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CERTAIN NEW FAMILIES FOR BI-UNIVALENT FUNCTIONS DEFINED BY A KNOWN OPERATOR

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Abstract. In this paper, we aim to introduce two new families of analytic and bi-univalent functions associated with the Attiya's operator, which is defined by the Hadamard product of a generalized Mittag-Leffler function and analytic functions on the open unit disk. Then we estimate the second and third coefficients of the Taylor-Maclaurin series expansions of functions belonging to these families. Also, we investigate Fekete-Szegö problem for these families. Some relevant connections of certain special cases of the main results with those in several earlier works are also pointed out. Two naturally-arisen problems are given for further investigation.

1. Introduction and preliminaries

Denote by $\mathcal{A}$ the family of all analytic functions $f(z)$ in the open unit disc $\mathsf{U} = \{z \in \mathbb{C} : |z| < 1\}$ whose Taylor series expansions are of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \tag{1}$$

Here and elsewhere, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{N}$, and $\mathbb{Z}_0^-$ be the sets of complex numbers, real numbers, positive integers, and non-positive integers, respectively. Also let $\mathcal{S}$ stand for the subfamily of the family $\mathcal{A}$ whose members are univalent in U.

The Hadamard product (or convolution) $*$ of f and g in $\mathcal{A}$ is defined by

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k \quad (z \in \mathsf{U}),$$

where

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad \text{and} \quad g(z) = z + \sum_{k=2}^{\infty} b_k z^k$$

Throughout this paper, if $f \in \mathcal{A}$ (or $\mathcal{S}$) and its Taylor-Maclaurin series expansion is considered, then f is assumed to have the form (1).

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Since Gösta Mittag-Leffler in 1903 (see [20, 21]) introduced so-called Mittag-Leffler function $E_\lambda(z)$ defined by

$$E_\lambda(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\lambda k + 1)} \quad (z \in \mathbb{C};\ \Re(\lambda) > 0), \tag{2}$$

Γ being the familiar gamma function, the function and a large number of its extensions (generalizations) have been introduced and applied in a variety of research subjects (see, e.g., [8, 10, 12, 25, 32] and the references therein). Among them, Srivastava and Tomovski [32] introduced and studied the following generalization $E^{\delta,\tau}_{\lambda,\eta}(z)$ defined by

$$E^{\delta,\tau}_{\lambda,\eta}(z) = \sum_{k=0}^{\infty} \frac{(\delta)_{k\tau}\, z^k}{\Gamma(\lambda k + \eta) k!} \tag{3}$$

$$(z,\ \delta,\ \eta \in \mathbb{C};\ \Re(\lambda) > \max\{0, \Re(\tau) - 1\};\ \Re(\tau) > 0)\,,$$

where the Pochhammer symbol $(\lambda)_\nu$ is defined (for $\lambda,\ \nu \in \mathbb{C}$), in terms of the Gamma function Γ, by

$$\begin{aligned}(\lambda)_\nu :&= \frac{\Gamma(\lambda+\nu)}{\Gamma(\lambda)} \quad \left(\lambda \in \mathbb{C} \setminus \mathbb{Z}_0^-\right) \\ &= \begin{cases} 1 & (\nu = 0;\ \lambda \in \mathbb{C} \setminus \{0\}) \\ \lambda(\lambda+1)\cdots(\lambda+n-1) & (\nu = n \in \mathbb{N};\ \lambda \in \mathbb{C}), \end{cases}\end{aligned} \tag{4}$$

it being understood conventionally that $(0)_0 := 1$ (see, e.g., [27, p. 2 and p. 5]).

Attiya [8] introduced and investigated an operator $\mathcal{H}^{\delta,\tau}_{\lambda,\eta} : \mathcal{A} \longrightarrow \mathcal{A}$ which, in terms of the Hadamard product (or convolution) $*$, is defined by

$$\mathcal{H}^{\delta,\tau}_{\lambda,\eta} f(z) = Q^{\delta,\tau}_{\lambda,\eta}(z) * f(z) \quad (z \in \mathsf{U}), \tag{5}$$

where

$$Q^{\delta,\tau}_{\lambda,\eta}(z) = \frac{\Gamma(\lambda+\eta)}{(\delta)_\tau} \left(E^{\delta,\tau}_{\lambda,\eta}(z) - \frac{1}{\Gamma(\eta)} \right).$$

Here

$$\big(\delta,\ \eta \in \mathbb{C};\ \Re(\lambda) > \max\{0, \Re(\tau) - 1\};\ \Re(\tau) > 0;$$
$$\Re(\lambda) = 0 \text{ when } \Re(\tau) = 1 \text{ with } \eta \neq 0\big).$$

It is easy to find that

$$\mathcal{H}^{\delta,\tau}_{\lambda,\eta} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(\delta + k\tau)\Gamma(\lambda+\eta)}{\Gamma(\delta+\tau)\Gamma(\lambda k + \eta)\Gamma(k+1)} a_k z^k, \tag{6}$$

where $f(z)$ is of the form in (1).

The convergent series $f(z)$ of the form

$$f(z) = \sum_{k=1}^{\infty} a_k z^k \quad (a_1 \neq 0)$$

is often called a *delta series* which has a compositional inverse $f^{-1}(z)$ gratifying $f\left(f^{-1}(z)\right) = f^{-1}(f(z)) = z$ (see, e.g., [17], [24, p. 4]). Koebe one-quarter theorem (see, e.g., [11, pages 31, 45, 69, 74] states that the range of every function of the class $\mathcal{S}$ contains the disk $\left\{w : |w| < \frac{1}{4}\right\}$. Accordingly, every function $f \in \mathcal{S}$ has an inverse f^{-1} which satisfies $f^{-1}(f(z)) = z$ $(z \in \mathsf{U})$ and $f\left(f^{-1}(w)\right) = w$ $(|w| < r_0(f), r_0(f) \geq \frac{1}{4})$, where

$$g(w) = f^{-1}(w) = w - a_2w^2 + \left(2a_2^2 - a_3\right)w^3 - \left(5a_2^3 - 5a_2a_3 + a_4\right)w^4 + \cdots . \quad (7)$$

A function $f \in \mathcal{A}$ is called *bi-univalent* in U if both $f(z)$ and $f^{-1}(z)$ are univalent in U. Let Σ denote the family of bi-univalent functions in U whose Taylor-Maclaurin series expansion are of the form (1). Srivastava et al. [30, p. 1189] presented some functions which belong to Σ and some other ones which are not in Σ. Lewin [18] studied the class of bi-univalent functions Σ and proved that $|a_2| < 1.51$. Subsequently, Netanyahu [22] showed that

$$\max_{f\in\Sigma} |a_2| = \frac{4}{3}.$$

Interestingly, Brannan and Clunie [9] conjectured that $|a_2| \leq \sqrt{2}$. commented that It is noted (see, e.g., [30]) that the coefficient estimate problem for each of the following TaylorMaclaurin coefficients: $|a_n|$ $(n \in \mathbb{N} \setminus \{1, 2\})$ may be an open question. Afterward a concise indication of a few subclasses of the function class Σ together with non-sharp estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ which were introduced by D. A. Brannan and T. S. Taha (see the references in [30]), Srivastava et al. [30] introduced two new subclasses of the function class Σ and found estimates on the coefficients $|a_2|$ and $|a_3|$ for functions in these new subclasses of the function class Σ. Beginning with Srivastava et al.'s pioneering work [30] on the subject, a large number of works related to the subject have been presented (see, e.g., [1, 2, 3, 4, 6, 14, 15, 16, 19, 26, 28, 29, 37, 38, 39, 40, 41, 42, 43, 44]). On this subject in geometric function theory, the so-called Fekete-Szegö type inequalities (or problems) which estimate some upper bounds for $\left|a_3 - \xi a_2^2\right|$ $(f \in \mathcal{S})$ have been actively investigated in almost parallel with those of $|a_2|$ and $|a_3|$ (for a very brief introduction among their affluent developments, see the preface in Section 4).

In this paper, we aim to introduce and investigate two new families of analytic and bi-univalent functions associated with the Attiya's operator (5), which is defined by the Hadamard product of the generalized Mittag-Leffler function in (3) and a function in $\mathcal{A}$. We present certain upper bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these families, We also investigate the Fekete-Szegö type inequalities for functions belonging to these families. Some relevant connections of certain special cases of the main results with those in several earlier works are pointed out. We present two naturally-arisen problems for further investigation.

The following lemma, first proved by Carathéodory, will be used in the proof of our main results.

Lemma 1.1. (see [11, p. 41]) *Let $\mathcal{P}$ be the class of all functions h analytic in U of the form*

$$h(z) = 1 + \sum_{k=1}^{\infty} c_k\, z^k$$

which satisfy

$$\Re\,(h(z)) > 0$$

for all $z \in \mathsf{U}$. Then if $h \in \mathcal{P}$, then $|c_k| \leq 2$ $(k \in \mathbb{N})$.

2. Two subclasses of the class Σ

Here two subclasses of the class Σ are introduced.

Definition 1. Let $\delta,\, \eta \in \mathbb{C}$, $\Re(\lambda) > \max\{0, \Re(\tau) - 1\}$, and $\Re(\tau) > 0$. Also let $z,\, w \in \mathsf{U}$, $0 < \alpha \leq 1$, $\mu \geq 0$, and $\gamma \geq 0$. Then the family denoted by $\mathcal{T}_{\Sigma}(\gamma, \mu, \lambda, \eta, \delta, \tau; \alpha)$ is defined by the class of functions $f \in \Sigma$ which satisfy the following conditions:

$$\left|\arg\left(\gamma z \left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z)\right)'' + (\mu - 2\gamma)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z)\right)' + (1 - \mu + 2\gamma)\,\frac{\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z)}{z}\right)\right| < \frac{\alpha\pi}{2} \tag{8}$$

and

$$\left|\arg\left(\gamma w \left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} g(w)\right)'' + (\mu - 2\gamma)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} g(w)\right)' + (1 - \mu + 2\gamma)\,\frac{\mathcal{H}_{\lambda,\eta}^{\delta,\tau} g(w)}{w}\right)\right| < \frac{\alpha\pi}{2}, \tag{9}$$

where $g = f^{-1}$ is given as in (7).

Definition 2. Let $\delta,\, \eta \in \mathbb{C}$, $\Re(\lambda) > \max\{0, \Re(\tau) - 1\}$, and $\Re(\tau) > 0$. Also let $z,\, w \in \mathsf{U}$, $0 \leq \beta < 1$, $\mu \geq 0$, and $\gamma \geq 0$. Then the family denoted by $\mathcal{T}_{\Sigma}^{*}(\gamma, \mu, \lambda, \eta, \delta, \tau; \beta)$ is defined by the class of functions $f \in \Sigma$ which satisfy the following conditions:

$$\Re\left\{\left(\gamma z \left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z)\right)'' + (\mu - 2\gamma)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z)\right)' + (1 - \mu + 2\gamma)\,\frac{\mathcal{H}_{\lambda,\eta}^{\delta,\tau} f(z)}{z}\right)\right\} > \beta \tag{10}$$

and

$$\Re\left\{\left(\gamma w \left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} g(w)\right)'' + (\mu - 2\gamma)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau} g(w)\right)' + (1 - \mu + 2\gamma)\,\frac{\mathcal{H}_{\lambda,\eta}^{\delta,\tau} g(w)}{w}\right)\right\} > \beta, \tag{11}$$

where $g = f^{-1}$ is given as in (7).

The two subfamilies of the bi-univalent family Σ are found to reduce to several known classes as in the following remarks.

Remark 1. The family $\mathcal{T}_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\alpha)$ may reduce to

(a) the family $\mathcal{H}_\Sigma(\alpha,\gamma)$ (see Frasin [14]) when $\mu = 1+2\gamma$, $\lambda = 0$ and $\eta = \delta = \tau = 1$;

(b) the family $\mathcal{B}_\Sigma(\alpha,\mu)$ (see Frasin and Aouf [15]) when $\gamma = \lambda = 0$ and $\eta = \delta = \tau = 1$;

(c) the family $\mathcal{H}_\Sigma^\alpha$ (see Srivastava et al. [30]) when $\gamma = \lambda = 0$ and $\mu = \eta = \delta = \tau = 1$.

Remark 2. The family $\mathcal{T}^*_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\beta)$ may reduce to

(c) the family $\mathcal{H}_\Sigma(\gamma,\beta)$ (see Frasin [14]) when $\mu = 1+2\gamma$, $\lambda = 0$ and $\eta = \delta = \tau = 1$;

(d) the family $\mathcal{B}_\Sigma(\beta,\mu)$ (see Frasin and Aouf [15]) when $\gamma = \lambda = 0$ and $\eta = \delta = \tau = 1$;

(f) the family $\mathcal{H}_\Sigma(\beta)$ (see Srivastava et al. [30]) when $\gamma = \lambda = 0$ and $\mu = \eta = \delta = \tau = 1$.

3. Coefficient estimates for the functions of the two families introduced in Section 2

We present the estimates for the second and third coefficients of the Taylor-Maclaurin series expansion of the bi-univalent functions in the families $\mathcal{T}_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\alpha)$ and $\mathcal{T}^*_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\beta)$ as in the following theorems.

Theorem 3.1. *Let δ, η, λ, $\tau \in \mathbb{R}$ be such that $\delta+\tau > 0$, $\lambda+\eta > 0$, $\lambda > \max\{0, \tau-1\}$, and $\tau > 0$. Also let $0 < \alpha \le 1$, $\mu \ge 0$, and $\gamma \ge 0$. If $f \in \mathcal{T}_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\alpha)$, then*

$$|a_2| \le \frac{4\alpha\,\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)}{(\mu+1)\Gamma(\delta+2\tau)\Gamma(\lambda+\eta)} \tag{12}$$

and

$$|a_3| \le \frac{12\alpha(2-\alpha)\,\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}. \tag{13}$$

Proof. It follows from conditions (8) and (9) that

$$\gamma z\left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta}f(z)\right)'' + (\mu-2\gamma)\left(\mathcal{H}^{\delta,\tau}_{\lambda,\eta}f(z)\right)' + (1-\mu+2\gamma)\,\frac{\mathcal{H}^{\delta,\tau}_{\lambda,\eta}f(z)}{z} = [p(z)]^\alpha \tag{14}$$

and

$$\gamma w\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)''+(\mu-2\gamma)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)'+(1-\mu+2\gamma)\frac{\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)}{w}=[q(w)]^{\alpha}, \tag{15}$$

where $g=f^{-1}$ and $p,\ q$ in $\mathcal{P}$ have the following series representations:

$$p(z)=1+p_1z+p_2z^2+p_3z^3+\cdots \tag{16}$$

and

$$q(w)=1+q_1w+q_2w^2+q_3w^3+\cdots. \tag{17}$$

Comparing the corresponding coefficients of (14) and (15) yields

$$\frac{(\mu+1)\Gamma(\delta+2\tau)\Gamma(\lambda+\eta)}{2\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)}a_2=\alpha p_1, \tag{18}$$

$$\frac{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}{6\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}a_3=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2, \tag{19}$$

$$-\frac{(\mu+1)\Gamma(\delta+2\tau)\Gamma(\lambda+\eta)}{2\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)}a_2=\alpha q_1 \tag{20}$$

and

$$\frac{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}{6\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}\left(2a_2^2-a_3\right)=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2. \tag{21}$$

Now, employing Lemma 1.1 in either (18) or (20), we obtain the inequality (12).

Also using Lemma 1.1 in (19), we get the inequality (13). This completes the proof.

□

Theorem 3.2. *Let $\delta,\ \eta,\ \lambda,\ \tau\in\mathbb{R}$ be such that $\delta+\tau>0,\ \lambda+\eta>0,\ \lambda>\max\{0,\tau-1\}$, and $\tau>0$. Also let $0\le\beta<1,\ \mu\ge 0$, and $\gamma\ge 0$. If $f\in\mathcal{T}_{\Sigma}^{*}(\gamma,\mu,\lambda,\eta,\delta,\tau;\beta)$, then*

$$|a_2|\le\frac{4(1-\beta)\,\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)}{(\mu+1)\Gamma(\delta+2\tau)\Gamma(\lambda+\eta)} \tag{22}$$

and

$$|a_3|\le\frac{12(1-\beta)\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}. \tag{23}$$

Proof. From (10) and (11), we find that there are $p,q\in\mathcal{P}$ such that

$$\gamma z\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}f(z)\right)''+(\mu-2\gamma)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}f(z)\right)'+(1-\mu+2\gamma)\frac{\mathcal{H}_{\lambda,\eta}^{\delta,\tau}f(z)}{z}=\beta+(1-\beta)p(z) \tag{24}$$

and

$$\begin{aligned}&\gamma w\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)''+(\mu-2\gamma)\left(\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)\right)'+(1-\mu+2\gamma)\,\frac{\mathcal{H}_{\lambda,\eta}^{\delta,\tau}g(w)}{w}\\&=\beta+(1-\beta)q(w),\end{aligned}\tag{25}$$

where $p(z)$ and $q(w)$ have the forms (16) and (17), respectively. Comparing the corresponding coefficients in (24) and (25) yields

$$\frac{(\mu+1)\Gamma(\delta+2\tau)\Gamma(\lambda+\eta)}{2\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)}a_2=(1-\beta)p_1,\tag{26}$$

$$\frac{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}{6\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}a_3=(1-\beta)p_2,\tag{27}$$

$$-\frac{(\mu+1)\Gamma(\delta+2\tau)\Gamma(\lambda+\eta)}{2\Gamma(\delta+\tau)\Gamma(2\lambda+\eta)}a_2=(1-\beta)q_1,\tag{28}$$

and

$$\frac{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}{6\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}\left(2a_2^2-a_3\right)=(1-\beta)q_2.\tag{29}$$

Now, using Lemma 1.1 in either (26) or (28), we get the inequality (22).

Also employing Lemma 1.1 in (27), we obtain the inequality (23). The proof is complete.

□

4. Fekete-Szegö type inequalities for the functions of the two families introduced in Section 2

Fekete and Szegö [13] (also see, e.g.,[7], [11, Theorem 3.8]) showed that if $f\in\mathcal{S}$, then

$$\left|a_3-\xi a_2^2\right|\leq\begin{cases}3-4\xi & (\xi\leq 0),\\ 1+2\,e^{-2\xi/(1-\xi)} & (0<\xi<1),\\ 4\xi-3 & (\xi\geq 1).\end{cases}\tag{30}$$

Obviously, the case $\xi=1$ of (30) produces the well-known inequality $\left|a_3-a_2^2\right|\leq 1$ for $f\in\mathcal{S}$. Since then, the estimation of an upper bound of $\left|a_3-\xi a_2^2\right|$ ($\xi\in\mathbb{R}$ or $\mathbb{C}$) for a function f which belongs to a variety of subclasses of $\mathcal{A}$, recently, in particular, associated with a number of newly-introduced operators, has been actively investigated (see, e.g., [1, 3, 5, 6, 7, 16, 23, 26, 31, 34, 35, 36, 40, 41, 43, 44]). Here the resulting upper bound estimations for $\left|a_3-\xi a_2^2\right|$ are called Fekete-Szegö type inequalities or Fekete-Szegö problems.

In this section, we provide the Fekete-Szegö type inequalities for the functions of the families $\mathcal{T}_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\alpha)$ and $\mathcal{T}_\Sigma^*(\gamma,\mu,\lambda,\eta,\delta,\tau;\beta)$ in the following theorems.

Theorem 4.1. *Let $\xi \in \mathbb{C}$. Also let δ, η, λ, $\tau \in \mathbb{R}$ be such that $\delta+\tau>0$, $\lambda+\eta>0$, $\lambda>\max\{0,\tau-1\}$, and $\tau>0$. Further let $0<\alpha\leq 1$, $\mu\geq 0$, and $\gamma\geq 0$. If $f\in\mathcal{T}_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\alpha)$, then*

$$\left|a_3-\xi a_2^2\right| \leq \frac{12\alpha(2-\alpha)\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}(1+|\xi|). \tag{31}$$

Proof. Using (19) in (21) gives

$$a_2^2=\left(\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right)\right)\frac{3\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}. \tag{32}$$

From (19) and (32) we obtain

$$\begin{aligned}a_3-\xi a_2^2=&\left\{2\alpha p_2+\alpha(\alpha-1)p_1^2-\xi\left(\alpha(p_2+q_2)+\frac{\alpha(\alpha-1)}{2}\left(p_1^2+q_1^2\right)\right)\right\}\\ &\times\frac{3\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}.\end{aligned} \tag{33}$$

Finally using Lemma 1.1 in (33) leads to the inequality (31). The proof is complete. ∎

Remark 3. A direct use of Lemma 1.1 in (21) gives

$$\left|a_3-2a_2^2\right| \leq \frac{12\alpha(2-\alpha)\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}. \tag{34}$$

It is interesting to observe that the inequality (34) is much sharper than the inequality (31) when $\xi=2$.

Theorem 4.2. *Let $\xi \in \mathbb{C}$. Also let δ, η, λ, $\tau \in \mathbb{R}$ be such that $\delta+\tau>0$, $\lambda+\eta>0$, $\lambda>\max\{0,\tau-1\}$, and $\tau>0$. Further let $0\leq\beta<1$, $\mu\geq 0$, and $\gamma\geq 0$. If $f\in\mathcal{T}^*_\Sigma(\gamma,\mu,\lambda,\eta,\delta,\tau;\beta)$, then*

$$\left|a_3-\xi a_2^2\right| \leq \frac{12(1-\beta)\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}\left(1+|\xi|\right). \tag{35}$$

Proof. From (26) and (28), we get

$$p_1=-q_1 \tag{36}$$

and

$$\frac{(\mu+1)^2\,\Gamma^2(\delta+2\tau)\Gamma^2(\lambda+\eta)}{4\Gamma^2(\delta+\tau)\Gamma^2(2\lambda+\eta)}a_2^2=(1-\beta)^2\,(p_1^2+q_1^2). \tag{37}$$

Adding (27) and (29), we obtain

$$\frac{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}{3\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}a_2^2=(1-\beta)(p_2+q_2).$$

Or, equivalently,

$$a_2^2 = \frac{3(1-\beta)\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)(p_2+q_2)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}. \tag{38}$$

From (27) and (38), we obtain

$$a_3 - \xi a_2^2 = \frac{3(1-\beta)\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}\big(2p_2 - \xi(p_2+q_2)\big). \tag{39}$$

Using Lemma 1.1 in (39) proves the inequality (35). This completes the proof. □

Remark 4. A direct application of Lemma 1.1 in (29) provides

$$\left|a_3 - 2a_2^2\right| \leq \frac{12(1-\beta)\Gamma(\delta+\tau)\Gamma(3\lambda+\eta)}{(2(\gamma+\mu)+1)\,\Gamma(\delta+3\tau)\Gamma(\lambda+\eta)}. \tag{40}$$

It is interesting to see that the inequality (40) is much sharper than the inequality (35) when $\xi = 2$.

5. Concluding remarks

We introduced two new families of analytic and bi-univalent functions associated with the Attiya's operator (5), which is defined by the Hadamard product of the generalized Mittag-Leffler function in (3) and a function in $\mathcal{A}$. Then we presented certain upper bounds for the Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$ for functions belonging to these families. We also investigated the Fekete-Szegö type inequalities for these families.

Our main results are found to reduce to several earlier results as follows: The results in Theorem 3.1 may reduce to

- [14, Theorem 2.2] when $\mu = 1 + 2\gamma$, $\lambda = 0$ and $\eta = \delta = \tau = 1$;
- [15, Theorem 2.2] when $\gamma = \lambda = 0$ and $\eta = \delta = \tau = 1$;
- [30, Theorem 1] when $\gamma = \lambda = 0$ and $\mu = \eta = \delta = \tau = 1$.

The results in Theorem 3.2 may reduce to

- [14, Theorem 3.2] when $\mu = 1 + 2\gamma$, $\lambda = 0$ and $\eta = \delta = \tau = 1$;
- [15, Theorem 3.2] when $\gamma = \lambda = 0$ and $\eta = \delta = \tau = 1$;
- [30, Theorem 2] when $\gamma = \lambda = 0$ and $\mu = \eta = \delta = \tau = 1$.

Since a number of generalizations of the Mittag-Leffler function (2) including (3) have been presented, for example, the generalized multiindex Mittag-Leffler function (see Saxena and Nishimoto [25]):

$$E^{\gamma,\kappa}_{(\alpha_j,\beta_j)_m}[z] = E_{\gamma,\kappa}\left[(\alpha_j,\beta_j)_{j=1}^{m};z\right] = \sum_{n=0}^{\infty}\frac{(\gamma)_{\kappa n}}{\prod\limits_{j=1}^{m}\Gamma(\alpha_j n+\beta_j)}.\frac{z^n}{n!}$$

$$(\alpha_j,\,\beta_j,\,\gamma,\,\kappa,\,z\in\mathbb{C},\ \Re(\beta_j)>0\ (j=1,\ldots,m);\ \Re\big(\sum_{j=1}^{m}\alpha_j\big)>\max\{0,\,\Re(\kappa)-1\}),\tag{41}$$

it seems natural to have some questions for further investigation as follows:

Two problems

(a) Using a more generalized Mittag-Leffler function than (3) or other generalized Mittag-Leffler function, give a corresponding operator like the Attiya's operator (5);

(b) With the new operator in (a) (if any), establish the corresponding results as those in Theorems 3.1, 3.2, 4.1 and 4.2.

For these problems, one of the referees is advised to consult [11, p. 41].

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