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APPROXIMATION OF ZEROS OF SUM OF MONOTONE MAPPINGS WITH APPLICATIONS TO VARIATIONAL INEQUALITY AND IMAGE RESTORATION PROBLEMS

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Abstract. In this paper, an inertial Halpern-type forward backward iterative algorithm for approximating solution of a monotone inclusion problem whose solution is also a fixed point of some nonlinear mapping is introduced and studied. Strong convergence theorem is established in a real Hilbert space. Furthermore, our theorem is applied to variational inequality problems, convex minimization problems and image restoration problems. Finally, numerical illustrations are presented to support the main theorem and its applications.

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1. Introduction

Let H be a real Hilbert space and let $T : H \to H$ be a map. The mapping T is called monotone if

$$\langle Tx - Ty, x - y\rangle \geq 0,\ \forall x, y \in H.$$

It is called α-inverse strongly monotone (α-ism) if

$$\langle Tx - Ty, x - y\rangle \geq \alpha\|Tx - Ty\|^2,\ \forall x, y \in H.$$

It is called maximal monotone if it is monotone, and in addition, the graph of T is not properly contained in the graph of any other monotone mapping.

Let $A : H \to H$ be an α-ism mapping, $B : H \multimap H$ be a set-valued maximal monotone mapping. A classical problem of interest in nonlinear operator theory is the following inclusion problem:

$$\text{find} \quad x \in H \quad \text{such that} \quad 0 \in (A + B)x. \tag{1.1}$$

Numerous problems in nonlinear analysis (for example, variational inequality problems, split feasibility problems, convex minimization problems, equilibrium problems) can be transformed in to the inclusion (1.1). In applications, some concrete problems in machine learning, signal processing, linear inverse problems and image restoration can be modeled as the inclusion (1.1).

In fact, if B is the *subdifferential,* $\partial f : H \multimap H$ of a proper, lower semicontinuous and convex function $f : H \to \mathbb{R} \cup \{\infty\}$, defined by

$$\partial f(u) := \{x \in H : f(y) - f(u) \geq \langle x, y - u\rangle,\ \forall y \in H\},$$

then the inclusion (1.1) is equivalent to the following problem:

$$\text{find} \quad u \in H \ \text{ such that } \ - Au \in Bu = \partial f(u), \tag{1.2}$$

that is,

$$f(y) - f(u) + \langle Au, y - u\rangle \geq 0,\ \forall y \in H.$$

Observe that if f is the *indicator function* of a nonempty closed and convex subset, say C, of H, problem (1.2) reduces to the so-called *variational inequality problem,* that is, find $u \in C$ such that

$$\langle Au, y - u\rangle \geq 0,\ \forall y \in C.$$

Iterative algorithms for approximating solution(s) of the inclusion (1.1) have been studied extensively by numerous authors (see, e.g., [3], [10], [11],[12], [13], [19], [21], [22], [26], [27], [28], [45], [46]). Assuming existence of solution, one of the classical methods for approximating solution(s) of (1.1) is the well-known *forward-backward splitting method* introduced independently by Lions and Mercier [30], and Passty [35] and studied extensively by Mercier [33], Gabay [23] and a host of other authors.

In a real Hilbert H, the forward-backward algorithm (FBA) for maximal monotone operators A and B is an iterative procedure that starts at a point $x_1 \in H$, and generates inductively the sequence $\{x_n\} \subset H$ by:

$$x_{n+1} = \big(I + \lambda_n B\big)^{-1}\big(I - \lambda_n A\big)x_n, \tag{1.3}$$

where $\{\lambda_n\}$ is a sequence of positive real numbers. Mercier and Gabay proved that if A^{-1} is *strongly monotone* with modulus $\alpha > 0$, and $\{\lambda_n\} \subset (0, 2\alpha)$, then, the sequence $\{x_n\}$ converges *weakly* to a solution of (1.1). Furthermore, if, in addition, A is strongly monotone, then $\{x_n\}$ converges strongly to the unique solution (see, e.g., [33]). Chen and Rockafellar [8] showed that if A is Lipschitz and $(A+B)$ is strongly monotone then the sequence $\{x_n\}$ converges strongly. Concerning the Lipschitz and strong monotonicity assumption on A, see, e.g., [33]. Due to its usefulness in applications, the problem of finding zeros of sum of two monotone operators in Hilbert spaces (problem (1.1)) is receiving a lot of research interest by a host of authors (see e.g., [21], [39], [40], [49]).

In 2012, Takahashi *et al.* [43] introduced and studied a generalization of the forward-backward splitting algorithm in real Hilbert spaces. They proved strong convergence of the sequence generated by their algorithm to a solution of (1.1).

Recently, Kitkuan *et al.* [28] introduced and studied a generalized Halpern-type forward-backward splitting algorithm in real Hilbert spaces. They proved the following theorem:

Theorem 1.1. *Let H be a real Hilbert space. Let $\alpha > 0$, $A : H \to H$ be an α-inverse strongly monotone mapping and $B : H \multimap H$ be a maximal monotone operator. Let $J^B_{\lambda_n} = (I + \lambda_n B)^{-1}$ be the resolvent of B for $\lambda_n > 0$. Suppose that*

$$\Omega := (A + B)^{-1}0 \neq \emptyset.$$

Let $u \in H$, $x_1 = x \in H$ and let $\{x_n\} \subset H$ defined by

$$\begin{cases} z_n = r_n x_n + (1 - r_n)J^B_{\lambda_n}(I - \lambda_n A)x_n, \\ y_n = s_n x_n + (1 - s_n)J^B_{\lambda_n}(I - \lambda_n A)z_n, \\ x_{n+1} = \alpha_n u + (1 - \alpha_n)y_n, \end{cases} \tag{1.4}$$

for all $n \in \mathbb{N}$, where $\{r_n\}, \{s_n\}, \{\alpha_n\} \subset (0, 1)$ satisfy the conditions:

(i) $\lim\limits_{n\to\infty} \alpha_n = 0$ *and* $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(ii) $0 < \liminf\limits_{n\to\infty} \lambda_n \leq \limsup\limits_{n\to\infty} \lambda_n < 2\alpha$;

(iii) $\liminf\limits_{n\to\infty}(1 - r_n)(1 - s_n) > 0$.

Then the sequence $\{x_n\}$ converges strongly to a point $z \in \Omega$.

In this paper, we are interested in the following split inclusion problem:

$$\text{find} \quad x \in H \text{ such that } 0 \in (A+B)x \text{ and } \quad Sx = x, \tag{1.5}$$

where $A : H \to H$ is an α-ism mapping, $B : H \multimap H$ is a set-valued maximal monotone mapping and $S : H \to H$ is a *nonexpansive* mapping (i.e., $\|Sx - Sy\| \leq \|x - y\|, \ \forall x, y \in H$).

This problem has been of interest to researchers in fixed point theory and applications over the years. Several hybrid algorithms have been proposed by many authors to solve problems of this form (see, e.g., [14], [15], [18], [20], [29], [34], [41], [42], [44], [47], [48]). In 2010, Takahashi *et al.* [44] introduced and studied a hybrid algorithm for approximating a solution of the inclusion (1.5) in real Hilbert spaces. They proved the following theorem:

Theorem 1.2. *Let C be a closed and convex subset of a real Hilbert space H. Let A be an α-inverse strongly monotone mapping of C into H and let B be a maximal monotone operator on H, such that the domain of B is included in C. Let $J_\lambda = (I + \lambda B)^{-1}$ be the resolvent of B for $\lambda > 0$ and let S be a nonexpansive mapping of C into itself, such that*

$$F(S) \cap (A+B)^{-1}0 \neq \emptyset.$$

Let $x_1 = x \in C$ and let $\{x_n\} \subset C$ be a sequence generated by

$$x_{n+1} = \beta_n x_n + (1 - \beta_n) S\big(\alpha_n x + (1 - \alpha_n) J_{\lambda_n}(x_n - \lambda_n A x_n)\big) \tag{1.6}$$

for all $n \in \mathbb{N}$, where $\{\lambda_n\} \subset (0, 2\alpha), \ \{\alpha_n\}, \ \{\beta_n\} \subset (0,1)$ satisfy

$$0 < a \leq \lambda_n \leq b < 2\alpha, \ \ 0 < c \leq \beta_n \leq d < 1,$$

$$\lim_{n\to\infty} (\lambda_n - \lambda_{n+1}) = 0, \ \ \lim_{n\to\infty} \alpha_n = 0, \quad \sum_{n=1}^{\infty} \alpha_n = \infty.$$

Then $\{x_n\}$ converges strongly to a point of $F(S) \cap (A+B)^{-1}0$.

It is well known that iterative algorithms for approximating zeros of monotone maps are generally slow. This is expected since monotone maps are, in general, not differentiable. In fact, monotone maps are, in general, not even continuous. Thus, fast converging algorithms such as the *Newton-Kantorovich* algorithm can not be used. Consequently, a lot of research effort is now devoted to improving speed of convergence of iterative algorithms for approximating zeros of monotone maps. One method that is now studied is to incorporate the *inertial extrapolation term* in the algorithms.

In [1], Alvarez introduced and studied an inertial *proximal point algorithm* in the context of convex minimization. Later, Attouch and Alvarez [2] considered its extension to maximal monotone operators. Using the idea of [36] and [2], Lorenz and Pock [31] introduced an inertial version of algorithm (1.3) in real Hilbert spaces. In [31], the authors showed numerically that the FBA (1.3) with inertial extrapolation step (accelerated version) converges faster than the unaccelerated version. Several alternatives or modifications of algorithm (1.3) with inertial extrapolation step have been proposed by many authors in real Hilbert spaces (see, for example, [4], [5], [7], [10], [16], [17], [25], [26], [34]).

Motivated by the results of Kitkuan *et al.* [28] and Takahashi *et al.* [44], in this paper, we introduce a new hybrid inertial Halpern-type forward-backward splitting method:

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = \gamma_n w_n + (1-\gamma_n)J^B_{\lambda_n}(I - \lambda_n A)w_n, \\ y_n = s_n w_n + (1-s_n)J^B_{\lambda_n}(I - \lambda_n A)z_n, \\ x_{n+1} = \tau_n u + \sigma_n w_n + \mu_n S y_n, \end{cases} \tag{1.7}$$

for approximating a solution of the inclusion (1.5) in a real Hilbert space. We prove strong convergence of the sequence generated by our algorithm and apply the convergence result obtained to variational inequality problem, convex minimization problem and image restoration problem.

Remark 1.3. Observe that setting $\alpha_n = \sigma_n = 0$, for all $n \in \mathbb{N}$ and $S = I$ in (1.7), we obtain the algorithm of Theorem 1.1, making our propose algorithm more general.

2. Preliminaries

The following definitions and lemmas will be needed in the proof of our main theorem.

Lemma 2.1. ([9]) *Let H be a real Hilbert space. Then the following identities hold:*

(i) $\|x+y\|^2 = \|x\|^2 + 2\langle y, x+y\rangle, \ \forall x, y \in H,$

(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \forall\ \lambda \in (0,1)$ *and* $x, y \in H$.

Remark 2.2. In the sequel we shall adopt the following notation:

$$T^{A,B}_\lambda := J^B_\lambda(I - \lambda A) = (I + \lambda B)^{-1}(I - \lambda A), \quad \lambda > 0.$$

Lemma 2.3. ([32]) *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone operator and $B : H \multimap H$ a maximal monotone operator. Then we have*

(i) *for* $\lambda > 0$, $F(T_\lambda^{A,B}) = (A+B)^{-1}0$,
(ii) *for* $0 < \lambda \le \epsilon$ *and* $x \in H$, $\|x - T_\lambda^{A,B}x\| \le 2\|x - T_\epsilon^{A,B}x\|$.

Lemma 2.4. ([32]) *Let H be a real Hilbert space. Let $A : H \to H$ be an α-inverse strongly monotone operator and $B : H \multimap H$ be a maximal monotone operator. Then for all $x, y \in B(0,r)$*

$$\|T_\lambda^{A,B}x - T_\lambda^{A,B}y\|^2 \le \|x-y\|^2 - \lambda(2\alpha - \lambda)\|Ax - Ay\|^2 - \|(I - J_\lambda)(I - \lambda A)x - (I - J_\lambda)(I - \lambda A)y\|^2.$$

Lemma 2.5. ([24]) *Let $\{d_n\}$ be a sequence of nonnegative real numbers such that*

$$d_{n+1} \le (1-\theta_n)d_n + \theta_n t_n \quad \text{and} \quad d_{n+1} \le d_n - \eta_n + \rho_n,$$

where $\{\theta_n\}$ is a sequence in $(0,1)$, $\{\eta_n\}$ is a sequence of of nonnegative real numbers, $\{t_n\}$ and $\{\rho_n\}$ are real sequences such that

(i) $\sum_{n=1}^{\infty} \theta_n = \infty$, (ii) $\lim_{n\to\infty} \rho_n = 0$,
(iii) $\lim_{k\to\infty} \eta_{n_k} = 0$ *implies* $\limsup_{k\to\infty} t_{n_k} \le 0$, *for any subsequence* $\{n_k\} \subset \{n\}$.

Then, $\lim_{n\to\infty} d_n = 0$.

Lemma 2.6. ([6]) *Let H be a real Hilbert space and let C be a nonempty closed and convex subset of H. Let $S : C \to C$ be a nonexpansive mapping. Then $I - S$ is demiclosed at zero.*

3. Main result

The following assumptions are central in the proof of our main theorem.

Assumption 3.1. *The space H is a real Hilbert space, the operator $A : H \to H$ is α-ism, $B : H \multimap H$ is a set-valued maximal monotone operator, the operator $S : H \to H$ is a nonexpansive mapping and the solution set*

$$\Omega := F(S) \cap (A+B)^{-1}0 \neq \emptyset.$$

Assumption 3.2. *The sequences $\{\tau_n\}, \{\sigma_n\}$, $\{\mu_n\}$, $\{\epsilon_n\}$, $\{\gamma_n\}$ and $\{s_n\}$ in $(0,1)$, and $\{\lambda_n\} \subset (0,\infty)$ are chosen such that*

(AS1) $\tau_n + \sigma_n + \mu_n = 1$, $\lim_{n\to\infty} \tau_n = 0$ *and* $\sum_{n=1}^{\infty} \tau_n = \infty$,
(AS2) $\lim_{n\to\infty} \epsilon_n = \lim_{n\to\infty} \gamma_n = \lim_{n\to\infty} s_n = 0$,
(AS3) $0 < \lambda \le \lambda_n < 2\alpha$.

Based on Assumptions 3.1 and 3.2, we now give our algorithm.

Algorithm 3.3. *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. *(Initialization) Choose arbitrary points* $x_0, x_1 \in H$, $a \in [0,1)$, $\epsilon_1, \gamma_1, \lambda_1, s_1, \tau_1, \sigma_1, \mu_1$ *and set* $n = 1$.

Step 1. *Choose* α_n *such that* $0 \le \alpha_n \le \bar{\alpha_n}$, *where*

$$\bar{\alpha_n} = \begin{cases} \min\left\{a, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1}, \\ a, & \text{otherwise.} \end{cases}$$

Step 2. *Compute*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = \gamma_n w_n + (1-\gamma_n) J^B_{\lambda_n}(I - \lambda_n A) w_n, \\ y_n = s_n w_n + (1-s_n) J^B_{\lambda_n}(I - \lambda_n A) z_n, \\ x_{n+1} = \tau_n u + \sigma_n w_n + \mu_n S y_n. \end{cases}$$

Step 3. *Update* $n = n+1$ *and go to Step 1.*

Remark 3.4. Observe that Assumption 3.2 and Step 1 imply $\lim\limits_{n\to\infty} \alpha_n \|x_n - x_{n-1}\| = 0$. Thus, there exists $M^* > 0$ such that

$$\alpha_n \|x_n - x_{n-1}\| \le M^*, \quad \forall n \ge 1. \tag{3.1}$$

Lemma 3.5. *Let* $\{x_n\}$ *be the sequence generated by Algorithm* 3.3*. Then* $\{x_n\}$ *is bounded.*

Proof. For each $n \ge 1$, let $T_n := J^B_{\lambda_n}(I - \lambda_n A)$. Then, T_n is nonexpansive and $F(T_n) = (A+B)^{-1}0$ (see, e.g., [28]). Let $z \in \Omega$, using the fact T_n is nonexpansive, we obtain the following estimation

$$\begin{aligned} \|z_n - z\| &= \|\beta_n w_n + (1-\beta_n) T_n w_n - z\| \\ &\le \beta_n \|w_n - z\| + (1-\beta_n)\|T_n w_n - z\| \\ &\le \beta_n \|w_n - z\| + (1-\beta_n)\|w_n - z\| \\ &\le \|x_n - z\| + \alpha_n \beta_n \|x_n - x_{n-1}\|. \end{aligned} \tag{3.2}$$

Similarly and using inequality (3.2), we have

$$\begin{aligned}\|y_n - z\| &= \|\gamma_n w_n + (1-\gamma_n)T_n z_n - z\| \\ &\le \gamma_n\|w_n - z\| + (1-\gamma_n)\|T_n z_n - z\| \\ &\le \gamma_n\|w_n - z\| + (1-\gamma_n)\|z_n - z\| \\ &\le \gamma_n\|w_n - z\| + (1-\gamma_n)\|x_n - z\| + (1-\gamma_n)\alpha_n\beta_n\|x_n - x_{n-1}\| \\ &\le \|x_n - z\| + \big((1-\gamma_n)\beta_n + \gamma_n\big)\alpha_n\|x_n - x_{n-1}\|. \end{aligned} \tag{3.3}$$

Now, using the fact that S is nonexpansive and inequality (3.3), we obtain

$$\begin{aligned}\|x_{n+1} - z\| &= \|\tau_n u + \sigma_n w_n + \mu_n S y_n - z\| \\ &\le \tau_n\|u - z\| + \sigma_n\|w_n - z\| + \mu_n\|S y_n - z\| \\ &\le \tau_n\|u - z\| + \sigma_n\|w_n - z\| + \mu_n\|y_n - z\| \\ &\le \tau_n\|u - z\| + \sigma_n\|x_n - z\| + \sigma_n\alpha_n\|x_n - x_{n-1}\| \\ &\quad + \mu_n\|x_n - z\| + \mu_n\big((1-\gamma_n)\beta_n + \gamma_n\big)\alpha_n\|x_n - x_{n-1}\| \\ &= \tau_n\|u - z\| + (1-\tau_n)\|x_n - z\| \\ &\quad + \big((1-\gamma_n)\beta_n + \gamma_n + \sigma_n\big)\alpha_n\|x_n - x_{n-1}\| \\ &\le \tau_n\|u - z\| + (1-\tau_n)\|x_n - z\| + \widehat{M} \\ &\le \max\big\{\|u - z\|, \|x_n - z\|\big\} + \widehat{M},\end{aligned}$$

where $\widehat{M} > 0$. Thus, by induction, we have

$$\|x_n - z\| \le \max\{\|u - z\|, \|x_1 - z\|\} + M \quad \text{for some } M > 0.$$

Hence, $\{x_n\}$ is bounded. Consequently, $\{z_n\}$ and $\{y_n\}$ are bounded. □

In the proof of theorem below, the operators satisfy Assumption 3.1 and the control sequences are assumed to satisfy the Assumption 3.2.

Theorem 3.6. *Let $\{x_n\}$ be the sequence generated by Algorithm 3.3. Then $\{x_n\}$ converges strongly to $z \in \Omega$.*

Proof. Let $z \in \Omega$. Using Lemma 2.4, we obtain the following estimation

$$\begin{aligned}\|T_n w_n - z\|^2 &= \|T_n w_n - T_n z\|^2 \\ &\le \|w_n - z\|^2 - \lambda_n(2\alpha - \lambda_n)\|Aw_n - Az\|^2 \\ &\quad - \|w_n - \lambda_n A w_n - T_n w_n + \lambda_n A z\|^2. \end{aligned} \tag{3.4}$$

Similarly,

$$
\begin{aligned}
\|T_n z_n - z\|^2 &= \|T_n z_n - T_n z\|^2 \\
&\leq \|z_n - z\|^2 - \lambda_n(2\alpha - \lambda_n)\|Az_n - Az\|^2 \\
&\quad - \|z_n - \lambda_n A z_n - T_n z_n + \lambda_n Az\|^2. \qquad (3.5)
\end{aligned}
$$

Now, using Lemma 2.1 and inequality (3.4), we have

$$
\begin{aligned}
\|z_n - z\|^2 &= \|\beta_n w_n - (1-\beta_n)T_n w_n - z\|^2 \\
&\leq \beta_n\|w_n - z\|^2 + (1-\beta_n)\|T_n w_n - z\|^2 \\
&\leq \|w_n - z\|^2 - (1-\beta_n)\lambda_n(2\alpha - \lambda_n)\|Aw_n - Az\|^2 \\
&\quad - \|w_n - \lambda_n A w_n - T_n w_n + \lambda_n Az\|^2. \qquad (3.6)
\end{aligned}
$$

Next, using Lemma 2.1 and inequalities (3.5) and (3.6), we obtain

$$
\begin{aligned}
\|y_n - z\|^2 &= \|\gamma_n w_n + (1-\gamma_n)T_n z_n - z\|^2 \\
&\leq \gamma_n\|w_n - z\|^2 + (1-\gamma_n)\|T_n z_n - z\|^2 \\
&\leq \gamma_n\|w_n - z\|^2 + (1-\gamma_n)\big(\|z_n - z\|^2 \\
&\quad - \lambda_n(2\alpha - \lambda_n)\|Az_n - Az\|^2 \\
&\quad - \|z_n - \lambda_n A z_n - T_n z_n + \lambda_n Az\|^2\big) \\
&= \gamma_n\|w_n - z\|^2 + (1-\gamma_n)\|z_n - z\|^2 \\
&\quad - (1-\gamma_n)\lambda_n(2\alpha - \lambda_n)\|Az_n - Az\|^2 \\
&\quad - (1-\gamma_n)\|z_n - \lambda_n A z_n - T_n z_n + \lambda_n Az\|^2 \\
&\leq \gamma_n\|w_n - z\|^2 + (1-\gamma_n)\big(\|w_n - z\|^2 \\
&\quad - (1-\beta_n)\lambda_n(2\alpha - \lambda_n)\|Aw_n - Az\|^2 \\
&\quad - \|w_n - \lambda_n A w_n - T_n w_n + \lambda_n Az\|^2\big) \\
&\quad - (1-\gamma_n)\lambda_n(2\alpha - \lambda_n)\|Az_n - Az\|^2 \\
&\quad - (1-\gamma_n)\|z_n - \lambda_n A z_n - T_n z_n + \lambda_n Az\|^2 \\
&= \|w_n - z\|^2 - (1-\gamma_n)(1-\beta_n)\lambda_n(2\alpha - \lambda_n)\|Aw_n - Az\|^2 \\
&\quad - (1-\gamma_n)\|w_n - \lambda_n A w_n - T_n w_n + \lambda_n Az\|^2 \\
&\quad - (1-\gamma_n)\lambda_n(2\alpha - \lambda_n)\|Az_n - Az\|^2 \\
&\quad - (1-\gamma_n)\|z_n - \lambda_n A z_n - T_n z_n + \lambda_n Az\|^2 \qquad (3.7)
\end{aligned}
$$

Furthermore, using Lemma 2.1 and inequality (3.7), we obtain

$$\begin{aligned}
\|x_{n+1}-z\|^2 &= \|\sigma_n(w_n-z)+\mu_n(Sy_n-z)+\tau_n(u-z)\|^2\\
&= \|\sigma_n(w_n-z)+\mu_n(Sy_n-z)\|^2+2\tau_n\langle u-z,x_{n+1}-z\rangle\\
&\le \sigma_n\|w_n-z\|^2+\mu_n\|y_n-z\|^2-\sigma_n\mu_n\|w_n-Sy_n\|^2\\
&\quad +2\tau_n\langle u-z,x_{n+1}-z\rangle\\
&\le \sigma_n\|w_n-z\|^2+\mu_n\big(\|w_n-z\|^2\\
&\quad -(1-\gamma_n)(1-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Aw_n-Az\|^2\\
&\quad -(1-\gamma_n)\|w_n-\lambda_nAw_n-T_nw_n+\lambda_nAz\|^2\\
&\quad -(1-\gamma_n)\lambda_n(2\alpha-\lambda_n)\|Az_n-Az\|^2\\
&\quad -(1-\gamma_n)\|z_n-\lambda_nAz_n-T_nz_n+\lambda_nAz\|^2\big)\\
&\quad -\sigma_n\mu_n\|w_n-Sy_n\|^2+2\tau_n\langle u-z,x_{n+1}-z\rangle\\
&\le (1-\tau_n)\|x_n-z\|^2+(1-\tau_n)2\alpha_n\langle x_n-x_{n-1},w_n-z\rangle\\
&\quad -\mu_n(1-\gamma_n)(1-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Aw_n-Az\|^2\\
&\quad -\mu_n(1-\gamma_n)\|w_n-\lambda_nAw_n-T_nw_n+\lambda_nAz\|^2\\
&\quad -\mu_n(1-\gamma_n)\lambda_n(2\alpha-\lambda_n)\|Az_n-Az\|^2\\
&\quad -\mu_n(1-\gamma_n)\|z_n-\lambda_nAz_n-T_nz_n+\lambda_nAz\|^2\\
&\quad -\sigma_n\mu_n\|w_n-Sy_n\|^2+2\tau_n\langle u-z,x_{n+1}-z\rangle \qquad (3.8)
\end{aligned}$$

From inequality (3.8), we deduce that

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le (1-\tau_n)\|x_n-z\|^2+(1-\tau_n)2\alpha_n\langle x_n-x_{n-1},w_n-z\rangle\\
&\quad +2\tau_n\langle u-z,x_{n+1}-z\rangle
\end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-z\|^2 &\le \|x_n-z\|^2+(1-\tau_n)2\alpha_n\langle x_n-x_{n-1},w_n-z\rangle\\
&\quad -\mu_n(1-\gamma_n)(1-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Aw_n-Az\|^2\\
&\quad -\mu_n(1-\gamma_n)\|w_n-\lambda_nAw_n-T_nw_n+\lambda_nAz\|^2\\
&\quad -\mu_n(1-\gamma_n)\lambda_n(2\alpha-\lambda_n)\|Az_n-Az\|^2\\
&\quad -\mu_n(1-\gamma_n)\|z_n-\lambda_nAz_n-T_nz_n+\lambda_nAz\|^2\\
&\quad -\sigma_n\mu_n\|w_n-Sy_n\|^2+2\tau_n\langle u-z,x_{n+1}-z\rangle
\end{aligned}$$

Setting $d_n=\|x_n-z\|^2, \quad \theta_n=\tau_n,$

$$t_n=\frac{2(1-\tau_n)}{\tau_n}\alpha_n\langle x_n-x_{n-1},w_n-z\rangle+2\langle u-z,x_{n+1}-z\rangle$$

$$\begin{aligned}\eta_n &= \mu_n(1-\gamma_n)(1-\beta_n)\lambda_n(2\alpha-\lambda_n)\|Aw_n - Az\|^2\\ &\quad + \mu_n(1-\gamma_n)\|w_n - \lambda_n Aw_n - T_n w_n + \lambda_n Az\|^2\\ &\quad + \mu_n(1-\gamma_n)\lambda_n(2\alpha-\lambda_n)\|Az_n - Az\|^2\\ &\quad + \mu_n(1-\gamma_n)\|z_n - \lambda_n Az_n - T_n z_n + \lambda_n Az\|^2\\ &\quad + \sigma_n\mu_n\|w_n - Sy_n\|^2\end{aligned}$$

and

$$\rho_n = 2(1-\tau_n)\alpha_n\langle x_n - x_{n-1}, w_n - z\rangle + 2\tau_n\langle u - z, x_{n+1} - z\rangle,$$

we obtain that

$$d_{n+1} \le (1-\theta_n)d_n + \theta_n t_n, \text{ and } d_{n+1} \le d_n - \eta_n + \rho_n, \ \ n \ge 1.$$

Since $\sum_{n=1}^{\infty}\tau_n = \infty$, $\sum_{n=1}^{\infty}\theta_n = \infty$. By Remark 3.4, the boundedness of $\{x_n\}$ and the fact that $\lim_{n\to\infty}\tau_n = 0$, we have $\lim_{n\to\infty}\rho_n = 0$.

In order to complete the proof, by using Lemma 2.5, it remains to show that $\lim_{k\to\infty}\eta_{n_k} = 0$ implies $\limsup_{k\to\infty} t_{n_k} \le 0$, for any subsequence $\{n_k\} \subset \{n\}$.

Indeed, if $\{n_k\}$ is a subsequence of $\{n\}$ such that $\lim_{k\to\infty}\eta_{n_k} = 0$, then we can deduce that

$$\begin{gathered}\lim_{k\to\infty}\|Aw_{n_k} - Az\| = 0, \quad \lim_{k\to\infty}\|Az_{n_k} - Az\| = 0,\\ \lim_{k\to\infty}\|w_{n_k} - Sy_{n_k}\| = 0,\\ \lim_{k\to\infty}\|w_{n_k} - \lambda_{n_k}(Aw_{n_k} - Az) - T_{n_k}w_{n_k}\| = 0 \quad \text{and}\\ \lim_{k\to\infty}\|z_{n_k} - \lambda_{n_k}(Az_{n_k} - Az) - T_{n_k}z_{n_k}\| = 0.\end{gathered} \tag{3.9}$$

Thus, by the triangle inequality, we have

$$\lim_{k\to\infty}\|T_{n_k}w_{n_k} - w_{n_k}\| = 0 \text{ and } \lim_{k\to\infty}\|T_{n_k}z_{n_k} - z_{n_k}\| = 0.$$

Observe that

$$\begin{aligned}\|T_{n_k}z_{n_k} - w_{n_k}\| &\le \|T_{n_k}z_{n_k} - z_{n_k}\| + \|w_{n_k} - z_{n_k}\|\\ &= \|T_{n_k}z_{n_k} - z_{n_k}\| + \|\beta_{n_k}w_{n_k} + (1-\beta_{n_k})T_{n_k}w_{n_k} - w_{n_k}\|\\ &= \|T_{n_k}z_{n_k} - z_{n_k}\| + (1-\beta_{n_k})\|T_{n_k}w_{n_k} - w_{n_k}\|,\end{aligned}$$

implies

$$\lim_{k\to\infty}\|T_{n_k}z_{n_k} - w_{n_k}\| = 0.$$

Also, we have

$$\|w_{n_k} - z_{n_k}\| \le \|T_{n_k}z_{n_k} - w_{n_k}\| + \|z_{n_k} - T_{n_k}z_{n_k}\|,$$

it implies that $\lim_{k\to\infty}\|w_{n_k} - z_{n_k}\| = 0$. Moreover,

$$\|y_{n_k} - w_{n_k}\| = (1-\sigma_{n_k})\|T_{n_k}z_{n_k} - w_{n_k}\|,$$

this implies that $\lim_{k\to\infty} \|y_{n_k} - w_{n_k}\| = 0$.

Furthermore,

$$\begin{aligned}\|w_{n_k} - Sw_{n_k}\| &\le \|w_{n_k} - Sy_{n_k}\| + \|Sy_{n_k} - Sw_{n_k}\| \\ &\le \|w_{n_k} - Sy_{n_k}\| + \|y_{n_k} - w_{n_k}\|,\end{aligned}$$

it implies that

$$\lim_{k\to\infty} \|w_{n_k} - Sw_{n_k}\| = 0. \tag{3.10}$$

Since $0 < \lambda \le \lambda_n$, for all $n \ge 1$, by Lemma 2.3, we have

$$\|T_\lambda^{A,B} w_{n_k} - w_{n_k}\| \le 2\|T_{n_k} w_{n_k} - w_{n_k}\|,$$

this implies that

$$\lim_{k\to\infty} \|T_\lambda^{A,B} w_{n_k} - w_{n_k}\| = 0.$$

Observe that

$$\|T_\lambda^{A,B} w_{n_k} - x_{n_k}\| \le \|T_\lambda^{A,B} w_{n_k} - w_{n_k}\| + \|w_{n_k} - x_{n_k}\|$$

implies

$$\lim_{k\to\infty} \|T_\lambda^{A,B} w_{n_k} - x_{n_k}\| = 0.$$

Also,

$$\|T_\lambda^{A,B} x_{n_k} - x_{n_k}\| \le \|T_\lambda^{A,B} x_{n_k} - T_\lambda^{A,B} w_{n_k}\| + \|T_\lambda^{A,B} w_{n_k} - x_{n_k}\|$$

implies

$$\lim_{k\to\infty} \|T_\lambda^{A,B} x_{n_k} - x_{n_k}\| = 0. \tag{3.11}$$

Let $z_t = tu + (1-t)T_\lambda^{A,B} z_t$ for all $t \in (0,1)$. Then, by a well-known theorem of Reich (see [37]), $\{z_t\}$ converges strongly to a point $z \in F(T_\lambda^{A,B})$.

Now, using Lemma 2.1 and the fact that $T_\lambda^{A,B}$ is nonexpansive, we have

$$\begin{aligned}\|z_t - x_{n_k}\|^2 &= \|t(u - x_{n_k}) + (1-t)(T_\lambda^{A,B} z_t - x_{n_k})\|^2 \\ &\le (1-t)^2 \|T_\lambda^{A,B} z_t - x_{n_k}\|^2 + 2t\langle u - x_{n_k}, z_t - x_{n_k}\rangle \\ &= (1-t)^2 \|T_\lambda^{A,B} z_t - x_{n_k}\|^2 + 2t\langle u - z_t, z_t - x_{n_k}\rangle \\ &\quad + 2t\langle z_t - x_{n_k}, z_t - x_{n_k}\rangle \\ &\le (1-t)^2 \big(\|T_\lambda^{A,B} z_t - T_\lambda^{A,B} x_{n_k}\| + \|T_\lambda^{A,B} x_{n_k} - x_{n_k}\|\big)^2 \\ &\quad + 2t\langle u - z_t, z_t - x_{n_k}\rangle + 2t\|z_t - x_{n_k}\|^2 \\ &\le (1-t)^2 \big(\|z_t - x_{n_k}\| + \|T_\lambda^{A,B} x_{n_k} - x_{n_k}\|\big)^2 \\ &\quad + 2t\langle u - z_t, z_t - x_{n_k}\rangle + 2t\|z_t - x_{n_k}\|^2.\end{aligned}$$

This implies that

$$\langle u - z_t, x_{n_k} - z_t\rangle \le \frac{(1-t)^2}{2t}\Big(\|z_t - x_{n_k}\| + \|T_\lambda^{A,B}x_{n_k} - x_{n_k}\|\Big)^2 + \frac{(2t-1)}{2t}\|z_t - x_{n_k}\|^2. \tag{3.12}$$

Thus, using (3.11), we have

$$\begin{aligned}\limsup_{k\to\infty}\langle u - z_t, x_{n_k} - z_t\rangle &\le \frac{(1-t)^2}{2t}M + \frac{(2t-1)}{2t}M\\ &= \Big(\frac{(1-t)^2+2t-1}{2t}\Big)M,\end{aligned} \tag{3.13}$$

where $M = \limsup_{k\to\infty}\|z_t - x_{n_k}\|^2$.

It is easy to see that $\frac{(1-t)^2+2t-1}{2t} \to 0$, as $t \to 0$. Hence,

$$\limsup_{k\to\infty}\langle z - u, z - x_{n_k}\rangle \le 0. \tag{3.14}$$

Furthermore, since

$$\begin{aligned}\|x_{n_k+1} - x_{n_k}\| &= \|\tau_{n_k}u + \sigma_{n_k}w_{n_k} + \mu_{n_k}Sy_{n_k} - x_{n_k}\|\\ &\le \tau_{n_k}\|u - x_{n_k}\| + \sigma_{n_k}\alpha_{n_k}\|x_{n_k} - x_{n_k-1}\|\\ &\quad + \mu_{n_k}\|Sy_{n_k} - x_{n_k}\|\\ &\le \tau_{n_k}\|u - x_{n_k}\| + \sigma_{n_k}\alpha_{n_k}\|x_{n_k} - x_{n_k-1}\|\\ &\quad + \mu_{n_k}\|Sy_{n_k} - w_{n_k}\| + \mu_{n_k}\|w_{n_k} - x_{n_k}\|\\ &\le \tau_{n_k}\|u - x_{n_k}\| + \mu_{n_k}\|Sy_{n_k} - w_{n_k}\|\\ &\quad + (1-\tau_{n_k})\alpha_{n_k}\|x_{n_k} - x_{n_k-1}\|,\end{aligned}$$

we have

$$\lim_{k\to\infty}\|x_{n_k+1} - x_{n_k}\| = 0. \tag{3.15}$$

Thus, by inequality (3.14) and equation (3.15), we deduce that

$$\limsup_{k\to\infty}\langle z - u, z - x_{n_k+1}\rangle \le 0.$$

Observe that

$$\frac{2(1-\tau_{n_k})\alpha_{n_k}}{\tau_{n_k}}\langle x_{n_k} - x_{n_k-1}, w_{n_k} - z\rangle \le \frac{2(1-\tau_{n_k})\alpha_{n_k}}{\tau_{n_k}}\|x_{n_k} - x_{n_k-1}\|\|w_{n_k} - z\|$$

implies

$$\limsup_{k\to\infty}\frac{2(1-\tau_{n_k})\alpha_{n_k}}{\tau_{n_k}}\langle x_{n_k} - x_{n_k-1}, w_{n_k} - z\rangle \le 0.$$

Hence, we obtain that $\limsup_{k\to\infty} t_{n_k} \leq 0$. Therefore, by Lemma 2.5, $\lim_{n\to\infty} d_n = 0$, that is,

$$\begin{aligned}\lim_{n\to\infty} x_n &= z \in F\big(T_\lambda^{A,B}\big)\\ &= (A+B)^{-1}0. \end{aligned} \tag{3.16}$$

Furthermore, by (3.10), (3.16) and Lemma 2.6, $z \in F(S)$. Thus, $z \in \Omega$. This completes the proof. □

4. Applications and Numerical Illustrations

In this section, we shall utilize the Halpern-type implicit rules presented in section 3 to study monotone variational inequality problem, convex minimization problem and convex constrained linear inverse problem.

4.1. **Application to monotone variational inequality problems.** A monotone variational inequality problem (VIP) is a problem of finding a point $u \in C$ such that

$$\langle Au, x-u\rangle \geq 0, \ \forall\, x \in C, \tag{4.1}$$

where C is a nonempty closed and convex subset of H and $A : C \to H$ is monotone. Assuming existence of solution, the VIP (4.1) is equivalent to the following inclusion problem:

$$\text{find} \quad u \in C \quad \text{such that} \quad 0 \in (A+B)u,$$

where $B : C \to 2^H$ is the subdifferential of the *indicator function* (see, e.g., [28]). By [38], in this case the resolvent of B is the metric projection P_C. Thus, the following algorithm can be deduced from Algorithm 3.3:

Algorithm 4.1. *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. *(Initialization) choose arbitrary points* $x_0, x_1 \in H$, $a \in [0,1)$, $\epsilon_1, \gamma_1, \lambda_1, s_1, \tau_1, \sigma_1, \mu_1$ *and set* $n = 1$.

Step 1. *Choose* α_n *such that* $0 \leq \alpha_n \leq \bar{\alpha_n}$, *where*

$$\bar{\alpha_n} = \begin{cases} \min\left\{a, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\right\}, & x_n \neq x_{n-1},\\ a, & \textit{otherwise}.\end{cases}$$

Step 2. *Compute*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}),\\ z_n = \gamma_n w_n + (1-\gamma_n)P_C(I - \lambda_n A)w_n,\\ y_n = s_n w_n + (1-s_n)P_C(I-\lambda_n A)z_n,\\ x_{n+1} = \tau_n u + \sigma_n w_n + \mu_n S y_n.\end{cases}$$

Step 3. *Update $n = n + 1$ and go to Step 1.*

The control parameters and the operators satisfy Assumptions 3.1 and 3.2.

Theorem 4.2. *Let $\{x_n\}$ be the sequence generated by Algorithm 4.1. Assume that the solution set is nonempty, then $\{x_n\}$ converges strongly to a point u, where u is a fixed point of S and solves the VIP (4.1).*

4.2. **Application to convex minimization problem.** Let $h : H \to \mathbb{R}$ be a convex smooth function and $g : H \to \mathbb{R} \cup \{\infty\}$ be a proper convex and lower-semicontinuous function. We consider the following convex minimization problem: find $x^* \in H$ such that

$$h(x^*) + g(x^*) = \min_{x \in H} \Big\{ h(x) + g(x) \Big\}. \tag{4.2}$$

Problem (4.2) is equivalent, by Fermat's rule, to the problem of finding $x^* \in H$ such that

$$0 \in \nabla h(x^*) + \partial g(x^*), \tag{4.3}$$

where ∇h is the gradient of h and ∂g is the subdifferential of g.

Set $A = \nabla h$ and $B = \partial g$ in Algorithm 3.3. It is well known that if ∇h is $(1/L)$-Lipschitz continuous, then it is L-inverse strongly monotone and ∂g is maximal monotone. Hence from Algorithm 3.3 we have the following algorithm.

Algorithm 4.3. *Inertial Halpern-type forward-backward splitting algorithm.*

Step 0. *(Initialization) choose arbitrary points $x_0, x_1 \in H$, $a \in [0, 1)$, $\epsilon_1, \gamma_1, \lambda_1, s_1, \tau_1, \sigma_1, \mu_1$ and set $n = 1$.*

Step 1. *Choose α_n such that $0 \leq \alpha_n \leq \bar{\alpha_n}$, where*

$$\bar{\alpha_n} = \begin{cases} \min\Big\{a, \frac{\epsilon_n}{\|x_n - x_{n-1}\|}\Big\}, & x_n \neq x_{n-1}, \\ a, & \textit{otherwise.} \end{cases}$$

Step 2. *Compute*

$$\begin{cases} w_n = x_n + \alpha_n(x_n - x_{n-1}), \\ z_n = \gamma_n w_n + (1 - \gamma_n) J_{\lambda_n}^{\partial g}(I - \lambda_n \nabla h) w_n, \\ y_n = s_n w_n + (1 - s_n) J_{\lambda_n}^{\partial g}(I - \lambda_n \nabla h) z_n, \\ x_{n+1} = \tau_n u + \sigma_n w_n + \mu_n S y_n. \end{cases}$$

Step 3. *Update $n = n + 1$ and go to Step 1.*

4.3. **Application to image restoration problems.** In this section, we apply our method to image deblurring and denoising. General image recovery problem can be formulated by the inversion of the following observation model:

$$b = Ax + v, \tag{4.4}$$

where $x \in \mathbb{R}^n$, x, v and b are unknown original image, unknown additive random noise and known degraded observation, respectively, and A is a linear operator that depends on the concerned image recovery problem.

This model (4.4), is approximately equivalent to several different formulations available for optimization problems. In the literature, there is a growing interest in using the l_1 norm in solving these types of problems. The l_1 regularization problem is given by

$$\min_x \left\{ \frac{1}{2}\|Ax - b\|_2^2 + \lambda_n \|x\|_1 \right\}, \tag{4.5}$$

where $x \in \mathbb{R}^n$, $b \in \mathbb{R}^k$, A is a $k \times n$ matrix and λ_n is a nonnegative parameter.

Next, we use our algorithm to approximate the solution of the following convex minimization problem:

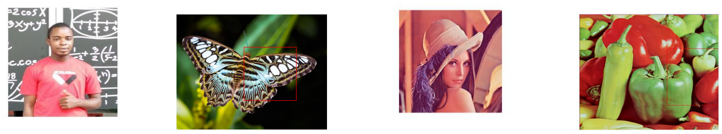
$$\text{find } x \in \operatorname{Argmin}_{x \in \mathbb{R}^n} \left\{ \frac{1}{2}\|Ax - b\|_2^2 + \lambda_n \|x\|_1 \right\}, \tag{4.6}$$

where b is the degraded image, and A is an operator representing the mask. Therefore, we use algorithms (1.3), (1.6) and Algorithm 4.3 to solve (4.6).

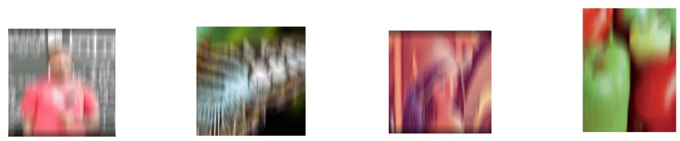
In algorithm (1.3), we set $\lambda_n = 0.01$, in algorithm (1.6), we set $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n+1}{2n}$, $\lambda_n = 0.01$, in Algorithm 4.3, we set $a = 0$, $\lambda_n = 0.01$, $\gamma_n = \frac{1}{n^2}$, $s_n = \frac{1}{(n+1)^4}$, $\tau_n = \frac{1}{(n+1)^3}$, $\sigma_n = 0$, $\mu_n = 1 - \tau_n$, $g(x) = \|x\|_1$, $h(x) = \frac{1}{2}\|Ax - b\|_2^2$, $S(x) = x$ and in all these algorithms, we set $A = \nabla h$ and $B = \partial g$. We define the gradient as:

$$\nabla h(x) = A^*(Ax - b).$$

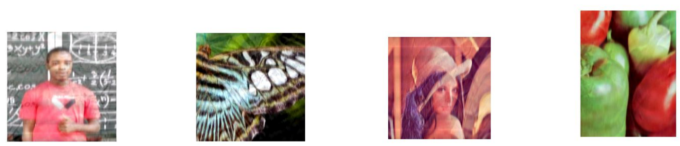
We consider the blur function in MATLAB “special (’motion’, 30, 60)” and add random noise. The test images are Abubakar, Barbra, butterfly and pepper (see Figure 1) and the stopping criterion of the algorithms is $\frac{\|x_{n+1} - x_n\|}{\|x_{n+1}\|} < 10^{-4}$. As we can see from Figure 1 and Table 1, our proposed algorithm is competitive and promising.



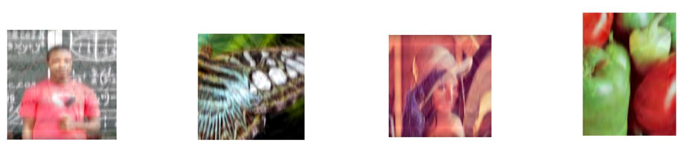
(A) original images



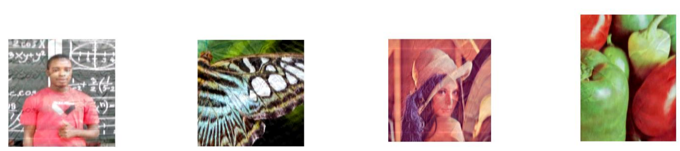
(B) images degraded by motion blur and random noise



(C) restored images with the FBA algorithm 1.3



(D) restored image with algorithm 1.6



(E) restored images with our algorithm 4.3

FIGURE 1. Test images and their restorations via algorithms 1.3, 1.6 and 4.3

The signal to noise ratio (SNR) and the improvement in signal to noise ratio (ISNR) are used to measure the quality of the restored images and they are defined as:

$$\text{SNR} := 10 \log \frac{\|x\|^2}{\|x - x_n\|} \quad \text{and} \quad \text{ISNR} := 10 \log \frac{\|x - b\|^2}{\|x - x_n\|},$$

where x, b and x_n are the original, observed and estimated image at iteration n, respectively. All algorithms were implemented with Ubuntu 64bits and MATLAB 2018b running on a Zinox laptop with Intel(R) Core(TM) i7 CPU and 4 GB of RAM.

TABLE 1. SNR and ISNR for the Test Images

	algorithm (1.3)		algorithm (1.6)		Algorithm 4.3	
Test image	SNR	ISNR	SNR	ISNR	SNR	ISNR
Abubakar	33.12	5.15	29.41	3.3	35.45	6.32
Barbra	41.12	6.73	37.18	4.76	43.69	8.02
butterfly	28.74	6.31	23.82	3.84	32.07	7.97
pepper	42.39	6.89	38.31	4.84	44.76	8.07

4.4. **An Example in $L_2([0,1])$.** Now, we present an example to compare the convergence of the sequence generated by our Algorithm 3.3 and that of algorithm (1.6).

Example 1.

In Theorems 1.2 and 3.6, set $H = L_2([0,1])$, and let $A : H \to H,\ B : H \to H,\ S : H \to H$, be defined as

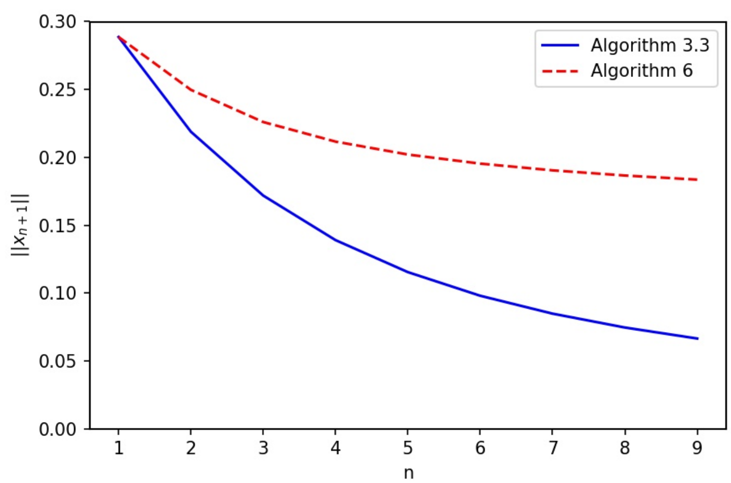
$$Ax(t) := 2x(t), \qquad Bx(t) := 5x(t), \qquad Sx(t) := tx(t).$$

Then, it is easy to see that A is $\frac{1}{2}$-ism, B is maximal monotone and S is nonexpansive. Furthermore, the solution set $\Omega = F(S) \cap (A+B)^{-1}0 = \{0\}$. In Theorem 1.2, we take $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{n+1}{2n}$ and $\lambda_n = \frac{1}{4}$, $x(t) = \frac{t}{2}$ and in Theorem 3.6, we take $\tau_n = \frac{1}{n}$, $\sigma_n = \mu_n = \frac{n-1}{2n}$, $\epsilon_n = \frac{1}{(n+1)^6}$, $\gamma_n = \frac{1}{(n+1)^8}$, $s_n = \frac{1}{(n+1)^9}$, $\lambda_n = \frac{1}{4}$ $a = 0.8$, $u(t) = \frac{t}{2}$, for all $n \in \mathbb{N}$, as our parameters. Clearly, these parameters satisfy the hypothesis of Theorems 1.2 and 3.6, respectively. Finally, we use a tolerance of 10^{-3} and set maximum number of iterations $n = 10$.

TABLE 2. Numerical results choosing $x_0 = t$ and $x_1(t) = 2t + 1$

N	algorithm (1.6) $\|x_{n+1} - z\|$	Algorithm 3.3 $\|x_{n+1} - z\|$
1	0.2886	0.2886
2	0.2497	0.2188
3	0.226	0.1717
4	0.2115	0.1389
5	0.202	0.1153
6	0.1953	0.0979
7	0.1904	0.0847
8	0.1865	0.0745
9	0.1835	0.0664
10	0.1811	0.0418

In the graph sketched below, the y-axis represents the values of $\|x_{n+1} - z\|$ while the x-axis represents the number of iterations n.



(A) Graph of the first 10 iterates of algorithms (1.6) and Algorithm 3.3 choosing $x_0 = 2t$, $x_1(t) = t$

FIGURE 2. Graph of iterates choosing $x_0 = t$ and $x_1 = 2t + 1$

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