euclidean space E^3 to determine position vectors of a line of curvature which is relatively normal-slant helix in terms of the normal curvature and geodesic curvature in E^3. Then, we can deduce the parametric representation of a line of curvature which is isophote curve.

2. Relatively normal-slant and line of curvature lying on a regular surface

In this section, we give the definition and a characterization of relatively normal-slant helix as well as a line of curvature lying on a regular surface, and we deduce an immediate property in the case where the curve satisfies the two conditions.

Let M be a regular surface, and $\varphi : I \subset \mathbb{R} \longrightarrow M$ be a regular curve with arc-length parametrization. The Frenet frame along the curve φ is denoted by (T, N, B, κ, τ), where T is unit tangent vector, N is principal normal vector, B is the binormal vector, κ and τ are the curvature and the torsion of φ, respectively. On the other hand, if we denote the Darboux frame along the curve φ by (T, V, U), we have the derivative formula of the Darboux frame as:

$$\begin{cases} T' = \kappa_g V + \kappa_n U, \\ V' = -\kappa_g T + \tau_g U, \\ U' = -\kappa_n T - \tau_g V, \end{cases} \tag{2.1}$$

where T is the unit tangent vector of the curve φ, U is the unit normal vector of the surface restricted to the curve φ, V is the unit vector given by $V = U \times T$, and $\kappa_g, \kappa_n, \tau_g$ denote the geodesic curvature, normal curvature, geodesic torsion of the curve φ, respectively [8].

The relations between geodesic curvature, normal curvature, geodesic torsion and κ, τ are given as follows:

$$\begin{cases} \kappa_n = \kappa \cos(\phi), \\ \kappa_g = \kappa \sin(\phi), \\ \tau_g = \tau + \dfrac{d\phi}{ds}, \end{cases} \tag{2.2}$$

where ϕ is the angle between the vectors N and U.

Remark 2.1. ([9]) For a curve φ lying on a surface, the following are well known:

(1) φ is a geodesic curve if the geodesic curvature κ_g vanishes.
(2) φ is an asymptotic line if the normal curvature κ_n vanishes.
(3) φ is a line of curvature if the geodesic torsion τ_g vanishes.

Definition 2.2. Let φ be a unit speed curve lying on a regular surface and (T, V, U) be the Darboux frame along φ. The curve φ is called a relatively normal-slant helix if the vector field V of φ makes a constant angle with a fixed direction, that is, there exists a fixed unit vector d and a constant angle θ such that

$$\langle V, d \rangle = \cos(\theta). \tag{2.3}$$

Theorem 2.3. ([7]) *A unit speed curve φ on a surface with $(\kappa_g, \tau_g) \neq (0,0)$ is a relatively normal-slant helix if and only if*

$$\sigma_r = \frac{1}{\left(\tau_g^2 + \kappa_g^2\right)^{\frac{3}{2}}} \left(\tau_g' \kappa_g - \kappa_g' \tau_g - \kappa_n \left(\tau_g^2 + \kappa_g^2\right)\right) \tag{2.4}$$

is constant.

From the above theorem and characterization of general helix $\left(\frac{\tau}{\kappa} \text{ is constant}\right)$ and slant helix $\left(\frac{\kappa^2}{(\kappa^2+\tau^2)^{\frac{3}{2}}} \left(\frac{\tau}{\kappa}\right)' \text{ is constant}\right)$ [5, 10] the following results follow:

Corollary 2.4. ([7]) *Let φ be a curve lying on a regular surface $\sum$:*

(1) *If φ is an asymptotic curve on $\sum$ with $\kappa_g \neq 0$, then φ is a relatively normal-slant helix on $\sum$ if and only if φ is a slant helix.*
(2) *If φ is a geodesic curve on $\sum$ with $\tau_g \neq 0$, then φ is a relatively normal-slant helix on $\sum$ if and only if φ is a general helix.*

(3) *If φ is a line of curvature on $\sum$ with $\kappa_g \neq 0$, then φ is a relatively normal-slant helix on $\sum$ if and only if* $\dfrac{\kappa_n}{\kappa_g}$ *is constant.*

According to [1, 2] and Corollary 2.4, we know the parametric representation of an asymptotic (resp. geodesic) curve on a regular surface, which is relatively normal-slant helix. In this work, we propose to give the position vector of a line of curvature which is relatively normal-slant helix with respect to Darboux frame and standard frame, respectively.

First, we give the following result:

Corollary 2.5. *Let φ be a line of curvature lying on a regular surface, which is a relatively normal-slant helix, with $\kappa_g \neq 0$. Then φ is a plane curve and its axis, noted d, belongs to the plane perpendicular to the vector tangent T.*

Proof. The fact that φ is a line of curvature and also a relatively normal-slant helix, we have $\tau_g = 0$ and $\frac{\kappa_n}{\kappa_g}$ is constant. By means formulas (2.2), we get that the torsion $\tau = 0$. On the other hand, differentiating the equation (2.3), and using the derivative formula of Darboux frame (2.1), we obtain the result as desired. □

3. Position vector of a line of curvature of a regular surface which is relatively normal-slant helix, with respect to Darboux frame

Theorem 3.1. *The position vector $\varphi(s)$ of a line of curvature of a regular surface with $\kappa_g \neq 0$, which is relatively normal-slant helix, with respect to Darboux frame (T, V, U) is given by :*

$$\varphi(s) = \alpha(s)\, T + \left(-\int \kappa_g \alpha(s)\, ds + c_1\right) V + \left(-\int \kappa_n \alpha(s)\, ds + c_2\right) U, \quad (3.1)$$

where

$$\begin{aligned} &\alpha(s) \\ &= \cos\left(\sqrt{1+\sigma_{rc}^2}\int \kappa_g ds\right)\left(c_3 - \tfrac{1}{\sqrt{1+\sigma_{rc}^2}}\int \tfrac{d}{ds}\left(\tfrac{1}{\kappa_g}\right)\sin\left(\sqrt{1+\sigma_{rc}^2}\int \kappa_g ds\right) ds\right) \\ &\quad + \sin\left(\sqrt{1+\sigma_{rc}^2}\int \kappa_g ds\right)\left(c_4 + \tfrac{1}{\sqrt{1+\sigma_{rc}^2}}\int \tfrac{d}{ds}\left(\tfrac{1}{\kappa_g}\right)\cos\left(\sqrt{1+\sigma_{rc}^2}\int \kappa_g ds\right) ds\right), \end{aligned}$$

while c_1, c_2, c_3, c_4 are arbitrary constants, and $\sigma_{rc} = \dfrac{\kappa_n}{\kappa_g}$.

Proof. Let $\varphi = \varphi(s)$ be a line of curvature, that is, $\tau_g = 0$, lying on a regular surface in Euclidean 3-space, which is relatively normal-slant helix, that is, $\frac{\kappa_n}{\kappa_g}$

is constant. Then, we may express its position vector as follows:

$$\varphi(s) = \alpha(s) T(s) + \beta(s) V(s) + \gamma(s) U(s), \tag{3.2}$$

where α, β and γ are differentiable functions of $s \in I \subset \mathbb{R}$. Differentiating the above equation with respect to s and using the derivative formula of Darboux frame (2.1), we get the following:

$$\begin{cases} \alpha' - \kappa_g \beta - \kappa_n \gamma = 1, \\ \beta' + \alpha \kappa_g = 0, \\ \gamma' + \alpha \kappa_n = 0. \end{cases} \tag{3.3}$$

By means of the change of variables $t = \int \kappa_g ds$, the system (3.3) becomes:

$$\begin{cases} \dot{\alpha} - \beta - \sigma_{rc} \gamma = \frac{1}{\kappa_g}, \\ \dot{\beta} + \alpha = 0, \\ \dot{\gamma} + \sigma_{rc} \alpha = 0, \end{cases} \tag{3.4}$$

where $\sigma_{rc} = \dfrac{\kappa_n}{\kappa_g}$ and dot denote the derivative with respect to t. The second and third equation of (3.4) leads to

$$\begin{aligned} \beta(t) &= -\int \alpha(t)\, dt + c_1, \\ \gamma(t) &= -\int \sigma_{rc} \alpha(t)\, dt + c_2, \end{aligned} \tag{3.5}$$

where c_1, c_2 are arbitrary constants. Differentiating the first equation of (3.4) and using the second and the third equation of (3.4), we get the following equation:

$$\ddot{\alpha} + \left(1 + \sigma_{rc}^2\right) \alpha = \frac{d}{dt}\left(\frac{1}{\kappa_g}\right). \tag{3.6}$$

The general solution of equation (3.6) is

$$\begin{aligned} \alpha(t) = \; & \cos\left(\sqrt{1+\sigma_{rc}^2}\, t\right)\left(c_3 - \tfrac{1}{\sqrt{1+\sigma_{rc}^2}} \int \tfrac{d}{dt}\left(\tfrac{1}{\kappa_g}\right) \sin\left(\sqrt{1+\sigma_{rc}^2}\, t\right) dt\right) \\ & + \sin\left(\sqrt{1+\sigma_{rc}^2}\, t\right)\left(c_4 + \tfrac{1}{\sqrt{1+\sigma_{rc}^2}} \int \tfrac{d}{dt}\left(\tfrac{1}{\kappa_g}\right) \cos\left(\sqrt{1+\sigma_{rc}^2}\, t\right) dt\right), \end{aligned} \tag{3.7}$$

where c_3, c_4 are arbitrary constants. Setting $t = \int \kappa_g ds$ and substituting equation (3.7) and (3.5) into (3.2) we get equation (3.1) which completes the proof. □

As a consequence of the above theorem we have the following corollary.

Corollary 3.2. *The position vector $\varphi(s)$ of a line of curvature lying on a regular surface with κ_g a non-zero constant, which is relatively normal-slant helix, with respect to Darboux frame (T, V, U) is given by*

$$\varphi(s) = \alpha(s) T + \beta(s) V + \gamma(s) U,$$

for

$$\begin{aligned}\alpha(s) &= c_3\cos\left(\kappa_g\sqrt{1+\sigma_{rc}^2}s\right)+c_4\sin\left(\kappa_g\sqrt{1+\sigma_{rc}^2}s\right),\\ \beta(s) &= \frac{-1}{\sqrt{1+\sigma_{rc}^2}}\left(c_3\sin\left(\kappa_g\sqrt{1+\sigma_{rc}^2}s\right)-c_4\cos\left(\kappa_g\sqrt{1+\sigma_{rc}^2}s\right)\right)+c_1,\\ \gamma(s) &= \frac{-\sigma_{rc}}{\sqrt{1+\sigma_{rc}^2}}\left(c_3\sin\left(\kappa_g\sqrt{1+\sigma_{rc}^2}s\right)-c_4\cos\left(\kappa_g\sqrt{1+\sigma_{rc}^2}s\right)\right)+c_2,\end{aligned}$$

while c_1, c_2, c_3, c_4 *are arbitrary constants, and* $\sigma_{rc}=\frac{\kappa_n}{\kappa_g}$.

4. Position vector of a line curvature of a regular surface which is relatively normal-slant helix, with respect to standard frame

Theorem 4.1. *Let* $\varphi=\varphi(s)$ *be a line of curvature, that is,* $\tau_g=0$, *lying on a regular surface in Euclidean 3-space with* $\kappa_n\neq 0$, $\kappa_g\neq 0$. *Then, the vector* V *satisfies a vector differential equation of third order as follows*:

$$V'''(t)+\left(1+\sigma_{rc}^2\right)V'(t)-\frac{\sigma_{rc}'(t)}{\sigma_{rc}(t)}\left(V''(t)+V(t)\right)=0, \tag{4.1}$$

where $t=\int\kappa_g ds$ *and* $\sigma_{rc}=\frac{\kappa_n}{\kappa_g}$.

Proof. Let $\varphi=\varphi(s)$ be a line of curvature, that is, $\tau_g=0$, lying on a regular surface with $\kappa_n\neq 0$ and $\kappa_g\neq 0$. By means of the change of variables $t=\int\kappa_g ds$ in (2.1), we have the new Darboux equations as follows:

$$\left\{\begin{array}{l}\frac{dT}{dt}=V+\sigma_{rc}U,\\ \frac{dV}{dt}=-T,\\ \frac{dU}{dt}=-\sigma_{rc}T,\end{array}\right. \tag{4.2}$$

where $\sigma_{rc}=\frac{\kappa_n}{\kappa_g}$. Differentiating the second equation of (4.2) and using the first equation of (4.2), we obtain

$$\frac{d^2V(t)}{dt^2}=-V(t)-\sigma_{rc}(t)U(t). \tag{4.3}$$

Differentiating (4.3), we have

$$\frac{d^3V(t)}{dt^3}=-\frac{dV(t)}{dt}-\frac{d\sigma_{rc}(t)}{dt}U(t)-\sigma_{rc}(t)\frac{dU(t)}{dt}. \tag{4.4}$$

By substituting the second equation of (4.2) in the third equation of (4.2), we give

$$\frac{dU}{dt}=\sigma_{rc}\frac{dV(t)}{dt}. \tag{4.5}$$

If we substitute (4.5) and (4.3) in the equation (4.4) , we have a vector differential equation of third order (4.1) as desired. □

The solution of equation (4.1) gives a position vector of a line of curvature, however, in the case where this curve is also a relatively normal-slant helix, we have the following theorem:

Theorem 4.2. *The position vector $\varphi(s)$ of a line of curvature of a regular surface, with $\kappa_n \neq 0$, $\kappa_g \neq 0$, which is relatively normal-slant helix, is computed in the natural representation form with respect to standard frame (e_1, e_2, e_3) by*

$$\begin{cases} \varphi_1(s) = \dfrac{n}{m}\sqrt{1+\sigma_{rc}^2}\int \sin\left(\sqrt{1+\sigma_{rc}^2}\int \kappa_g ds\right) ds, \\ \varphi_2(s) = -\dfrac{n}{m}\sqrt{1+\sigma_{rc}^2}\int \cos\left(\sqrt{1+\sigma_{rc}^2}\int \kappa_g ds\right) ds, \\ \varphi_3(s) = c, \end{cases}$$

or in the parametric form

$$\begin{cases} \varphi_1(t) = \frac{n}{m}\sqrt{1+\sigma_{rc}^2}\int \frac{1}{\kappa_g}\sin\left(\sqrt{1+\sigma_{rc}^2}t\right) dt, \\ \varphi_2(t) = -\frac{n}{m}\sqrt{1+\sigma_{rc}^2}\int \frac{1}{\kappa_g}\cos\left(\sqrt{1+\sigma_{rc}^2}t\right) dt, \\ \varphi_3(t) = c, \end{cases}$$

where $t = \int \kappa_g ds$, $\sigma_{rc} = \frac{\kappa_n}{\kappa_g}$ and c is a constant, $m = \frac{n}{\sqrt{1-n^2}}$, $n = cos(\theta)$ and θ is the angle between the fixed straight line e_3 (axis of relatively normal-slant helix) and the vector V of the curve φ.

Proof. Let $s \longrightarrow \varphi(s)$ be the arc-length parametrization of a line of curvature lying on a regular surface with $\kappa_n \neq 0$ and $\kappa_g \neq 0$. As φ is a relatively normal-slant helix, then $\sigma_{rc} = \frac{\kappa_n}{\kappa_g}$ is constant. Therefore the Eq.(4.1) becomes

$$V'''(t) + \left(1+\sigma_{rc}^2\right)V'(t) = 0, \tag{4.6}$$

where $t = \int \kappa_g ds$.

If we write the vector V in (e_1, e_2, e_3) , as the following:

$$V(t) = V_1(t)\,e_1 + V_2(t)\,e_2 + V_3(t)\,e_3, \tag{4.7}$$

by reason of the curve φ is a relatively normal-slant helix, that is, the vector V makes a constant angle θ, with the constant vector called the axis of the relatively normal-slant helix, so, without loss of generality, we can take the axis of a relatively normal-slant helix parallel to e_3. Then

$$V_3 = \langle V, e_3\rangle = \cos(\theta) = n. \tag{4.8}$$

Also, the vector V is a unit vector, so the following condition is satisfied

$$V_1^2(t) + V_2^2(t) = 1 - n^2. \tag{4.9}$$

The general solution of equation (4.9) is given by

$$\begin{cases} V_1(t) = \dfrac{n}{m}\cos(\lambda(t)), \\ V_2(t) = \dfrac{n}{m}\sin(\lambda(t)), \end{cases} \tag{4.10}$$

where $m = \frac{n}{\sqrt{1-n^2}}$ and λ is an arbitrary function of t. Each one of the components of the vector V satisfies the equation (4.6). So, substituting the components $V_1(t)$ and $V_2(t)$ in the equation(4.6), we get the following differential equations of the function $\lambda(t)$

$$\left(\lambda''' - \lambda'^3 + \left(1 + \sigma_{rc}^2\right)\lambda'\right)\sin(\lambda) + 3\lambda'\lambda''\cos(\lambda) = 0, \tag{4.11}$$

$$\left(\lambda''' - \lambda'^3 + \left(1 + \sigma_{rc}^2\right)\lambda'\right)\cos(\lambda) - 3\lambda'\lambda''\sin(\lambda) = 0. \tag{4.12}$$

It is easy to prove that the above two equations lead to the following two equations

$$3\lambda'\lambda'' = 0, \tag{4.13}$$

$$\lambda''' - \lambda'^3 + \left(1 + \sigma_{rc}^2\right)\lambda' = 0. \tag{4.14}$$

As λ is not constant, then $\lambda' \neq 0$. The equation (4.14) becomes

$$-\lambda'^2 + \left(1 + \sigma_{rc}^2\right) = 0. \tag{4.15}$$

The general solution of the equation (4.15) is

$$\lambda = \sqrt{1 + \sigma_{rc}^2}\,t + c_0, \tag{4.16}$$

where c_0 is a constant of integration. The constant c_0 can be disappear if we change the parameter $\lambda \longrightarrow \lambda + c_0$. Now, the vector V take the following form

$$V(t) = \left(\frac{n}{m}\cos\left(\sqrt{1+\sigma_{rc}^2}\,t\right), \frac{n}{m}\sin\left(\sqrt{1+\sigma_{rc}^2}\,t\right), n\right). \tag{4.17}$$

On the other hand, as $\frac{d\varphi}{ds} = T$ and using the second equation of (4.2), we have

$$\varphi(t) = -\int \frac{1}{\kappa_g}\left(\frac{dV}{dt}\right)dt. \tag{4.18}$$

Substituting the solution (4.17) in the equation (4.18) and setting $t = \int \kappa_g ds$, which completes the proof. $\square$

Corollary 4.3. *Let φ be a line of curvature lying on a regular surface, with $\kappa_n \neq 0$ and $\kappa_g \neq 0$. We denote by (T, N, B) and (T, V, U), the Frenet frame and Darboux frame along the curve φ, respectively. If φ is a relatively normal-slant helix, that is, its vector V makes a constant angle with a fixed direction, noted d, then the vectors d and B are collinear.*

Proof. If φ is a line of curvature ($\tau_g = 0$) and relatively normal-slant helix $\left(\frac{\kappa_n}{\kappa_g} = constant\right)$, according to Corollary 2.5, φ belongs to a fixed plane (zero torsion), hence the binormal vector B is constant. According to Theorem 4.2, and as the curvature of φ is not zero, we have $e_3 = \pm B$, which completes the proof. □

Now, we may give the following lemmas for the special cases of a line of curvature lying on a regular surface which is relatively normal-slant helix.

Lemma 4.4. *The position vector $\varphi(s)$ of a line of curvature which is relatively normal-slant helix with $\kappa_g(s) = \kappa_g$, where $\kappa_g \in \mathbb{R}^*$, is expressed in the natural representation form, with respect to standard frame (e_1, e_2, e_3) by*

$$\begin{cases} \varphi_1(s) = -\dfrac{n}{m\kappa_g}\cos\left(\sqrt{1+\sigma_{rc}^2}\kappa_g s\right), \\ \varphi_2(s) = \dfrac{n}{m\kappa_g}\sin\left(\sqrt{1+\sigma_{rc}^2}\kappa_g s\right), \\ \varphi_3(s) = c, \end{cases}$$

where $\sigma_{rc} = \frac{\kappa_n}{\kappa_g}$, c is a constant, $m = \frac{n}{\sqrt{1-n^2}}$, $n = cos(\theta)$ and θ is the angle between the fixed straight line e_3 (axis of relatively normal-slant helix) and the vector V of the curve φ.

We can see a special example of such curve when $\kappa_n = 7, \kappa_g = 1, n = \frac{2}{3}$, in the Figure 1-(A).

Lemma 4.5. *The position vector $\varphi(s)$ of a line of curvature which is relatively normal-slant helix with $\kappa_n(s) = \frac{a}{s}$ and $\kappa_g(s) = \frac{b}{s}$, is expressed in the natural representation form, with respect to standard frame (e_1, e_2, e_3) by*

$$\begin{cases} \varphi_1(s) \\ = \dfrac{ns\sqrt{b^2+a^2}}{mb(b^2+a^2+1)}\left[\sin\left(\sqrt{b^2+a^2}\ln(s)\right) - \sqrt{b^2+a^2}\cos\left(\sqrt{b^2+a^2}\ln(s)\right)\right], \\ \\ \varphi_2(s) \\ = \dfrac{-ns\sqrt{b^2+a^2}}{mb(b^2+a^2+1)}\left[\cos\left(\sqrt{b^2+a^2}\ln(s)\right) + \sqrt{b^2+a^2}\sin\left(\sqrt{b^2+a^2}\ln(s)\right)\right], \\ \\ \varphi_3(s) = c, \end{cases}$$

where $\sigma_{rc} = \frac{\kappa_n}{\kappa_g}$, c is a constant, $m = \frac{n}{\sqrt{1-n^2}}$, $n = cos(\theta)$ and θ is the angle between the fixed straight line e_3 (axis of relatively normal-slant helix) and the vector V of the curve φ.

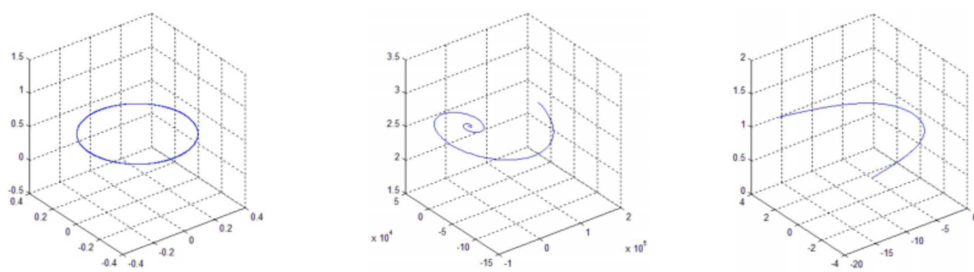
We can see a special example of such curve when $a = 2$, $b = 3$ and $n = \frac{1}{5}$, in the Figure 1-(B).

Lemma 4.6. *The position vector $\varphi(s)$ of a line of curvature which is relatively normal-slant helix with $\kappa_n(s) = \frac{\sqrt{1-a^2}}{1+s^2}$ and $\kappa_g(s) = \frac{a}{1+s^2}$, is expressed in the natural representation form, with respect to standard frame (e_1, e_2, e_3) by*

$$\begin{cases} \varphi_1(s) = \dfrac{n}{ma\cos(\omega)}, \\ \varphi_2(s) = \dfrac{n}{2ma}\ln\left(\dfrac{1-\sin(\omega)}{1+\sin(\omega)}\right), \\ \varphi_3(s) = c, \end{cases}$$

where $\omega = \arctan(s)$, c is a constant, $m = \frac{n}{\sqrt{1-n^2}}$, $n = cos(\theta)$ and θ is the angle between the fixed straight line e_3 (axis of relatively normal-slant helix) and the vector V of the curve φ.

We can see a special example of such curve when $n = \frac{1}{2}$ and $a = \frac{1}{2}$, in the Figure 1-(C).



(A) (B) (C)

Figure 1

5. Application for isophote curves

In this section we will deduce the position vector of special curves such as, line of curvature of a regular surface which is an isophote curve. First, we give the definition.

Definition 5.1. ([2]) Let φ be a unit speed curve lying on a regular surface and (T, V, U) be the Darboux frame along φ. The curve φ is called an isophote curve if the vector field U of φ makes a constant angle with a fixed direction, that is, there exists a fixed unit vector d and a constant angle η such that

$$\langle U, d\rangle = \cos(\eta). \tag{5.1}$$

Theorem 5.2. ([2]) *A unit speed curve φ on a regular surface with $(\kappa_n, \tau_g) \neq (0,0)$, is an isophote curve if and only if*

$$\mu_i(s) = \frac{1}{\left(\tau_g^2 + \kappa_n^2\right)^{\frac{3}{2}}} \left(\kappa_g \left(\tau_g^2 + \kappa_n^2\right) + \tau_g' \kappa_n - \kappa_n' \tau_g\right)(s). \tag{5.2}$$

is a constant function.

Corollary 5.3. ([2]) *Let φ be a curve lying on an oriented surface $\sum$:*

(1) *If φ is an asymptotic curve on $\sum$ with $\tau_g \neq 0$, then φ is an isophote curve on $\sum$ if and only if φ is a general helix.*
(2) *If φ is a geodesic curve on $\sum$ with $\kappa_n \neq 0$, then φ is an isophote curve on $\sum$ if and only if φ is a slant helix.*
(3) *If φ is a line of curvature on $\sum$ with $\kappa_n \neq 0$, then φ is an isophote curve on $\sum$ if and only if $\frac{\kappa_g}{\kappa_n}$ is constant.*

Similar to the previous section, we can also give the following characterizations for a line of curvature which is an isophote curve without proof.

Theorem 5.4. *Let $\varphi = \varphi(s)$ be a line of curvature lying on a regular surface in Euclidean 3-space with $\kappa_n \neq 0$ and $\kappa_g \neq 0$. Then the vector U satisfies a vector differential equation of third order as follows*

$$U'''(t) + \left(1 + \mu_{ic}^2\right) U'(t) - \frac{\mu_{ic}'(t)}{\mu_{ic}(t)} \left(U''(t) + U(t)\right) = 0, \tag{5.3}$$

where $t = \int \kappa_n ds$ and $\mu_{ic} = \frac{\kappa_g}{\kappa_n}$.

Theorem 5.5. *The position vector $\varphi(s)$ of a line of curvature lying on a regular surface with $\kappa_n \neq 0$, $\kappa_g \neq 0$, which is an isophote curve, is computed in the natural representation form, with respect to standard frame (e_1, e_2, e_3) by*

$$\begin{cases} \varphi_1(s) = \dfrac{n}{m} \sqrt{1 + \mu_{ic}^2} \int \sin\left(\sqrt{1 + \mu_{ic}^2} \int \kappa_n ds\right) ds, \\ \varphi_2(s) = -\dfrac{n}{m} \sqrt{1 + \mu_{ic}^2} \int \cos\left(\sqrt{1 + \mu_{ic}^2} \int \kappa_n ds\right) ds, \\ \varphi_3(s) = c, \end{cases} \tag{5.4}$$

or in the parametric form

$$\begin{cases} \varphi_1(t) = \dfrac{n}{m} \sqrt{1 + \mu_{ic}^2} \int \dfrac{1}{\kappa_n} \sin\left(\sqrt{1 + \mu_{ic}^2}\, t\right) dt, \\ \varphi_2(t) = -\dfrac{n}{m} \sqrt{1 + \mu_{ic}^2} \int \dfrac{1}{\kappa_n} \cos\left(\sqrt{1 + \mu_{ic}^2}\, t\right) dt, \\ \varphi_3(t) = c, \end{cases} \tag{5.5}$$

where $t = \int \kappa_n ds$, $\mu_{ic} = \frac{\kappa_g}{\kappa_n}$ and c is a constant, $m = \frac{n}{\sqrt{1-n^2}}$, $n = cos(\eta)$ and η is the angle between the fixed straight line e_3 (axis of isophote curve) and the vector U of the curve φ.

Data Availability: The data used to support the findings of this study are available from the corresponding author upon request. The articles used to support the findings of this study are included within the article and are cited at relevant places within the text as references.

References

[1] A.T. Ali, *Position vector of general helices in Euclidean 3-space,* Bull. Math. Anal. Appl., **3**(2) (2011), 198-205.

[2] A.T. Ali, *Position vector of slant helices in Euclidean 3-space,* J. Egyp. Math. Soc., **20**(1) (2012), 1-6.

[3] F. Dogan and Y. Yayl, *On isophote curves and their characterizations,* Turkish J. Math., **39** (2015), 650-664.

[4] A. Elhaimi, M. Izid and A. Ouazzani Chahdi, *Parametric Equations for space curves whose spherical images are slant helices,* J. Math. Research, Canadian Center of Science and Education, **11**(5) (2019), 82-88.

[5] S. Izumiya and N. Takeuchi, *New special curves and developable surfaces,* Turkish J. Math., **28** (2004), 153-163,.

[6] M.A. Lancret, Mémoire sur les courbes à double courbure, Mémoires présentés à l'Institut, (1806), 416-454.

[7] N. Macit and M. Duldul, *Relatively normal-slant helices lying on a surface and their characterizations,* Hacettepe J. Math. Statis., **46**(7) (2017), 397-408.

[8] B. O'Neill, *Elementary differential geometry,* Academic Press, 1996.

[9] B. Özcan and S. Yüce, *Special smarandache curves according to Darboux frame In E^3,* arXiv:1203.4830 [math.GM] (or arXiv:1203.4830v1 [math.GM] for this version (2012), 15 pages.

[10] D.J. Struik, *Lectures in Classical Differential Geometry,* Addison-Wesley, Reading, Ma, 1961.