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SOME L^q INEQUALITIES FOR POLYNOMIAL

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Abstract. Let $p(z)$be a polynomial of degree n. Then Bernstein's inequality [12,18] is

$$\max_{|z|=1}|p^{'}(z)| \leq n \max_{|z|=1}|(z)|.$$

For $q>0$, we denote

$$\|p\|_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}},$$

and a well-known fact from analysis [17] gives

$$\lim_{q\to\infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} = \max_{|z|=1}|p(z)|.$$

Above Bernstein's inequality was extended by Zygmund [19] into L^q norm by proving

$$\|p^{'}\|_q \leq n\|p\|_q, \quad q \geq 1.$$

Let $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, be a polynomial of degree n having no zero in $|z| < k, k \geq 1$. Then for $0 < r \leq R \leq k$, Aziz and Zargar [4] proved

$$\max_{|z|=R}|p^{'}(z)| \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}} \max_{|z|=r}|p(z)|.$$

In this paper, we obtain the L^q version of the above inequality for $q > 0$. Further, we extend a result of Aziz and Shah [3] into L^q analogue for $q > 0$. Our results not only extend some known polynomial inequalities, but also reduce to some interesting results as particular cases.

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1. Introduction

Let $p(z)$ be a polynomial of degree n. We define

$$\|p\|_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}}, \quad 0 < q < \infty. \tag{1.1}$$

If we let $q \to \infty$ in the above equality and make use of the well-known fact from analysis [17] that

$$\lim_{q \to \infty} \left\{ \frac{1}{2\pi} \int_0^{2\pi} |p(e^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} = \max_{|z|=1} |p(z)|,$$

we can suitably denote

$$\|p\|_\infty = \max_{|z|=1} |p(z)|.$$

Similarly, we can define $\|p\|_0 = \exp\left\{ \frac{1}{2\pi} \int_0^{2\pi} log|p(e^{i\theta})| d\theta \right\}$ and show that $\lim_{q \to 0^+} \|p\|_q = \|p\|_0$. It would be of further interest that by taking limits as $\lim_{q \to 0^+}$ that the stated result holding for $q > 0$, holds for $q = 0$ as well.

For $r > 0$, we denote by $M(p, r) = \max_{|z|=r} |p(z)|$.

A famous result due to Bernstein [12 or also see 18] states that if $p(z)$ is a polynomial of degree n, then

$$\|p'\|_\infty \leq n \|p\|_\infty. \tag{1.2}$$

Inequality (1.2) can be obtained by letting $q \to \infty$ in the inequality

$$\|p'\|_q \leq n \|p\|_q, \quad q > 0. \tag{1.3}$$

Inequality (1.3) for $q \geq 1$ is due to Zygmund [19]. Arestov [1] proved that (1.3) remains valid for $0 < q < 1$ as well. If we restrict ourselves to the class of polynomials having no zeros in $|z| < 1$, then inequality (1.2) and (1.3) can be respectively improved by

$$\|p'\|_\infty \leq \frac{n}{2} \|p\|_\infty \tag{1.4}$$

and

$$\|p'\|_q \leq \frac{n}{\|1 + z\|_q} \|p\|_q, \quad q > 0. \tag{1.5}$$

Inequality (1.4) was conjectured by Erdös and later verified by Lax [10], whereas, inequality (1.5) was proved by de-Bruijn [6] for $q \geq 1$. Rahman and Schmeisser [15] showed that (1.5) remains true for $0 < q < 1$.

As a generalization of (1.4), Malik [11] proved that if $p(z)$ does not vanish in $|z| < k$, $k \geq 1$, then

$$\|p'\|_\infty \leq \frac{n}{1 + k} \|p\|_\infty. \tag{1.6}$$

Under the same hypotheses of the polynomial $p(z)$, Govil and Rahman [9] extended inequality (1.6) to L^q norm by showing that

$$\|p'\|_q \leq \frac{n}{\|k+z\|_q}\|p\|_q, \quad q \geq 1. \tag{1.7}$$

It was shown by Gardner and Weems [8] and independently by Rather [16] that (1.7) also holds for $0 < q < 1$. Further, as a generalization of (1.6) Bidkham and Dewan [5] proved that if $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n having no zeros in $|z| < k$, $k \geq 1$, then

$$\|p'(rz)\|_\infty \leq \frac{n(r+k)^{n-1}}{(1+k)^n}\|p\|_\infty \quad \text{for } 1 \leq r \leq k. \tag{1.8}$$

As a generalization of (1.8), Aziz and Zargar [4] proved the following theorem.

Theorem 1.1. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, has no zeros in $|z| < k$, $k \geq 1$, then for $0 < r \leq R \leq k$,*

$$\|p'(Rz)\|_\infty \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\|p(rz)\|_\infty. \tag{1.9}$$

The result is best possible and equality in (1.9) holds for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

Further, as an improvement and generalization of (1.8), Aziz and Shah [3] proved the following theorem.

Theorem 1.2. *If $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, is a polynomial of degree n having no zeros in the disk $|z| < k$, $k \geq 0$, then for $0 < r \leq R \leq k$,*

$$\|p'(Rz)\|_\infty \leq \frac{nR^{\mu-1}(R^\mu + k^\mu)^{\frac{n}{\mu}-1}}{(r^\mu + k^\mu)^{\frac{n}{\mu}}}\left\{\|p(rz)\|_\infty - m\right\}, \tag{1.10}$$

where

$$m = \min_{|z|=k} |p(z)|.$$

The result is best possible and equality in (1.10) holds for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ.

2. Lemmas

For the proofs of the theorems, we require the following lemmas.

Lemma 2.1. ([14]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu,\ 1 \leq \mu \leq n$, *having no zeros in* $|z| < k,\ k \geq 1$, *then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^\mu} \max_{|z|=1} |p(z)|. \tag{2.1}$$

Lemma 2.2. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu,\ 1 \leq \mu \leq n$, *has no zeros in* $|z| < k$, $k > 0$ *then for* $0 < r \leq R \leq k$,

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt \tag{2.2}$$

and

$$M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt \leq \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}} M(p,r). \tag{2.3}$$

Proof. Since $p(z) \neq 0$ for $|z| < 1$, $p(tz) \neq 0$ for $|z| < \frac{1}{t}$ and so by Lemma 2.1,

$$\max_{|z|=1} t|p'(tz)| \leq \frac{n}{k^\mu + t^{-\mu}} \max |p(tz)|,$$

this gives

$$M(p',t) \leq \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t). \tag{2.4}$$

Now, for $0 \leq r < R \leq k$ and $\theta \in [0, 2\pi)$ we have,

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p'(te^{i\theta})|dt,$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R M(p',t)dt. \tag{2.5}$$

Using (2.4) in (2.5) we obtain

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt. \tag{2.6}$$

Which completes the first inequality (2.2).

Further, taking maximum over θ in (2.6), we have

$$M(p,R) \leq M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt. \tag{2.7}$$

Now let us denote the right hand side of inequality (2.7) by $\phi(R)$. Then

$$\phi^{'}(R) \leq \frac{nR^{\mu-1}}{R^{\mu}+k^{\mu}}\phi(R)$$

or

$$\phi^{'}(R) - \frac{nR^{\mu-1}}{R^{\mu}+k^{\mu}}\phi(R) \leq 0. \tag{2.8}$$

Multiplying both side of (2.8) by $(R^{\mu}+k^{\mu})^{\frac{-n}{\mu}}$, we obtain

$$\frac{d}{dR}(R^{\mu}+k^{\mu})^{\frac{-n}{\mu}}\phi(R) \leq 0,$$

which implies that $(R^{\mu}+k^{\mu})^{\frac{-n}{\mu}}\phi(R)$ is a nonincreasing function of R in $(0,k]$. Thus for $0 < r \leq R \leq k$,

$$\phi(r) \geq \left(\frac{k^{\mu}+r^{\mu}}{k^{\mu}+R^{\mu}}\right)^{\frac{n}{\mu}}\phi(R). \tag{2.9}$$

Since $\phi(r) = M(p,r)$ and using the value of $\phi(R)$ in (2.9), we get

$$M(p,r) \geq \left(\frac{k^{\mu}+r^{\mu}}{k^{\mu}+R^{\mu}}\right)^{\frac{n}{\mu}}\left[M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}M(p,t)dt\right].$$

This completes the proof of inequality (2.3). □

Lemma 2.3. ([14]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_{\nu}z^{\nu}$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zeros in* $|z| < k$, $k \geq 1$, *then on* $|z| = 1$

$$|q^{'}(z)| \geq k^{\mu}|p^{'}(z)|, \;\; \textit{where } q(z) = z^n\overline{p\left(\frac{1}{\bar{z}}\right)}. \tag{2.10}$$

Lemma 2.4. ([2]) *If* $p(z)$ *is a polynomial of degree* n *and* $q(z) = z^n\overline{p(\frac{1}{\bar{z}})}$, *then for each* α, $0 \leq \alpha < 2\pi$ *and* $q > 0$,

$$\int_0^{2\pi}\int_0^{2\pi} |q^{'}(e^{i\theta}) + e^{i\alpha}p^{'}(e^{i\theta})|^q d\theta d\alpha \leq 2\pi n^q \int_0^{2\pi} |p(e^{i\theta})|^q d\theta. \tag{2.11}$$

Lemma 2.5. ([7]) *Let* z *be complex and independent of* α, *where* α *is real, then for* $q > 0$,

$$\int_0^{2\pi} |1 + ze^{i\alpha}|^q d\alpha = \int_0^{2\pi} |e^{i\alpha} + |z||^q d\alpha. \tag{2.12}$$

Lemma 2.6. ([13]) *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *which does not vanish in* $|z| < k$, $k \geq 1$, *then*

$$\max_{|z|=1} |p'(z)| \leq \frac{n}{1+k^\mu}\left\{\max_{|z|=1}|p(z)| - \min_{|z|=k}|p(z)|\right\}. \tag{2.13}$$

Lemma 2.7. *Let* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *be a polynomial of degree* n *having no zeros in* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$,

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} m dt\right] \tag{2.14}$$

and

$$\begin{aligned} M(p,r) &+ \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} m dt \\ &\leq \left[M(p,r) - \left\{1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}\right\} m\right]\left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}, \end{aligned} \tag{2.15}$$

where $m = \min_{|z|=k} |p(z)|$.

Proof. By hypotheses, $p(z)$ has no zeros in $|z| < k$, therefore, the polynomial $F(z) = p(tz)$ has no zeros in $|z| < \frac{k}{t}$, $\frac{k}{t} \geq 1$, where $0 < t \leq k$. Since $\frac{k}{t} \geq 1$, by applying Lemma 2.6 to $F(z)$, it follows that

$$\max_{|z|=1} |F'(z)| \leq \frac{n}{1+\frac{k^\mu}{t^\mu}}\left\{\max_{|z|=1}|F(z)| - \min_{|z|=\frac{k}{t}}|F(z)|\right\},$$

this gives

$$\max_{|z|=t} |p'(z)| \leq \frac{nt^{\mu-1}}{t^\mu + k^\mu}\left\{\max_{|z|=t}|p(z)| - \min_{|z|=k}|p(z)|\right\}. \tag{2.16}$$

Now, for $0 < r \leq R \leq k$, and $0 \leq \theta < 2\pi$, we have

$$|p(Re^{i\theta}) - p(re^{i\theta})| = \left|\int_r^R e^{i\theta} p'(te^{i\theta})dt\right| \leq \int_r^R |p'(te^{i\theta})|dt,$$

from which it follows

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R |p'(te^{i\theta})|dt,$$

which implies

$$|p(Re^{i\theta})| \leq |p(re^{i\theta})| + \int_r^R M(p',t)dt. \tag{2.17}$$

Using (2.16) in (2.17), we obtain

$$|p(Re^{i\theta})| \le |p(re^{i\theta})| + n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} m dt\right], \quad (2.18)$$

which is the first inequality of Lemma 2.7.

Further, taking maximum over θ in (2.18), we have

$$M(p,R) \le M(p,r) + n\left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} m dt\right]. \quad (2.19)$$

Now let us denote the right hand side of inequality (2.19) by $\phi(R)$. Then

$$\phi^{'}(R) = \frac{nR^{\mu-1}}{R^\mu + k^\mu} M(p,R) - \frac{nR^{\mu-1}}{R^\mu + k^\mu} m. \quad (2.20)$$

Using $M(p,R) \le \phi(R)$, equality (2.20) can be written as

$$\phi^{'}(R) - \frac{nR^{\mu-1}}{R^\mu + k^\mu}\{\phi(R) - m\} \le 0. \quad (2.21)$$

Multiplying both sides of (2.21) by $(R^\mu + k^\mu)^{\frac{-n}{\mu}}$, we get

$$\phi^{'}(R)(R^\mu + k^\mu)^{\frac{-n}{\mu}} - n(\phi(R) - m)(R^\mu + k^\mu)^{\frac{-n}{\mu}-1} R^{\mu-1} \le 0,$$

which implies

$$\frac{d}{dR}\left\{(\phi(R) - m)(R^\mu + k^\mu)^{\frac{-n}{\mu}}\right\} \le 0. \quad (2.22)$$

From (2.22) we conclude that the function,

$$\{\phi(R) - m\}(R^\mu + k^\mu)^{\frac{-n}{\mu}}$$

is a nonincreasing function of R in $(0,k]$. Hence for $0 < r \le R \le k$,

$$\phi(r) \ge \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}} \phi(R) + \left\{1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}\right\} m. \quad (2.23)$$

Since $\phi(r) = M(p,r)$ and using the value of $\phi(R)$ in (2.23), we get

$$\begin{aligned} &M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} m dt \\ &\le \left[M(p,r) - \left\{1 - \left(\frac{k^\mu + r^\mu}{k^\mu + R^\mu}\right)^{\frac{n}{\mu}}\right\} m\right]\left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu}\right)^{\frac{n}{\mu}}. \end{aligned}$$

This completes the proof of inequality (2.15) of Lemma 2.7. $\square$

3. Main results

In this paper, first we extend Theorem 1.1 into L^q norm with the value of $k > 0$ instead of just $k \geq 1$. More precisely, we prove:

Theorem 3.1. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 1 \leq \mu \leq n$, *has no zeros in* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$, *and* $q > 0$,

$$\left\{ \int_0^{2\pi} |p'(Re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \leq \frac{n}{R} T_q \left\{ \int_0^{2\pi} \left| |p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t) dt \right|^q d\theta \right\}^{\frac{1}{q}}, \tag{3.1}$$

where

$$T_q = \left\{ \frac{1}{2\pi} \int_0^{2\pi} \left| \left(\frac{k}{R}\right)^\mu + e^{i\alpha} \right|^q d\alpha \right\}^{\frac{-1}{q}}.$$

Letting $q \to \infty$ *on both sides of* (3.1), *we obtain inequality* (1.9) *of Theorem* 1.1.

Proof. By hypothesis the polynomial $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$ has no zero in $|z| < k$, $k > 0$, therefore the polynomial $P(z) = p(Rz)$ has no zero in $|z| < \frac{k}{R}$, $\frac{k}{R} \geq 1$. By applying Lemma 2.3 to the polynomial $P(z)$, we have

$$A|P'(z)| \leq |Q'(z)| \text{ for } |z| = 1, \text{ where } Q(z) = z^n \overline{P(\frac{1}{\bar{z}})} \tag{3.2}$$

and

$$A = \left(\frac{k}{R}\right)^\mu \geq 1. \tag{3.3}$$

We can easily verify that for every real number α and $R \geq r \geq 1$,

$$|R + e^{i\alpha}| \geq |r + e^{i\alpha}|.$$

This implies for each $q > 0$,

$$\int_0^{2\pi} |R + e^{i\alpha}|^q d\alpha \geq \int_0^{2\pi} |r + e^{i\alpha}|^q d\alpha. \tag{3.4}$$

For point $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, for which $P'(e^{i\theta}) \neq 0$, we denote

$$R = \left| \frac{Q'(e^{i\theta})}{P'(e^{i\theta})} \right|,$$

and $r = A$, then from (3.2) and (3.3), $R \geq r \geq 1$.

Now, for each $q > 0$, by Lemma 2.5 and (3.4), we have

$$\begin{aligned}\int_0^{2\pi} |Q'(e^{i\theta}) + e^{i\alpha}P'(e^{i\theta})|^q d\alpha &= |P'(e^{i\theta})|^q \int_0^{2\pi} \left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})} + e^{i\alpha}\right|^q d\alpha \\ &= |P'(e^{i\theta})|^q \int_0^{2\pi} \left|\left|\frac{Q'(e^{i\theta})}{P'(e^{i\theta})}\right| + e^{i\alpha}\right|^q d\alpha \\ &\geq |P'(e^{i\theta})|^q \int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha. \qquad (3.5)\end{aligned}$$

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P'(e^{i\theta}) = 0$, inequality (3.5) trivially holds.

Now using (3.5) in Lemma 2.4, we obtain for each $q > 0$,

$$\int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \int_0^{2\pi} |P'(e^{i\theta})|^q d\theta \leq 2\pi n^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \qquad (3.6)$$

Since $P(z) = p(Rz)$,

$$P'(z) = Rp'(Rz).$$

Thus inequality (3.6) can be written as

$$\int_0^{2\pi} \left|\left(\frac{k}{R}\right)^\mu + e^{i\alpha}\right|^q d\alpha \int_0^{2\pi} |Rp'(Re^{i\theta})|^q d\theta \leq 2\pi n^q \int_0^{2\pi} |p(Re^{i\theta})|^q d\theta. \qquad (3.7)$$

Now applying inequality (2.2) of Lemma 2.2 in (3.7), we have

$$\begin{aligned}&\int_0^{2\pi} \left|\left(\frac{k}{R}\right)^\mu + e^{i\alpha}\right|^q d\alpha \int_0^{2\pi} |Rp'(Re^{i\theta})|^q d\theta \\ &\leq 2\pi n^q \int_0^{2\pi} \left\{|p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt\right\}^q d\theta \qquad (3.8)\end{aligned}$$

or equivalently

$$\begin{aligned}&\left\{\int_0^{2\pi} |Rp'(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}} \\ &\leq \frac{nT_q}{R} \left[\int_0^{2\pi} \left\{|p(re^{i\theta})| + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt\right\}^q d\theta\right]^{\frac{1}{q}}.\end{aligned}$$

This completes the proof of Theorem 3.1. □

Remark 3.2. Both the ordinary inequalities (1.9) and (1.10) of Theorems 1.1 and 1.2 are best possible for the polynomial $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ. It may be expected that inequality (3.1) of Theorem 3.1 is sharp for this polynomial. We discuss it as follows:

For $p(z) = (z^\mu+k^\mu)^{\frac{n}{\mu}}$, where n is a multiple of μ, inequality (3.1) of Theorem 3.1 equivalently takes

$$\left\{\int_0^{2\pi} \left|k^\mu + R^\mu e^{i\alpha}\right|^q d\alpha\right\}\left\{\int_0^{2\pi} \left|R^\mu e^{i\theta\mu} + k^\mu\right|^{q(\frac{n}{\mu}-1)} d\theta\right\}$$
$$\leq \left[\int_0^{2\pi} \left\{\left|r^\mu e^{i\theta\mu} + k^\mu\right|^{\frac{n}{\mu}} + (R^\mu + k^\mu)^{\frac{n}{\mu}} - (r^\mu + k^\mu)^{\frac{n}{\mu}}\right\}^q d\theta\right]. \tag{3.9}$$

In particular, if we set $k = R = r$, and $\mu = 1$, then inequality (3.9) assumes

$$\left\{\int_0^{2\pi} \left|1 + e^{i\alpha}\right|^q d\alpha\right\}\left\{\int_0^{2\pi} \left|e^{i\theta} + 1\right|^{q(n-1)} d\theta\right\} \leq \left\{\int_0^{2\pi} \left|e^{i\theta} + 1\right|^{nq} d\theta\right\}. \tag{3.10}$$

Now, we have for $p > -1$,

$$\int_0^{\frac{\pi}{2}} \cos^p \theta d\theta = \frac{\sqrt{\pi}\Gamma(\frac{p}{2} + \frac{1}{2})}{2\Gamma(\frac{p}{2} + 1)}. \tag{3.11}$$

For $q > 0$, by a simple calculation, we have

$$\int_0^{2\pi} \left|1 + e^{i\alpha}\right|^q d\alpha = 2^{q+2} \int_0^{\frac{\pi}{2}} \cos^q \alpha d\alpha,$$

which on using (3.11) gives

$$\int_0^{2\pi} \left|1 + e^{i\alpha}\right|^q d\alpha = 2^{q+1}\sqrt{\pi}\frac{\Gamma(\frac{q}{2} + \frac{1}{2})}{\Gamma(\frac{q}{2} + 1)}. \tag{3.12}$$

Applying equality (3.12) in inequality (3.10), we have

$$2^{q(n-1)+1}\sqrt{\pi}\frac{\Gamma(\frac{q(n-1)}{2} + \frac{1}{2})}{\Gamma(\frac{q(n-1)}{2} + 1)} \times 2^{q+1}\sqrt{\pi}\frac{\Gamma(\frac{q}{2} + \frac{1}{2})}{\Gamma(\frac{q}{2} + 1)} \leq 2^{nq+1}\sqrt{\pi}\frac{\Gamma(\frac{nq}{2} + \frac{1}{2})}{\Gamma(\frac{nq}{2} + 1)},$$

that is,

$$2\sqrt{\pi}\frac{\Gamma(\frac{q(n-1)}{2} + \frac{1}{2})}{\Gamma(\frac{q(n-1)}{2} + 1)} \times \frac{\Gamma(\frac{q}{2} + \frac{1}{2})}{\Gamma(\frac{q}{2} + 1)} \leq \frac{\Gamma(\frac{nq}{2} + \frac{1}{2})}{\Gamma(\frac{nq}{2} + 1)}. \tag{3.13}$$

Further, when $n = 3, q = 4$, inequality (3.13) becomes

$$2\sqrt{\pi}\frac{\Gamma(4 + \frac{1}{2})}{\Gamma(5)} \times \frac{\Gamma(2 + \frac{1}{2})}{\Gamma(3)} \leq \frac{\Gamma(6 + \frac{1}{2})}{\Gamma(7)}$$

which on simplification gives

$$10\pi \leq 11,$$

which is absurd. This shows that inequality (3.1) of Theorem 3.1 is not sharp.

Remark 3.3. Using $|p(re^{i\theta})| \leq M(p, r)$ in Theorem 3.1, we have the following result.

Corollary 3.4. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu, 1 \le \mu \le n$, *has no zeros in* $|z| < k$, $k > 0$ *then for* $0 < r \le R \le k$, *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \le \frac{n}{R} T_q \left| M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt \right|, \tag{3.14}$$

where T_q *is as defined in Theorem* 3.1.

Further, using inequality (2.3) of Lemma 2.2 in the inequality (3.11) of Corollary 3.4, we have the L^q version of Theorem 1.1, which has some interesting consequences as discussed below.

Corollary 3.5. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *has no zeros in* $|z| < k$, $k > 0$ *then for* $0 < r \le R \le k$ *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \le \frac{n}{R} T_q \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu} \right)^{\frac{n}{\mu}} M(p,r), \tag{3.15}$$

where T_q *is as defined in Theorem* 3.1.

Letting $q \to \infty$ in inequality (3.15) we get inequality (1.9) of Theorem 1.1. Further, if we let $\mu = 1$ and $r = 1$ in Corollary 3.5, it matches the L^q analogue of inequality (1.8) proved by Bidkham and Dewan [5].

In addition to the above, when $\mu = 1 = R = r$, Corollary 3.5 gives inequality (1.7) which is the L^q inequality of the famous inequality (1.6) due to Malik [11].

Further, we extend inequality (1.10) of Theorem 1.2 due to Aziz and Shah [3] to integral mean inequality. In fact, we obtain the following theorem.

Theorem 3.6. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \le \mu \le n$, *is a polynomial of degree* n *having no zeros in the disk* $|z| < k$, $k > 0$, *then for* $0 < r \le R \le k$, *and* $q > 0$,

$$\left\{ \int_0^{2\pi} |p^{'}(Re^{i\theta})|^q d\theta \right\}^{\frac{1}{q}} \tag{3.16}$$

$$\le \frac{n}{R} T_q \left\{ \int_0^{2\pi} \left| |p(re^{i\theta})| + n \left[\int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{t^{\mu-1}}{t^\mu + k^\mu} m dt \right] - m \right|^q d\theta \right\}^{\frac{1}{q}},$$

where T_q *is as in Theorem* 3.1 *and* $m = \min_{|z|=k} |p(z)|$. *Letting* $q \to \infty$ *on both sides of* (3.16), *we obtain inequality* (1.10) *of Theorem* 1.2.

Proof. Since the polynomial $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$ has no zero in $|z| < k$, $k > 0$, the polynomial $p(Rz)$ has no zero in $|z| < \frac{k}{R}$, $\frac{k}{R} \geq 1$.

Take $P(z) = p(Rz) + \alpha m$ where $|\alpha| < 1$ and $m = \min_{|z|=k} |p(z)$. By applying Lemma 2.3 to the polynomial $P(z)$, we have

$$A|P^{'}(z)| \leq |Q^{'}(z)| \text{ for } |z| = 1, \text{ where } Q(z) = z^n \overline{P(\frac{1}{\bar{z}})} \tag{3.17}$$

and

$$A = \left(\frac{k}{R}\right)^\mu \geq 1. \tag{3.18}$$

We can easily verify that for every real number α and $R \geq r \geq 1$,

$$|R + e^{i\alpha}| \geq |r + e^{i\alpha}|.$$

This implies for each $q > 0$,

$$\int_0^{2\pi} |R + e^{i\alpha}|^q d\alpha \geq \int_0^{2\pi} |r + e^{i\alpha}|^q d\alpha. \tag{3.19}$$

For point $e^{i\theta}$, $0 \leq \theta \leq 2\pi$, for which $P^{'}(e^{i\theta}) \neq 0$, we denote

$$R = \left| \frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})} \right|$$

and $r = A$, then from (3.17) and (3.18), $R \geq r \geq 1$.

Now, for each $q > 0$, by Lemma 2.5 and (3.19), we have

$$\begin{aligned}
\int_0^{2\pi} |Q^{'}(e^{i\theta}) + e^{i\alpha} P^{'}(e^{i\theta})|^q d\alpha &= |P^{'}(e^{i\theta})|^q \int_0^{2\pi} \left| \frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})} + e^{i\alpha} \right|^q d\alpha \\
&= |P^{'}(e^{i\theta})|^q \int_0^{2\pi} \left| \left| \frac{Q^{'}(e^{i\theta})}{P^{'}(e^{i\theta})} \right| + e^{i\alpha} \right|^q d\alpha \\
&\geq |P^{'}(e^{i\theta})|^q \int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha. \qquad (3.20)
\end{aligned}$$

For points $e^{i\theta}$, $0 \leq \theta < 2\pi$, for which $P^{'}(e^{i\theta}) = 0$, inequality (3.20) trivially holds.

Now using (3.20) in Lemma 2.4, we obtain for each $q > 0$,

$$\int_0^{2\pi} |A + e^{i\alpha}|^q d\alpha \int_0^{2\pi} |P^{'}(e^{i\theta})|^q d\theta \leq 2\pi n^q \int_0^{2\pi} |P(e^{i\theta})|^q d\theta. \tag{3.21}$$

Since $P(z) = p(Rz) + \alpha m$,

$$P^{'}(z) = R(p^{'}(Rz)).$$

Thus inequality (3.21) can be written as

$$\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^{\mu}+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta$$
$$\leq 2\pi n^q\int_0^{2\pi}|p(Re^{i\theta})+\alpha m|^q d\theta. \tag{3.22}$$

Now, in $|p(Re^{i\theta})+\alpha m|$, if we choose suitable argument of α, we have

$$|p(Re^{i\theta})+\alpha m|=|p(Re^{i\theta})|-|\alpha|m.$$

By letting $|\alpha|\to 1$, we obtain

$$|p(Re^{i\theta})+\alpha m|=|p(Re^{i\theta})|-m. \tag{3.23}$$

Using (3.23) in (3.22), we have

$$\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^{\mu}+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta$$
$$\leq 2\pi n^q\int_0^{2\pi}||p(Re^{i\theta})|-m|^q d\theta. \tag{3.24}$$

Now applying inequality (2.14) of Lemma 2.7 in (3.24), we have

$$\int_0^{2\pi}\left|\left(\frac{k}{R}\right)^{\mu}+e^{i\alpha}\right|^q d\alpha\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta$$

$$\leq 2\pi n^q\int_0^{2\pi}\left||p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}M(p,t)dt-\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}mdt-m\right|^q d\theta. \tag{3.25}$$

or equivalently

$$\left\{\int_0^{2\pi}|Rp^{'}(Re^{i\theta})|^q d\theta\right\}^{\frac{1}{q}}$$

$$\leq\frac{nT_q}{R}\left\{\int_0^{2\pi}\left||p(re^{i\theta})|+\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}M(p,t)dt-\int_r^R\frac{nt^{\mu-1}}{t^{\mu}+k^{\mu}}mdt-m\right|^q d\theta\right\}^{\frac{1}{q}}.$$

This completes the proof of Theorem 3.6. □

Remark 3.7. As is noticed earlier that inequality (1.10) of Theorem 1.2 is sharp for $p(z)=(z^{\mu}+k^{\mu})^{\frac{n}{\mu}}$ where n is a multiple of μ, we examine the sharpness of inequality (3.16) of Theorem 3.6 for this polynomial.

It is obvious that for $p(z) = (z^\mu + k^\mu)^{\frac{n}{\mu}}$ where n is a multiple of μ,

$$m = \min_{|z|=k} |p(z)| = 0$$

and hence by Remark 3.2, inequality (3.16) of Theorem 3.6 is not sharp.

Remark 3.8. Using $|p(re^{i\theta})| \leq M(p,r)$ in Theorem 3.6, we have the following result.

Corollary 3.9. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zeros in the disk* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$ *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \leq \frac{n}{R} T_q \left\{ \left| M(p,r) + \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} M(p,t)dt - \int_r^R \frac{nt^{\mu-1}}{t^\mu + k^\mu} m dt - m \right| \right\}, \quad (3.26)$$

where T_q *is as in Theorem* 3.1 *and* $m = \min_{|z|=k} |p(z)|$.

Further, using inequality (2.15) of Lemma 2.7 in the inequality (3.26) of Corollary 3.9, we have, the L^q version of Theorem 1.2:

Corollary 3.10. *If* $p(z) = a_0 + \sum_{\nu=\mu}^{n} a_\nu z^\nu$, $1 \leq \mu \leq n$, *is a polynomial of degree* n *having no zeros in the disk* $|z| < k$, $k > 0$, *then for* $0 < r \leq R \leq k$ *and* $q > 0$,

$$\|p^{'}(Rz)\|_q \leq \frac{n}{R} T_q \left(\frac{k^\mu + R^\mu}{k^\mu + r^\mu} \right)^{\frac{n}{\mu}} \{M(p,r) - m\}. \quad (3.27)$$

where T_q *is as in Theorem* 3.1 *and* $m = \min_{|z|=k} |p(z)|$.

Letting $q \to \infty$ in inequality (3.27), we get inequality (1.10) of Theorem 1.2. Further, if we let $\mu = 1$ and $r = 1$ in Corollary 3.10, we obtain an improvement in L^q version of inequality (1.8) proved by Bidkham and Dewan [5].

Also, when $\mu = 1 = R = r$ in Corollary 3.10, it gives an improvement of L^q inequality (1.7) due to Govil and Rahman [9] of the ordinary inequality (1.6) proved by Malik [11].

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