Nonlinear Functional Analysis and Applications
Vol. 26, No. 2 (2021), pp. 393-409
ISSN: 1229-1595(print), 2466-0973(online)

https://doi.org/10.22771/nfaa.2021.26.02.11
http://nfaa.kyungnam.ac.kr/journal-nfaa

ON COMMON FIXED POINT THEOREMS OF WEAKLY COMPATIBLE MAPPINGS SATISFYING CONTRACTIVE INEQUALITIES OF INTEGRAL TYPE

Chi Feng[1], Na Liu[2], Soo Hak Shim[3] and Chahn Yong Jung[4]

[1]College of Science, Dalian Ocean University
Dalian, Liaoning 116023, China
e-mail: chifeng@dlou.edu.cn

[2]Department of Mathematics, Liaoning Normal University
Dalian, Liaoning 116029, China
e-mail: naliu5@163.com

[3]Department of Refrigeration and Air Conditioning Engineering
Chonnam National University, Yeosu 59626, Korea
e-mail: zeno1213@daum.net

[4]Department of Business Administration, Gyeongsang National University
Jinju 52828, Korea
e-mail: bb5734@gnu.ac.kr

Abstract. Three common fixed point theorems for weakly compatible mappings satisfying three classes of contractive inequalities of integral type are proved. Three examples are included. The results obtained in this paper extend and improve a few results existing in literature.

1. Introduction and preliminaries

Throughout this paper, we assume that $\mathbb{R}^+ = [0, +\infty)$, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, where $\mathbb{N}$ denotes the set of all positive integers and

[0]Received October 5, 2020. Revised December 24, 2020. Accepted February 5, 2021.
[0]2010 Mathematics Subject Classification: 54H25.
[0]Keywords: Common fixed point, weakly compatible mappings, contractive inequalities of integral type.
[0]Corresponding author: Chahn Yong Jung(bb5734@gnu.ac.kr).

- $\Phi_1 = \{\varphi \mid \varphi : \mathbb{R}^+ \to \mathbb{R}^+$ satisfies that φ is Lebesgue integrable, summable on each compact subset of $\mathbb{R}^+$ and $\int_0^\varepsilon \varphi(t)dt > 0$ for each $\varepsilon > 0\}$,
- $\Phi_2 = \{\alpha \mid \alpha : \mathbb{R}^+ \to [0,1)$ satisfies that $\limsup_{s\to t} \alpha(s) < 1$ for each $t \in \mathbb{R}^+\}$,
- $\Phi_3 = \{\alpha \mid \alpha \in \Phi_2$ and $\limsup_{s\to+\infty} \alpha(s) < 1\}$.

In 2002, Branciari [2] was the first to introduce the concept of contractive mapping of integral type and obtained the following fixed point result for the mapping.

Theorem 1.1. ([2]) *Let T be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x,y \in X,$$

where $c \in (0,1)$ is a constant and $\varphi \in \Phi_1$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Afterwards several researchers in [1, 3, 5, 6, 7, 8, 9, 10, 11, 12] discussed the existence of fixed points and common fixed points for a few contractive mappings of integral type. In particular, Rhoades [11] and Liu et al. [9] proved the following fixed point theorems.

Theorem 1.2. ([11]) *Let T be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le c \int_0^{m(x,y)} \varphi(t)dt, \quad \forall x,y \in X,$$

where $c \in (0,1)$ is a constant, $\varphi \in \Phi_1$ and

$$m(x,y) = \max\left\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\right\}.$$

Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

Theorem 1.3. ([9]) *Let T be a mapping from a complete metric space (X,d) into itself satisfying*

$$\int_0^{d(Tx,Ty)} \varphi(t)dt \le \alpha\big(d(x,y)\big) \int_0^{d(x,y)} \varphi(t)dt, \quad \forall x,y \in X,$$

where $(\varphi,\alpha) \in \Phi_1 \times \Phi_2$. Then T has a unique fixed point $a \in X$ such that $\lim_{n\to\infty} T^n x = a$ for each $x \in X$.

The aim of this paper is to establish three common fixed point theorems for weakly compatible mappings satisfying three classes of contractive inequalities of integral type. Three examples are constructed to illustrate that the results obtained in this paper generalize Theorems 1.1 and 1.2 and differ from Theorem 1.3.

Definition 1.4. ([4]) Let (X,d) be a metric space and $A,S\colon X\to X$ be two mappings. A and S are called *weakly compatible* if they commute at their coincidence points.

Lemma 1.5. ([9]) *Let $\varphi\in\Phi_1$ and $\{r_n\}_{n\in\mathbb{N}}$ be a nonnegative sequence with $\lim_{n\to\infty} r_n = a$. Then*

$$\lim_{n\to\infty}\int_0^{r_n}\varphi(t)dt=\int_0^{a}\varphi(t)dt.$$

2. Common fixed point theorems

Our main results are as follows:

Theorem 2.1. *Let A,B,S and T be self mappings in a metric space (X,d) such that*

$$\{A,T\} \text{ and } \{B,S\} \text{ are weakly compatible;} \tag{2.1}$$

$$T(X)\subseteq B(X) \text{ and } S(X)\subseteq A(X); \tag{2.2}$$

$$\text{one of } A(X),B(X),S(X) \text{ and } T(X) \text{ is complete;} \tag{2.3}$$

$$\int_0^{d(Tx,Sy)}\varphi(t)dt\le\alpha\big(d(x,y)\big)\int_0^{M_1(x,y)}\varphi(t)dt,\quad\forall x,y\in X, \tag{2.4}$$

where $(\varphi,\alpha)\in\Phi_1\times\Phi_3$ and for all $x,y\in X$,

$$\begin{aligned}M_1(x,y)=\max\Bigg\{&d(Ax,By),d(Ax,Tx),d(By,Sy),\\&\frac{1}{2}[d(Ax,Sy)+d(Tx,By)],\frac{1+d(Ax,By)}{1+d(By,Sy)}d(Ax,Tx),\\&\frac{1+d(Ax,By)}{1+d(Ax,Tx)}d(By,Sy),\frac{d^2(Ax,Tx)}{1+d(Tx,Sy)},\frac{d^2(By,Sy)}{1+d(Tx,Sy)},\\&\frac{1+d(Ax,Sy)+d(Tx,By)}{1+d(Ax,By)+d(Tx,Sy)}d(Ax,Tx),\\&\frac{1+d(Ax,Sy)+d(Tx,By)}{1+d(Ax,By)+d(Tx,Sy)}d(By,Sy)\Bigg\}.\end{aligned} \tag{2.5}$$

Then, we have the following statements:

(1) *There exist* $w, u \in X$ *such that* $Aw = Tw = Bu = Su$;
(2) A, B, S *and* T *have a unique common fixed point in* X *if* T *and* A *as well as* S *and* B *are weakly compatible.*

Proof. Let $x_0 \in X$. It follows from (2.2) that there exist two sequences $\{x_n\}_{n\in\mathbb{N}_0}$ and $\{y_n\}_{n\in\mathbb{N}}$ in X such that

$$y_{2n+1} = Bx_{2n+1} = Tx_{2n}, \quad y_{2n+2} = Ax_{2n+2} = Sx_{2n+1}, \quad \forall n \in \mathbb{N}_0. \tag{2.6}$$

Put $d_n = d(y_n, y_{n+1})$ for each $n \in \mathbb{N}$.

Assume that $d_{2n} < d_{2n+1}$ for some $n \in \mathbb{N}$. Because of (2.4)-(2.6) and $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$, we derive that

$$\begin{aligned}
&M_1(x_{2n}, x_{2n+1})\\
&= \max\Big\{d(Ax_{2n}, Bx_{2n+1}), d(Ax_{2n}, Tx_{2n}), d(Bx_{2n+1}, Sx_{2n+1}),\\
&\qquad \frac{1}{2}[d(Ax_{2n}, Sx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})],\\
&\qquad \frac{1 + d(Ax_{2n}, Bx_{2n+1})}{1 + d(Bx_{2n+1}, Sx_{2n+1})} d(Ax_{2n}, Tx_{2n}),\\
&\qquad \frac{1 + d(Ax_{2n}, Bx_{2n+1})}{1 + d(Ax_{2n}, Tx_{2n})} d(Bx_{2n+1}, Sx_{2n+1}),\\
&\qquad \frac{d^2(Ax_{2n}, Tx_{2n})}{1 + d(Tx_{2n}, Sx_{2n+1})}, \frac{d^2(Bx_{2n+1}, Sx_{2n+1})}{1 + d(Tx_{2n}, Sx_{2n+1})},\\
&\qquad \frac{1 + d(Ax_{2n}, Sx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})}{1 + d(Ax_{2n}, Bx_{2n+1}) + d(Tx_{2n}, Sx_{2n+1})} d(Ax_{2n}, Tx_{2n}),\\
&\qquad \frac{1 + d(Ax_{2n}, Sx_{2n+1}) + d(Tx_{2n}, Bx_{2n+1})}{1 + d(Ax_{2n}, Bx_{2n+1}) + d(Tx_{2n}, Sx_{2n+1})} d(Bx_{2n+1}, Sx_{2n+1})\Big\}\\
&= \max\Big\{d(y_{2n}, y_{2n+1}), d(y_{2n}, y_{2n+1}), d(y_{2n+1}, y_{2n+2}),\\
&\qquad \frac{1}{2}[d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})],\\
&\qquad \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n+2})} d(y_{2n}, y_{2n+1}), \frac{1 + d(y_{2n}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1})} d(y_{2n+1}, y_{2n+2}),\\
&\qquad \frac{d^2(y_{2n}, y_{2n+1})}{1 + d(y_{2n+1}, y_{2n+2})}, \frac{d^2(y_{2n+1}, y_{2n+2})}{1 + d(y_{2n+1}, y_{2n+2})},\\
&\qquad \frac{1 + d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})} d(y_{2n}, y_{2n+1}),\\
&\qquad \frac{1 + d(y_{2n}, y_{2n+2}) + d(y_{2n+1}, y_{2n+1})}{1 + d(y_{2n}, y_{2n+1}) + d(y_{2n+1}, y_{2n+2})} d(y_{2n+1}, y_{2n+2})\Big\}
\end{aligned}$$

$$= \max\Big\{d_{2n}, d_{2n}, d_{2n+1}, \frac{1}{2}d(y_{2n}, y_{2n+2}), \frac{1+d_{2n}}{1+d_{2n+1}}d_{2n}, \frac{1+d_{2n}}{1+d_{2n}}d_{2n+1},$$
$$\frac{d_{2n}^2}{1+d_{2n+1}}, \frac{d_{2n+1}^2}{1+d_{2n+1}}, \frac{1+d(y_{2n}, y_{2n+2})}{1+d_{2n}+d_{2n+1}}d_{2n}, \frac{1+d(y_{2n}, y_{2n+2})}{1+d_{2n}+d_{2n+1}}d_{2n+1}\Big\}$$
$$= \max\{d_{2n}, d_{2n+1}\} = d_{2n+1}$$

and

$$\begin{aligned} 0 &< \int_0^{d_{2n+1}} \varphi(t)dt = \int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt \\ &\le \alpha\big(d(x_{2n}, x_{2n+1})\big) \int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt \\ &= \alpha\big(d(x_{2n}, x_{2n+1})\big) \int_0^{d_{2n+1}} \varphi(t)dt < \int_0^{d_{2n+1}} \varphi(t)dt, \end{aligned}$$

which is a contradiction. Hence

$$d_{2n+1} \le d_{2n} = M_1(x_{2n}, x_{2n+1}), \quad \forall n \in \mathbb{N}.$$

Similarly,

$$d_{2n} \le d_{2n-1} = M_1(x_{2n}, x_{2n-1}), \quad \forall n \in \mathbb{N}.$$

That is, for all $n \in \mathbb{N}$,

$$d_{n+1} \le d_n, \quad d_{2n} = M_1(x_{2n}, x_{2n+1}), \quad d_{2n-1} = M_1(x_{2n}, x_{2n-1}), \tag{2.7}$$

which implies that $\{d_n\}_{n\epsilon\mathbb{N}}$ is nonincreasing sequence and there exists a constant c with $\lim_{n\to\infty} d_n = c \ge 0$.

Suppose that $c > 0$. In light of (2.4), (2.7), $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$ and Lemma 1.5, we get that

$$\begin{aligned} 0 &< \int_0^{c} \varphi(t)dt = \limsup_{n\to\infty} \int_0^{d_{2n+1}} \varphi(t)dt \\ &= \limsup_{n\to\infty} \int_0^{d(Tx_{2n}, Sx_{2n+1})} \varphi(t)dt \\ &\le \limsup_{n\to\infty} \left[\alpha\big(d(x_{2n}, x_{2n+1})\big) \int_0^{M_1(x_{2n}, x_{2n+1})} \varphi(t)dt\right] \\ &= \limsup_{n\to\infty} \left[\alpha\big(d(x_{2n}, x_{2n+1})\big) \int_0^{d_{2n}} \varphi(t)dt\right] \\ &\le \limsup_{n\to\infty} \alpha\big(d(x_{2n}, x_{2n+1})\big) \limsup_{n\to\infty} \int_0^{d_{2n}} \varphi(t)dt \\ &< \int_0^{c} \varphi(t)dt, \end{aligned}$$

which is absurd. Thus $c = 0$, which means that

$$\lim_{n\to\infty} d_n = 0. \tag{2.8}$$

Next we prove that $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Because of (2.8), it is sufficient to verify that $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence.

Suppose that $\{y_{2n}\}_{n\in\mathbb{N}}$ is not a Cauchy sequence. It follows that there exist $\varepsilon > 0$ and two subsequences $\{y_{2m(k)}\}_{k\in\mathbb{N}}$ and $\{y_{2n(k)}\}_{k\in\mathbb{N}}$ of $\{y_{2n}\}_{n\in\mathbb{N}}$ with $2m(k) > 2n(k) > 2k$ satisfying

$$d(y_{2n(k)}, y_{2m(k)}) \geq \varepsilon, \quad \forall k \in \mathbb{N}, \tag{2.9}$$

where $2m(k)$ is the least integer exceeding $2n(k)$ satisfying (2.9). It follows that

$$d(y_{2n(k)}, y_{2m(k)-2}) < \varepsilon, \quad \forall k \in \mathbb{N},$$

which together with (2.9) and the triangle inequality give that

$$\begin{aligned} \varepsilon &\leq d(y_{2n(k)}, y_{2m(k)}) \\ &\leq d(y_{2n(k)}, y_{2m(k)-2}) + d(y_{2m(k)-2}, y_{2m(k)-1}) + d(y_{2m(k)-1}, y_{2m(k)}) \\ &< \varepsilon + d_{2m(k)-2} + d_{2m(k)-1}, \quad \forall k \in \mathbb{N} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} &|d(y_{2n(k)}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2m(k)-1}, \quad \forall k \in \mathbb{N}; \\ &|d(y_{2n(k)+1}, y_{2m(k)}) - d(y_{2n(k)}, y_{2m(k)})| \leq d_{2n(k)}, \quad \forall k \in \mathbb{N}; \\ &|d(y_{2n(k)+1}, y_{2m(k)-1}) - d(y_{2n(k)}, y_{2m(k)-1})| \leq d_{2n(k)}, \quad \forall k \in \mathbb{N}. \end{aligned} \tag{2.11}$$

Letting $k \to \infty$ in (2.10) and (2.11) and using (2.8), we deduce that

$$\begin{aligned} \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)}) &= \lim_{k\to\infty} d(y_{2n(k)}, y_{2m(k)-1}) \\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)}) \\ &= \lim_{k\to\infty} d(y_{2n(k)+1}, y_{2m(k)-1}) = \varepsilon. \end{aligned} \tag{2.12}$$

In view of (2.4), (2.5), (2.12), $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$ and Lemma 1.5, we obtain that

$$\begin{aligned} &M_1(x_{2n(k)}, x_{2m(k)-1}) \\ &= \max\Big\{ d(Ax_{2n(k)}, Bx_{2m(k)-1}), d(Ax_{2n(k)}, Tx_{2n(k)}), d(Bx_{2m(k)-1}, Sx_{2m(k)-1}), \\ &\qquad \frac{1}{2}[d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Tx_{2n(k)}, Bx_{2m(k)-1})], \\ &\qquad \frac{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Bx_{2m(k)-1}, Sx_{2m(k)-1})} d(Ax_{2n(k)}, Tx_{2n(k)}), \\ &\qquad \frac{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Ax_{2n(k)}, Tx_{2n(k)})} d(Bx_{2m(k)-1}, Sx_{2m(k)-1}), \end{aligned}$$

$$\frac{d^2(Ax_{2n(k)}, Tx_{2n(k)})}{1 + d(Tx_{2n(k)}, Sx_{2m(k)-1})}, \frac{d^2(Bx_{2m(k)-1}, Sx_{2m(k)-1})}{1 + d(Tx_{2n(k)}, Sx_{2m(k)-1})},$$

$$\frac{1 + d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Tx_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1}) + d(Tx_{2n(k)}, Sx_{2m(k)-1})} \times d(Ax_{2n(k)}, Tx_{2n(k)}),$$

$$\frac{1 + d(Ax_{2n(k)}, Sx_{2m(k)-1}) + d(Tx_{2n(k)}, Bx_{2m(k)-1})}{1 + d(Ax_{2n(k)}, Bx_{2m(k)-1}) + d(Tx_{2n(k)}, Sx_{2m(k)-1})} \times d(Bx_{2m(k)-1}, Sx_{2m(k)-1})\Bigg\}$$

$$= \max\Bigg\{d(y_{2n(k)}, y_{2m(k)-1}), d(y_{2n(k)}, y_{2n(k)+1}), d(y_{2m(k)-1}, y_{2m(k)}),$$

$$\frac{1}{2}[d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})],$$

$$\frac{1 + d(y_{2n(k)}, y_{2m(k)-1})}{1 + d(y_{2m(k)-1}, y_{2m(k)})} d(y_{2n(k)}, y_{2n(k)+1}),$$

$$\frac{1 + d(y_{2n(k)}, y_{2m(k)-1})}{1 + d(y_{2n(k)}, y_{2n(k)+1})} d(y_{2m(k)-1}, y_{2m(k)}),$$

$$\frac{d^2(y_{2n(k)}, y_{2n(k)+1})}{1 + d(y_{2n(k)+1}, y_{2m(k)})}, \frac{d^2(y_{2m(k)-1}, y_{2m(k)})}{1 + d(y_{2n(k)+1}, y_{2m(k)})},$$

$$\frac{1 + d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})}{1 + d(y_{2n(k)}, y_{2m(k)-1}) + d(y_{2n(k)+1}, y_{2m(k)})} d(y_{2n(k)}, y_{2n(k)+1}),$$

$$\frac{1 + d(y_{2n(k)}, y_{2m(k)}) + d(y_{2n(k)+1}, y_{2m(k)-1})}{1 + d(y_{2n(k)}, y_{2m(k)-1}) + d(y_{2n(k)+1}, y_{2m(k)})} d(y_{2m(k)-1}, y_{2m(k)})\Bigg\}$$

$$\to \max\left\{\varepsilon, 0, 0, \frac{1}{2}(\varepsilon + \varepsilon), 0, 0, 0, 0, 0, 0\right\}$$

$$= \varepsilon \quad \text{as } k \to \infty$$

and

$$0 < \int_0^{\varepsilon} \varphi(t)dt = \limsup_{k\to\infty} \int_0^{d(y_{2n(k)+1}, y_{2m(k)})} \varphi(t)dt$$

$$= \limsup_{k\to\infty} \int_0^{d(Tx_{2n(k)}, Sx_{2m(k)-1})} \varphi(t)dt$$

$$\le \limsup_{k\to\infty} \left[\alpha\big(d(x_{2n(k)}, x_{2m(k)-1})\big) \int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt\right]$$

$$\leq \limsup_{k\to\infty} \alpha\big(d(x_{2n(k)}, x_{2m(k)-1})\big) \limsup_{k\to\infty} \int_0^{M_1(x_{2n(k)}, x_{2m(k)-1})} \varphi(t)dt$$
$$< \int_0^{\varepsilon} \varphi(t)dt,$$

which is impossible. Hence $\{y_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Without loss of generality, we suppose that $A(X)$ is complete. Obviously, $\{y_{2n}\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $A(X)$. Consequently, there exists $(z, w) \in A(X) \times X$ with $\lim_{n\to\infty} y_{2n} = z = Aw$. It is easy to see that

$$\begin{aligned} z &= \lim_{n\to\infty} y_n \\ &= \lim_{n\to\infty} Tx_{2n} \\ &= \lim_{n\to\infty} Bx_{2n+1} \\ &= \lim_{n\to\infty} Sx_{2n-1} \\ &= \lim_{n\to\infty} Ax_{2n}. \end{aligned} \tag{2.13}$$

Suppose that $Tw \neq z$. Note that (2.4), (2.5), (2.13), $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$ and Lemma 1.5 imply that

$$\begin{aligned} &M_1(w, x_{2n+1}) \\ &= \max\Big\{ d(Aw, Bx_{2n+1}), d(Aw, Tw), d(Bx_{2n+1}, Sx_{2n+1}), \\ &\qquad \frac{1}{2}[d(Aw, Sx_{2n+1}) + d(Tw, Bx_{2n+1})], \\ &\qquad \frac{1 + d(Aw, Bx_{2n+1})}{1 + d(Bx_{2n+1}, Sx_{2n+1})} d(Aw, Tw), \\ &\qquad \frac{1 + d(Aw, Bx_{2n+1})}{1 + d(Aw, Tw)} d(Bx_{2n+1}, Sx_{2n+1}), \\ &\qquad \frac{d^2(Aw, Tw)}{1 + d(Tw, Sx_{2n+1})}, \frac{d^2(Bx_{2n+1}, Sx_{2n+1})}{1 + d(Tw, Sx_{2n+1})}, \\ &\qquad \frac{1 + d(Aw, Sx_{2n+1}) + d(Tw, Bx_{2n+1})}{1 + d(Aw, Bx_{2n+1}) + d(Tw, Sx_{2n+1})} d(Aw, Tw), \\ &\qquad \frac{1 + d(Aw, Sx_{2n+1}) + d(Tw, Bx_{2n+1})}{1 + d(Aw, Bx_{2n+1}) + d(Tw, Sx_{2n+1})} d(Bx_{2n+1}, Sx_{2n+1}) \Big\} \\ &\to \max\Big\{ d(Aw, z), d(Aw, Tw), d(z, z), \frac{1}{2}[d(Aw, z) + d(Tw, z)], \\ &\qquad \frac{1 + d(Aw, z)}{1 + d(z, z)} d(Aw, Tw), \frac{1 + d(Aw, z)}{1 + d(Aw, Tw)} d(z, z), \end{aligned}$$

$$\frac{d^2(Aw,Tw)}{1+d(Tw,z)}, \frac{d^2(z,z)}{1+d(Tw,z)}, \frac{1+d(Aw,z)+d(Tw,z)}{1+d(Aw,z)+d(Tw,z)}d(Aw,Tw),$$
$$\frac{1+d(Aw,z)+d(Tw,z)}{1+d(Aw,z)+d(Tw,z)}d(z,z)\Bigg\}$$
$$= \max\left\{0, d(z,Tw), 0, \frac{1}{2}d(Tw,z), d(z,Tw), 0, \frac{d^2(z,Tw)}{1+d(Tw,z)}, 0, d(z,Tw), 0\right\}$$
$$= d(Tw,z) \text{ as } n\to\infty$$

and

$$\begin{aligned}
0 < \int_0^{d(Tw,z)} \varphi(t)dt &= \limsup_{n\to\infty} \int_0^{d(Tw,Sx_{2n+1})} \varphi(t)dt \\
&\le \limsup_{n\to\infty} \left[\alpha\big(d(w,x_{2n+1})\big) \int_0^{M_1(w,x_{2n+1})} \varphi(t)dt\right] \\
&\le \limsup_{n\to\infty} \alpha\big(d(w,x_{2n+1})\big) \limsup_{n\to\infty} \int_0^{M_1(w,x_{2n+1})} \varphi(t)dt \\
&< \int_0^{d(Tw,z)} \varphi(t)dt,
\end{aligned}$$

which is a contradiction. Hence $Tw = z$. It follows from (2.2) that there exists a point $u \in X$ with $z = Bu = Tw$. Suppose that $Su \neq z$. In light of (2.4), (2.5), (2.13), $(\varphi,\alpha) \in \Phi_1 \times \Phi_3$ and Lemma 1.5, we deduce that

$$\begin{aligned}
&M_1(x_{2n},u) \\
&= \max\Bigg\{d(Ax_{2n},Bu), d(Ax_{2n},Tx_{2n}), d(Bu,Su), \\
&\qquad \frac{1}{2}[d(Ax_{2n},Su)+d(Tx_{2n},Bu)], \frac{1+d(Ax_{2n},Bu)}{1+d(Bu,Su)}d(Ax_{2n},Tx_{2n}), \\
&\qquad \frac{1+d(Ax_{2n},Bu)}{1+d(Ax_{2n},Tx_{2n})}d(Bu,Su), \frac{d^2(Ax_{2n},Tx_{2n})}{1+d(Tx_{2n},Su)}, \frac{d^2(Bu,Su)}{1+d(Tx_{2n},Su)}, \\
&\qquad \frac{1+d(Ax_{2n},Su)+d(Tx_{2n},Bu)}{1+d(Ax_{2n},Bu)+d(Tx_{2n},Su)}d(Ax_{2n},Tx_{2n}), \\
&\qquad \frac{1+d(Ax_{2n},Su)+d(Tx_{2n},Bu)}{1+d(Ax_{2n},Bu)+d(Tx_{2n},Su)}d(Bu,Su)\Bigg\} \\
&\to \max\Bigg\{d(z,Bu), d(z,z), d(Bu,Su), \frac{1}{2}[d(z,Su)+d(z,Bu)], \\
&\qquad \frac{1+d(z,Bu)}{1+d(Bu,Su)}d(z,z), \frac{1+d(z,Bu)}{1+d(z,z)}d(Bu,Su),
\end{aligned}$$

$$
\begin{aligned}
&\frac{d^2(z,z)}{1+d(z,Su)}, \frac{d^2(Bu,Su)}{1+d(z,Su)}, \frac{1+d(z,Su)+d(z,Bu)}{1+d(z,Bu)+d(z,Su)}d(z,z),\\
&\frac{1+d(z,Su)+d(z,Bu)}{1+d(z,Bu)+d(z,Su)}d(Bu,Su)\Bigg\}\\
= \max &\left\{0,0,d(z,Su),\frac{1}{2}d(z,Su),0,d(z,Su),0,\frac{d^2(z,Su)}{1+d(z,Su)},0,d(z,Su)\right\}\\
= d&(z,Su) \quad \text{as } n\to\infty
\end{aligned}
$$

and

$$
\begin{aligned}
0 < \int_0^{d(z,Su)}\varphi(t)dt &= \limsup_{n\to\infty}\int_0^{d(Tx_{2n},Su)}\varphi(t)dt\\
&\le \limsup_{n\to\infty}\left[\alpha\big(d(x_{2n},u)\big)\int_0^{M_1(x_{2n},u)}\varphi(t)dt\right]\\
&\le \limsup_{n\to\infty}\alpha\big(d(x_{2n},u)\big)\limsup_{n\to\infty}\int_0^{M_1(x_{2n},u)}\varphi(t)dt\\
&< \int_0^{d(z,Su)}\varphi(t)dt,
\end{aligned}
$$

which is absurd. Hence $Su = z$.

Next we prove (2). By means of (2.1), we know that $Az = ATw = TAw = Tz$ and $Bz = BSu = SBu = Sz$. Assume that $Tz \neq Sz$. It follows from (2.4), (2.5) and $(\varphi,\alpha)\in\Phi_1\times\Phi_3$ that

$$
\begin{aligned}
&M_1(z,z)\\
&= \max\Bigg\{d(Az,Bz),d(Az,Tz),d(Bz,Sz),\frac{1}{2}[d(Az,Sz)+d(Tz,Bz)],\\
&\qquad \frac{1+d(Az,Bz)}{1+d(Bz,Sz)}d(Az,Tz),\frac{1+d(Az,Bz)}{1+d(Az,Tz)}d(Bz,Sz),\\
&\qquad \frac{d^2(Az,Tz)}{1+d(Tz,Sz)},\frac{d^2(Bz,Sz)}{1+d(Tz,Sz)},\frac{1+d(Az,Sz)+d(Tz,Bz)}{1+d(Az,Bz)+d(Tz,Sz)}d(Az,Tz),\\
&\qquad \frac{1+d(Az,Sz)+d(Tz,Bz)}{1+d(Az,Bz)+d(Tz,Sz)}d(Bz,Sz)\Bigg\}\\
&= \max\left\{d(Tz,Sz),0,0,\frac{1}{2}[d(Tz,Sz)+d(Tz,Sz)],0,0,0,0,0,0\right\}\\
&= d(Tz,Sz)
\end{aligned}
$$

and

$$\begin{aligned} 0 &< \int_0^{d(Tz,Sz)} \varphi(t)dt \le \alpha\big(d(z,z)\big) \int_0^{M_1(z,z)} \varphi(t)dt \\ &= \alpha(0) \int_0^{d(Tz,Sz)} \varphi(t)dt < \int_0^{d(Tz,Sz)} \varphi(t)dt, \end{aligned}$$

which is a contradiction. Hence $Tz = Sz$. That is, $Az = Tz = Bz = Sz$.

Suppose that $Tz \neq z$. On account of (2.4), (2.5) and $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$, we attain that

$$\begin{aligned} &M_1(z,u) \\ &= \max\Big\{ d(Az,Bu), d(Az,Tz), d(Bu,Su), \frac{1}{2}[d(Az,Su) + d(Tz,Bu)], \\ &\qquad \frac{1+d(Az,Bu)}{1+d(Bu,Su)} d(Az,Tz), \frac{1+d(Az,Bu)}{1+d(Az,Tz)} d(Bu,Su), \\ &\qquad \frac{d^2(Az,Tz)}{1+d(Tz,Su)}, \frac{d^2(Bu,Su)}{1+d(Tz,Su)}, \frac{1+d(Az,Su)+d(Tz,Bu)}{1+d(Az,Bu)+d(Tz,Su)} d(Az,Tz), \\ &\qquad \frac{1+d(Az,Su)+d(Tz,Bu)}{1+d(Az,Bu)+d(Tz,Su)} d(Bu,Su) \Big\} \\ &= \max\Big\{ d(Tz,z), 0, 0, \frac{1}{2}[d(Tz,z) + d(Tz,z)], 0, 0, 0, 0, 0, 0 \Big\} \\ &= d(Tz,z) \end{aligned}$$

and

$$\begin{aligned} 0 &< \int_0^{d(Tz,z)} \varphi(t)dt = \int_0^{d(Tz,Su)} \varphi(t)dt \le \alpha\big(d(z,u)\big) \int_0^{M_1(z,u)} \varphi(t)dt \\ &= \alpha\big(d(z,u)\big) \int_0^{d(Tz,z)} \varphi(t)dt < \int_0^{d(Tz,z)} \varphi(t)dt, \end{aligned}$$

which is ridiculous. Therefore, $Tz = z$, which implies that z is a common fixed point of A, B, S and T.

Suppose that A, B, S and T have another common fixed point $b \in X \setminus \{z\}$. It follows from (2.4), (2.5) and $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$ that

$$\begin{aligned} &M_1(b,z) \\ &= \max\Big\{ d(Ab,Bz), d(Ab,Tb), d(Bz,Sz), \frac{1}{2}[d(Ab,Sz) + d(Tb,Bz)], \\ &\qquad \frac{1+d(Ab,Bz)}{1+d(Bz,Sz)} d(Ab,Tb), \frac{1+d(Ab,Bz)}{1+d(Ab,Tb)} d(Bz,Sz), \end{aligned}$$

$$\frac{d^2(Ab,Tb)}{1+d(Tb,Sz)}, \frac{d^2(Bz,Sz)}{1+d(Tb,Sz)}, \frac{1+d(Ab,Sz)+d(Tb,Bz)}{1+d(Ab,Bz)+d(Tb,Sz)}d(Ab,Tb),$$
$$\left.\frac{1+d(Ab,Sz)+d(Tb,Bz)}{1+d(Ab,Bz)+d(Tb,Sz)}d(Bz,Sz)\right\}$$
$$= \max\left\{d(b,z),0,0,\frac{1}{2}[d(b,z)+d(b,z)],0,0,0,0,0,0\right\}$$
$$= d(b,z)$$

and

$$\begin{aligned} 0 &< \int_0^{d(b,z)} \varphi(t)dt \\ &= \int_0^{d(Tb,Sz)} \varphi(t)dt \\ &\le \alpha\big(d(b,z)\big) \int_0^{M_1(b,z)} \varphi(t)dt \\ &= \alpha\big(d(b,z)\big) \int_0^{d(b,z)} \varphi(t)dt \\ &< \int_0^{d(b,z)} \varphi(t)dt, \end{aligned}$$

which is a contradiction. Hence A, B, S and T have a unique common fixed point in X. This completes the proof. □

Similar to the proof of Theorem 2.1, we have the following results and omit their proofs.

Theorem 2.2. *Let A, B, S and T be self mappings in a metric space (X, d) satisfying* (2.1)-(2.3) *and*

$$\int_0^{d(Tx,Sy)} \varphi(t)dt \le \alpha\big(d(x,y)\big) \int_0^{M_2(x,y)} \varphi(t)dt, \quad \forall x,y \in X, \tag{2.14}$$

where $(\varphi,\alpha) \in \Phi_1 \times \Phi_3$ and

$$M_2(x,y) = \max\left\{d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy)+d(Tx,By)],\right.$$
$$\frac{1+d(Tx,By)}{2+d(Tx,Sy)}d(Ax,Sy), \frac{1+d(Ax,Sy)}{2+d(Tx,Sy)}d(Tx,By),$$
$$\frac{1+d(Ax,Sy)+d(Tx,By)}{1+2d(Tx,Sy)}d(Ax,Tx),$$
$$\left.\frac{1+d(Ax,Sy)+d(Tx,By)}{1+2d(Tx,Sy)}d(By,Sy)\right\}, \quad \forall x,y \in X.$$

Then (1) *and* (2) *of Theorem* 2.1 *hold.*

Theorem 2.3. *Let* A, B, S *and* T *be self mappings in a metric space* (X, d) *satisfying* (2.1)-(2.3) *and*

$$\int_0^{d(Tx,Sy)} \varphi(t)dt \le \alpha\big(d(x,y)\big) \int_0^{M_3(x,y)} \varphi(t)dt, \quad \forall x, y \in X, \tag{2.15}$$

where $(\varphi, \alpha) \in \Phi_1 \times \Phi_3$ *and*

$$\begin{aligned} M_3(x,y) = \max\Big\{ & d(Ax,By), d(Ax,Tx), d(By,Sy), \frac{1}{2}[d(Ax,Sy) + d(Tx,By)], \\ & \frac{1+d(Tx,By)}{1+2d(Tx,Sy)} d(Ax,Tx), \frac{1+d(Ax,Sy)}{1+2d(Tx,Sy)} d(By,Sy), \\ & \frac{1+d(Ax,Sy)d(Tx,By)}{1+d(Ax,By)d(Tx,Sy)} d(Ax,Tx), \\ & \frac{1+d(Ax,Sy)d(Tx,By)}{1+d(Ax,By)d(Tx,Sy)} d(By,Sy) \Big\}, \quad \forall x, y \in X. \end{aligned}$$

Then (1) *and* (2) *of the Theorem* 2.1 *hold.*

3. Examples

Now we construct three examples with uncountably many points to explain the common fixed point theorems obtained in Section 2.

Remark 3.1. Theorems 2.1-2.3 are generalizations of Theorem 1.2, which, in turns, extends Theorem 1.1. Examples 3.2-3.4 show that Theorems 2.1- 2.3 extend substantially Theorem 1.1 and differ from Theorem 1.3.

Example 3.2. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d(x, y) = |x - y|$ for all $x, y \in X$. Let $A, B, S, T : X \to X$, $\alpha : \mathbb{R}^+ \to [0, 1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$Ax = \frac{1}{2}x + \frac{1}{2}, \quad Bx = x^3, \quad Sx = 1, \quad \forall x \in X,$$

$$Tx = \begin{cases} 1, & \forall x \in X \setminus \{\frac{1}{4}\}, \\ \frac{15}{16}, & x = \frac{1}{4}, \end{cases}$$

$$\alpha(t) = \frac{1+t}{2+3t}, \quad \varphi(t) = 2t, \quad \forall t \in \mathbb{R}^+.$$

Evidently, (2.1)-(2.3) hold and $\alpha(t) \in (\frac{1}{3}, \frac{1}{2}]$, for all $t \in \mathbb{R}^+$. Let $x, y \in X$. In order to verify (2.4), we consider two cases as follows:

Case 1. $x \in X \setminus \{\frac{1}{4}\}$. It is obvious that

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = 0 \le \alpha\big(d(x,y)\big) \int_0^{M_1(x,y)} \varphi(t)dt;$$

Case 2. $x = \frac{1}{4}$. Clearly

$$M_1(x,y) \ge d(Ax,Tx) = \left|\frac{5}{8} - \frac{15}{16}\right| = \frac{5}{16}$$

and

$$\begin{aligned}\int_0^{d(Tx,Sy)} \varphi(t)dt &= \int_0^{\frac{1}{16}} \varphi(t)dt = \frac{1}{256} < \frac{1}{3} \cdot \frac{25}{256} \\ &\le \alpha\left(d\left(\frac{1}{4}, y\right)\right) \int_0^{\frac{5}{16}} \varphi(t)dt \\ &\le \alpha\big(d(x,y)\big) \int_0^{M_1(x,y)} \varphi(t)dt.\end{aligned}$$

That is, (2.4) holds. It follows from Theorem 2.1 that the mappings A, B, S and T have a unique common fixed point $1 \in X$.

Note that Theorem 1.3 generalizes Theorem 1.1. Now we need to prove that Theorem 1.3 is useless in proving the existence of fixed points of T in X.

Suppose that there exists $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$ satisfies the conditions of Theorem 1.3. By Theorem 1.3, we get that

$$\begin{aligned}0 < \int_0^{\frac{1}{16}} \varphi(t)dt &= \int_0^{d(T\frac{1}{4}, T\frac{5}{16})} \varphi(t)dt \\ &\le \alpha\left(d\left(\frac{1}{4}, \frac{5}{16}\right)\right) \int_0^{d(\frac{1}{4}, \frac{5}{16})} \varphi(t)dt < \int_0^{\frac{1}{16}} \varphi(t)dt,\end{aligned}$$

which is a contradiction.

Example 3.3. Let $X = \mathbb{R}^+$ be endowed with the Euclidean metric $d(x,y) = |x - y|$ for all $x, y \in X$. Let $A, B, S, T : X \to X$, $\alpha : \mathbb{R}^+ \to [0,1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$Bx = \frac{1}{5}x^2, \quad Sx = 0, \quad \forall x \in X,$$

$$Ax = \begin{cases} 0, & \forall x \in X \setminus \{\frac{1}{6}\}, \\ \frac{1}{3}, & x = \frac{1}{6}, \end{cases} \qquad Tx = \begin{cases} 0, & \forall x \in X \setminus \{\frac{1}{6}\}, \\ \frac{1}{12}, & x = \frac{1}{6}, \end{cases}$$

$$\alpha(t) = \frac{t^2 + 4t + 3}{5t^2 + 5t + 4}, \qquad \varphi(t) = 3t^2, \quad \forall t \in \mathbb{R}^+.$$

Obviously, (2.1)-(2.3) hold and $\alpha(t) \in (\frac{1}{5}, \frac{4}{5})$, for all $t \in \mathbb{R}^+$. Put $x, y \in X$. To prove (2.14), we have to consider two possible cases as follows:

Case 1. $x \in X \setminus \{\frac{1}{6}\}$. It is clear that

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = 0 \leq \alpha\big(d(x,y)\big) \int_0^{M_2(x,y)} \varphi(t)dt;$$

Case 2. $x = \frac{1}{6}$. Obviously

$$M_2(x,y) \geq d(Ax,Tx) = \left|\frac{1}{3} - \frac{1}{12}\right| = \frac{1}{4}$$

and

$$\begin{aligned}\int_0^{d(Tx,Sy)} \varphi(t)dt &= \int_0^{\frac{1}{12}} \varphi(t)dt = \frac{1}{1728} < \frac{1}{5} \cdot \frac{1}{64} \\ &\leq \alpha\left(d\left(\frac{1}{6}, y\right)\right) \int_0^{\frac{1}{4}} \varphi(t)dt \\ &\leq \alpha\big(d(x,y)\big) \int_0^{M_2(x,y)} \varphi(t)dt.\end{aligned}$$

It means that (2.14) holds. It follows from Theorem 2.2 that the mappings A, B, S and T have a unique common fixed point $0 \in X$. Observe that Theorem 1.3 is a generalization of Theorem 1.1.

Next we assert that Theorem 1.3 is unapplicable in ensuring the existence of fixed points of T in X Suppose that there exists $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$ satisfies the conditions of Theorem 1.3. Using Theorem 1.3, we gain that

$$\begin{aligned}0 &< \int_0^{\frac{1}{12}} \varphi(t)dt = \int_0^{d(T\frac{1}{6}, T\frac{1}{4})} \varphi(t)dt \\ &\leq \alpha\left(d\left(\frac{1}{6}, \frac{1}{4}\right)\right) \int_0^{d(\frac{1}{6}, \frac{1}{4})} \varphi(t)dt < \int_0^{\frac{1}{12}} \varphi(t)dt,\end{aligned}$$

which is impossible.

Example 3.4. Let $X = [0,1]$ be endowed with the Euclidean metric $d(x,y) = |x-y|$ for all $x, y \in X$. Let $A, B, S, T : X \to X$, $\alpha : \mathbb{R}^+ \to [0,1)$ and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be defined by

$$Ax = x, \quad Bx = x^3, \quad Sx = 1, \quad \forall x \in X,$$

$$Tx = \begin{cases} \frac{2}{3}, & \forall x \in [0, \frac{1}{9}], \\ 1, & \forall x \in (\frac{1}{9}, 1], \end{cases}$$

$$\alpha(t) = \frac{2^{t+1} - 1}{2^{t+2}}, \quad \varphi(t) = 729^t \ln 729, \quad \forall t \in \mathbb{R}^+.$$

It is easy to verify that (2.1)-(2.3) are valid and $\alpha(t) \in [\frac{1}{4}, \frac{1}{2})$, for all $t \in \mathbb{R}^+$. Put $x, y \in X$. For the sake of verifying (2.15), we consider the following two possible cases:

Case 1. $x \in [0, \frac{1}{9}]$. It is follows that

$$M_3(x, y) \geq d(Ax, Tx) = \left| x - \frac{2}{3} \right| \geq \frac{5}{9}$$

and

$$\begin{aligned}\int_0^{d(Tx,Sy)} \varphi(t)dt = \int_0^{\frac{1}{3}} \varphi(t)dt = 9 < \frac{1}{4} \cdot 729^{\frac{5}{9}} \\ \leq \alpha\big(d(x, y)\big) \int_0^{\frac{5}{9}} \varphi(t)dt \\ \leq \alpha\big(d(x, y)\big) \int_0^{M_3(x,y)} \varphi(t)dt;\end{aligned}$$

Case 2. $x \in (\frac{1}{9}, 1]$. Evidently

$$\int_0^{d(Tx,Sy)} \varphi(t)dt = 0 \leq \alpha\big(d(x, y)\big) \int_0^{M_3(x,y)} \varphi(t)dt.$$

Hence (2.15) holds. It follows from Theorem 2.3 that the mappings A, B, S and T have a unique common fixed point $1 \in X$. Notice that Theorem 1.3 generalizes Theorem 1.1. Now we prove that Theorem 1.3 is useless in proving the existence of fixed points of T in X.

Suppose that there exists $(\varphi, \alpha) \in \Phi_1 \times \Phi_2$ satisfies the conditions of Theorem 1.3. According to the Theorem 1.3, we conclude that

$$\begin{aligned}0 < \int_0^{\frac{1}{3}} \varphi(t)dt = \int_0^{d(T\frac{1}{9}, T\frac{4}{9})} \varphi(t)dt \\ \leq \alpha\left(d\left(\frac{1}{9}, \frac{4}{9}\right)\right) \int_0^{d(\frac{1}{9}, \frac{4}{9})} \varphi(t)dt < \int_0^{\frac{1}{3}} \varphi(t)dt,\end{aligned}$$

which is absurd.

References

[1] A. Aliouche, *A common fixed point theorem for weakly compatible mappings in symmetric spaces satisfying a contractive condition of integral type*, J. Math. Anal. Appl., **322** (2006), 796–802.

[2] A. Branciari, *A fixed point theorem for mappings satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **29** (2002), 531–536.

[3] A. Djoudi and A. Aliouche, *Common fixed point theorems of Gregus type for weakly compatible mappings satisfying contractive conditions of integral type*, J. Math. Anal. Appl., **329** (2007), 31–45.

[4] G. Jungck and B.E. Rhoades, *Fixed points for set valued functions without continuity*, Indian. J. Pure. Appl. Math., **29** (1998), 227–238.

[5] Z. Liu, Y. Han, S.M. Kang and J.S. Ume, *Common fixed point theorems for weakly compatible mappings satisfying contractive conditions of integral type*, Fixed Point Theory Appl., **2014** (2014), Paper No. 132, 16 pages.

[6] Z. Liu, M. He and C.Y. Jung, *Common fixed point for two pairs of mappings satisfying contractive inequalities of integral type*, Nonlinear Funct. Anal. Appl., **24**(2) (2019), 361–387.

[7] Z. Liu, M. He, X. Liu and L. Zhao, *Common fixed point theorems for four mappings satisfying contractive inequalities of integral type*, Nonlinear Funct. Anal. Appl., **23**(3) (2018), 473–501.

[8] Z. Liu, J.L. Li and S.M. Kang, *Fixed point theorems of contractive mappings of integral type*, Fixed Point Theory Appl., **2013** (2013), Paper No. 300, 17 pages.

[9] Z. Liu, X. Li, S.M. Kang and S.Y. Cho, *Fixed point theorems for mappings satisfying contractive conditions of integral type and applications*, Fixed Point Theory Appl., **2011** (2011), Paper No. 64, 18 pages.

[10] Z. Liu, L. Meng, N. Liu and C.Y. Jung, *Common fixed point theorems for a pair of mappings satisfying contractive inequalities of integral type*, Nonlinear Funct. Anal. Appl., **25** (2020), 69–100.

[11] B.E. Rhoades, *Two fixed-point theorems for mappings satisfying a general contractive contraction of integral type*, Int. J. Math. Math. Sci., **2003** (2003), 4007–4013.

[12] P. Vijayaraju, B.E. Rhoades and R. Mohanraj, *A fixed point theorem for a pair of maps satisfying a general contractive condition of integral type*, Int. J. Math. Math. Sci., **2005** (2005), 2359–2364.