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AN INTRINSIC CRITERION FOR THE TYPE OF AUTOMORPHISMS OF THE UNIT DISC

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Abstract. In this paper, we deal with a problem to determine the type of automorphisms of the unit disc in $\mathbb{C}$ in terms of intrinsic geometry. We will characterize the hyperbolicity and parabolicity of automorphism by the distance function of the Poincaré metric.

1. Introduction

As a fundamental consequence of the unifomization theorem for Riemann surfaces due to H. Poincaré [9] and P. Koebe [6], a compact Riemann surface S of genus $g \geq 2$ is a quotient space of the unit disc,

$$\Delta = \{z \in \mathbb{C} : |z| < 1\} .$$

This means that there is a discrete subgroup Γ of the automorphism group $\mathrm{Aut}(\Delta)$ such that the quotient Δ/Γ is biholomorphic to S. Therefore the Lie group structure of $\mathrm{Aut}(\Delta)$ and its action on Δ has been studied in wide fields of Mathematics (see [7]).

In the study of $\mathrm{Aut}(\Delta)$, there is a typical trichotomy for non-trivial elements of $\mathrm{Aut}(\Delta)$ given by the action on Δ. A non-trivial automorphism f of Δ is called

(1) *elliptic* if f has a fixed point in Δ;
(2) *parabolic* if f has only one fixed point in the boundary $\partial\Delta$;
(3) *hyperbolic* if f has only two fixed points in the boundary $\partial\Delta$.

An elliptic automorphism is conjugate to a Euclidean rotation $z \mapsto e^{i\theta}z$ of $\mathbb{C}$. A parabolic automorphism acts as an ideal rotation centered at the boundary fixed point. A hyperbolic automorphism acts as a squeezing mapping attracting at a boundary fixed point and repelling at another fixed point (see [3]). This trichotomy is also characterized by the representation of f in $\mathrm{PSL}(2,\mathbb{R})$ which is isomorphic to $\mathrm{Aut}(\Delta)$. An element $\mathfrak{H}$ of $\mathrm{PSL}(2,\mathbb{R})$ is elliptic, parabolic or

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hyperbolic if $|\mathrm{Tr}\,\mathfrak{H}| < 2$, $= 2$ or > 2, with respectively. Then the type of an automorphism f of the unit disc is the same as that of its representation $\mathfrak{H}_f$ in $\mathrm{PSL}(2, \mathbb{R})$.

In this paper, we would characterize intrinsically this classification of $\mathrm{Aut}(\Delta)$; especially not using the boundary extension of the automorphisms on $\overline{\Delta}$. Let f be an automorphism of Δ without fixed point on Δ, so not elliptic. In order to determine intrinsically whether f is parabolic or hyperbolic, it is natural to consider the *Poincaré metric*,

$$ds_\Delta^2 = \frac{1}{\left(1 - |z|^2\right)^2} |dz|^2 \ ,$$

of the unit disc which is a complete hermitian metric of a negative constant curvature and the unique, holomorphically invariant metric of Δ (see [4]). The aim of this research is to distinguish the parabolic and hyperbolic automorphisms in terms of the Poincaré metric.

In the previous work [8] by the author, we gave an intrinsic characterization of automorphisms of Δ in terms of the Poincaré metric and iterating limit of automorphisms. For the intrinsic argument, we dealt with a simply connected Riemann surface S admitting a complete hermitian metric g of constant curvature $\kappa \equiv -4$. Then (S, g) is holomorphically isometric to the Poincaré disc model (Δ, ds_Δ^2). For a point $p \in S$, let $\varphi_{g,p} : S \to \mathbb{R}$ be a negatively valued function defined by

$$\varphi_{g,p}(z) = \tanh^2(d_g(p, z)) - 1 \tag{1.1}$$

for $z \in S$ where d_g is the distance function with respect to g. Let $f \in \mathrm{Aut}(S)$ be without fixed point and denote by $f^{(n)}$ the n-th iteration of f:

$$f^{(1)} = f \ , \quad f^{(n+1)} = f^{(n)} \circ f \ .$$

In [8], the author consider the sequence (φ_n) of functions defined by

$$\varphi_n(z) = \frac{\varphi_{g,p} \circ f^{(n)}}{\left(\varphi_{g,p} \circ f^{(n)}\right)(p)} \ . \tag{1.2}$$

Then it follows

Theorem 1.1 (Lee [8]). *For an automorphism f of S without fixed point, the sequence (φ_n) of functions in* (1.2) *converges to a positive function $\hat{\varphi}$ and*

$$\frac{\hat{\varphi} \circ f}{\hat{\varphi}} \equiv c$$

for some positive c. Moreover f is parabolic if and only if $c = 1$.

The function $\varphi_{g,p}$ in (1.2) is purely geometric quantity of $S \simeq \Delta$; thus this theorem is an intrinsic characterization of automorphisms of Δ. But by the limit procedure involved in the theorem, the type cannot be determined in the finite number of calculations.

The main result of this research is give another characterization of parabolicity and hyperbolicity by finite terms of iterations in (1.2).

Theorem 1.2. *Let S be a simply connected Riemann surface admitting a hermitian metric g of constant curvature -4. For a point $p \in S$, let $\varphi_{g,p} : S \to \mathbb{R}$ be a negative function defined as* (1.1)*. Let*

$$c_n = \frac{\left(\varphi_{g,p} \circ f^{(n+1)}\right)(p)}{\left(\varphi_{g,p} \circ f^{(n)}\right)(p)} = \frac{\tanh^2\left(d_g\left(p, f^{(n+1)}(p)\right)\right) - 1}{\tanh^2\left(d_g\left(p, f^{(n)}(p)\right)\right) - 1} .$$

Then f is parabolic if and only if

$$8c_2 - 5c_1c_2 = 3 .$$

As mentioned in Propositions 3.2 and 3.3, f is hyperbolic if and only if $8c_2 - 5c_1c_2 < 3$. Note that c_n is the value of φ_n in (1.2) evaluated by $f(p)$:

$$c_n = \varphi_n(f(p)) .$$

In Section 2, we will introduce a basic expression of parabolic and hyperbolic automorphisms of Δ. Then we will devote to confirm the main theorem for $S = \Delta$ and $p = 0$ in Section 3. A short proof for the theorem will be in Section 4.

2. Automorphisms of the unit disc

Let f be an automorphism of Δ. Then there are $\theta \in \mathbb{R}$ and $\alpha \in \Delta$ such that

$$f(z) = e^{i\theta}\frac{z + \alpha}{1 + \bar{\alpha}z}$$

When we assume that f leaves the boundary point $1 \in \partial\Delta$ fixed so f is parabolic, hyperbolic or the identity, then the rotational factor $e^{i\theta}$ is uniquely determined by α in the sense of $e^{i\theta} = (1 + \bar{\alpha})/(1 + \alpha)$. Therefore $f = f_\alpha$ where f_α is defined by

$$f_\alpha(z) = \frac{1 + \bar{\alpha}}{1 + \alpha}\frac{z + \alpha}{1 + \bar{\alpha}z} . \tag{2.1}$$

In order to find another fixed point of f_α, we can write the equation $f_\alpha(z) = z$ by

$$0 = (z - 1)(\bar{\alpha}(1 + \alpha)z + \alpha(1 + \bar{\alpha})) .$$

Thus another fixed point of f_α is only

$$q_\alpha = -\frac{\bar{\alpha}(1 + \alpha)}{\alpha(1 + \bar{\alpha})} \tag{2.2}$$

which also belongs to $\partial\Delta$. Thus f_α is parabolic if and only if $q_\alpha = 1$, equivalently,

$$\frac{1 - |\alpha|^2}{|1 + \alpha|^2} = 1 .$$

2.1. The parabolic case

From Proposition 2.3 in [8], the automorphism f_α in (2.1) is parabolic if and only if there is some nonzero $t \in \mathbb{R}$ such that

$$\alpha = p_t := \frac{-it}{2+it} .$$

In this case, the automorphism $f_\alpha = f_{p_t}$ can be written by

$$f_{p_t}(z) = \frac{1+\bar{p}_t}{1+p_t} \frac{z+p_t}{1+\bar{p}_t z} = \frac{(2+it)z - it}{itz + (2-it)} .$$

Moreover t is the parameter of the *parabolic subgroup*

$$\mathcal{P} = \{f_{p_t} : t \in \mathbb{R}\}$$

of $\mathrm{Aut}(\Delta)$ leaving 1 fixed in the sense that

$$f_{p_s} \circ f_{p_t} = f_{p_{s+t}} \quad \text{for any } s, t \in \mathbb{R}.$$

Therefore

$$f_{p_t}^{(n)} = f_{p_{nt}} . \tag{2.3}$$

Remark 2.1. The image of -1 under f_{p_t} is

$$f_{p_t}(-1) = \frac{-(2+it) - it}{-it + (2-it)} = -\frac{1+it}{1-it} = -\frac{1-t^2-2it}{1+t^2} = \frac{t^2-1}{t^2+1} + i\frac{2t}{t^2+1} .$$

Take any point $e^{i\theta} = \cos\theta + i\sin\theta$ on $\partial\Delta \setminus \{1\}$ where $0 < \theta < 2\pi$ and let $\eta = \theta/2$. For $t = \cot\eta$,

$$\begin{aligned} \frac{t^2-1}{t^2+1} &= \frac{\cot^2\eta - 1}{\cot^2\eta + 1} = \cos^2\eta - \sin^2\eta = \cos(2\eta) = \cos\theta , \\ \frac{2t}{t^2+1} &= \frac{2\cot^2\eta}{\cot^2\eta + 1} = 2\cos^2\eta\sin^2\eta = \sin(2\eta) = \sin\theta . \end{aligned}$$

This implies that $f_{p_t}(-1) = e^{i\theta}$. As a conclusion, the parabolic subgroup $\mathcal{P}$ acts on $\partial\Delta \setminus \{1\}$ transitively.

2.2. The hyperbolic case

Let f_α be a hyperbolic automorphism of Δ. The initial fixed point of f_α is 1. Suppose that f_α also leaves -1 fixed. Then (2.2) can be written by

$$1 = \frac{\bar{\alpha}(1+\alpha)}{\alpha(1+\bar{\alpha})} ,$$

equivalently, $\alpha = \bar{\alpha}$. Thus α is a nonzero real number with $-1 < \alpha < 1$. Therefore there is a unique nonzero t with $\tanh(t/2) = \alpha$, equivalently

$$\alpha = h_t := \tanh(t/2) = \frac{e^t - 1}{e^t + 1}$$

for some nonzero $t \in \mathbb{R}$. Note that for real α and β,

$$(f_\alpha \circ f_\beta)(z) = \frac{\frac{z+\beta}{1+\beta z} + \alpha}{1 + \alpha \frac{z+\beta}{1+\beta z}} = \frac{z + \beta + \alpha + \alpha\beta z}{1 + \beta z + \alpha z + \alpha\beta}$$

$$= \frac{(1+\alpha\beta)z + \beta + \alpha}{1 + \alpha\beta + (\alpha+\beta)z} = \frac{z + \frac{\alpha+\beta}{1+\alpha\beta}}{1 + \frac{\alpha+\beta}{1+\alpha\beta} z} = f_{\frac{\alpha+\beta}{1+\alpha\beta}} \ .$$

Applying $\alpha = h_s$ and $\beta = h_t$, then we have

$$f_{h_s} \circ f_{h_t} = f_{\frac{h_s+h_t}{1+h_s h_t}} = f_{h_{s+t}}$$

since

$$\frac{h_s + h_t}{1 + h_s h_t} = \frac{\tanh(s/2) + \tanh(t/2)}{1 + \tanh(s/2)\tanh(t/2)} = \tanh\left(\frac{s+t}{2}\right) = h_{s+t} \ .$$

Therefore we have

$$f_{h_t}^{(n)} = f_{h_{nt}} \ . \tag{2.4}$$

The hyperbolic group $\mathcal{H} = \{f_{h_t} : t \in \mathbb{R}\}$ is the set of automorphisms of Δ leaving 1, -1 fixed.

3. Testing the type of automorphisms for the unit disc

In this section, we will prove Theorem 1.2 for $S = \Delta$. In this case, the hermitian metric g of the theorem is the Poincaré metric ds^2_Δ. The distance function of ds^2_Δ is given by

$$d_{ds^2_\Delta}(z, w) = \tanh^{-1}\left|\frac{z - w}{1 - \bar{w}z}\right|$$

for $z, w \in \Delta$ (see [5, 2]). For a point $q \in \Delta$, the function in (1.1) is of the form

$$\varphi_q(z) := \varphi_{ds^2_\Delta, q}(z) = \tanh^2(d_{ds^2_\Delta}(q, z)) - 1 = \left|\frac{z - q}{1 - \bar{q}z}\right|^2 - 1 \ .$$

Throughout this section, we will mainly consider the case of $q = 0$. The test function above in this case is of the form,

$$\varphi_0(z) = \tanh^2(d_{ds^2}(0, z)) - 1 = |z|^2 - 1 \ . \tag{3.1}$$

For an automorphism f of Δ, the sequence of functions

$$\frac{\varphi_0 \circ f^{(n)}}{(\varphi_0 \circ f^{(n)})(0)}$$

converges some $\hat{\varphi}$ if f is parabolic or hyperbolic as shown in [8]. In this section, we will deal with

$$\frac{\varphi_0 \circ f}{(\varphi_0 \circ f)(0)} \quad \text{and} \quad \frac{\varphi_0 \circ f^{(2)}}{(\varphi_0 \circ f^{(2)})(0)}$$

and their values at $f(0)$. Before looking at this terms, we introduce widely used formulae in this section: for an automorphism f_α in (2.1) it follows that

$$(\varphi_0 \circ f_\alpha)(z) = \left|\frac{1+\bar\alpha}{1+\alpha}\frac{z+\alpha}{1+\bar\alpha z}\right|^2 - 1 = \frac{(|z|^2-1)(1-|\alpha|^2)}{|1+\bar\alpha z|^2}\ , \tag{3.2}$$

so

$$(\varphi_0 \circ f_\alpha)(0) = |\alpha|^2 - 1\ . \tag{3.3}$$

3.1. Parabolic automorphisms leaving 1 fixed

Let f be a parabolic automorphism of Δ leaving 1 fixed. Then there is a unique $t \in \mathbb{R} \setminus \{0\}$ such that $f = f_{p_t}$ where

$$p_t = \frac{-it}{2+it}$$

as we saw in Section 2.1.

We can write Equation (3.3) by

$$(\varphi_0 \circ f_{p_t})(0) = |p_t|^2 - 1 = \left|\frac{-it}{2+it}\right|^2 - 1 = \frac{t^2}{4+t^2} - 1 = -\frac{1}{4+t^2}\ .$$

For a fixed nonzero t, let

$$\begin{aligned} c_n &= \frac{\left(\varphi_0 \circ f_{p_t}^{(n+1)}\right)(0)}{\left(\varphi_0 \circ f_{pt}^{(n)}\right)(0)} = \frac{\left(\varphi \circ f_{p_{(n+1)t}}\right)(0)}{\left(\varphi \circ f_{p_{nt}}\right)(0)} \\ &= \left(-\frac{1}{4+(n+1)^2t^2}\right) \Big/ \left(-\frac{1}{4+n^2t^2}\right) = \frac{4+n^2t^2}{4+(n+1)^2t^2}\ . \end{aligned}$$

Since

$$8c_2 - 5c_1c_2 = 8\frac{4+4t^2}{4+9t^2} - 5\frac{4+t^2}{4+4t^2}\frac{4+4t^2}{4+9t^2} = \frac{12+27t^2}{4+9t^2} = 3\ ,$$

we have

Proposition 3.1. *Let f be a parabolic automorphism of Δ leaving 1 fixed. Then two numbers c_1 and c_2 defined by*

$$c_n = \frac{\left(\varphi_0 \circ f^{(n+1)}\right)(0)}{\left(\varphi_0 \circ f^{(n)}\right)(0)}\ .$$

satisfies $8c_2 - 5c_1c_2 = 3$.

3.2. Hyperbolic automorphisms leaving 1 and −1 fixed

If f is a hyperbolic automorphism of Δ with fixed point 1 and -1, then there is a unique nonzero t such that $f = f_{h_t}$ where

$$h_t = \tanh(t/2) = \frac{e^t - 1}{e^t + 1}$$

as in Section 2.2. Apply f_{h_t} to (3.3), we have

$$(\varphi_0 \circ f_{h_t})(0) = h_t^2 - 1 = \left(\frac{e^t - 1}{e^t + 1}\right)^2 - 1 = -\frac{4e^t}{(e^t + 1)^2} \ .$$

As the same way as Section 3.1, we can define a sequence of positive real numbers,

$$c_n = \frac{\left(\varphi_0 \circ f_{h_{(n+1)t}}\right)(0)}{\left(\varphi_0 \circ f_{h_{nt}}\right)(0)} = \frac{h_{(n+1)t}^2 - 1}{h_{nt}^2 - 1} = \frac{e^t(e^{nt} + 1)^2}{(e^{(n+1)t} + 1)^2} \tag{3.4}$$

for a fixed t. Note that c_1 and c_2 in this case do not satisfy the condition,

$$8c_2 - 5c_1c_2 = 3 \ ,$$

as in Proposition 3.1. From easy calculations as

$$\begin{aligned} \text{(3.5)} \quad 8c_2 - 5c_1c_2 - 3 &= 8\frac{e^t(e^{2t} + 1)^2}{(e^{3t} + 1)^2} - 5\frac{e^{2t}(e^t + 1)^2}{(e^{3t} + 1)^2} - 3 \\ &= \frac{-3e^{6t} + 8e^{5t} - 5e^{4t} - 5e^{2t} + 8e^t - 3}{(e^{3t} + 1)^2} \\ &= \frac{-(e^t - 1)^4(3e^{2t} + 4e^t + 3)}{(e^{3t} + 1)^2} \ , \end{aligned}$$

the number $8c_2 - 5c_1c_2 - 3$ is always negative for any nonzero t.

Proposition 3.2. *Let f be a hyperbolic automorphism of Δ leaving 1 and -1 fixed. For the numbers*

$$c_n = \frac{\left(\varphi_0 \circ f^{(n+1)}\right)(0)}{\left(\varphi_0 \circ f^{(n)}\right)(0)} \ ,$$

it satisfies that $8c_2 - 5c_1c_2 < 3$.

3.3. Hyperbolic automorphisms in general

Continuing from the previous subsection, we consider the hyperbolic automorphism f_{h_t} for some nonzero t. Here we will consider a point q in Δ and the sequence of numbers

$$c_n^q = \frac{\left(\varphi_q \circ f_{h_t}^{(n+1)}\right)(q)}{\left(\varphi_q \circ f_{h_t}^{(n)}\right)(q)} = \frac{\left(\varphi_q \circ f_{h_{(n+1)t}}\right)(q)}{\left(\varphi_q \circ f_{h_{nt}}\right)(q)} \tag{3.6}$$

where

$$\varphi_q(z) = \varphi_{ds^2_\Delta,q}(z) = \tanh^2(d_{ds^2}(q,z)) - 1 = \left|\frac{z-q}{1-\bar{q}z}\right|^2 - 1$$
$$= (|z|^2 - 1)\frac{1-|q|^2}{|1-\bar{q}z|^2} = \varphi_0(z)\frac{1-|q|^2}{|1-\bar{q}z|^2} .$$

Since

$$(\varphi_0 \circ f_{h_t})(z) = \frac{(|z|^2-1)(1-h_t^2)}{|1+h_tz|^2} ,$$

from Equation (3.2), it follows that

$$(\varphi_q \circ f_{h_{nt}})(q) = \frac{(|q|^2-1)(1-h_{nt}^2)}{|1+h_{nt}q|^2}\frac{1-|q|^2}{|1-\bar{q}f_{h_{nt}}(q)|^2}$$
$$= \frac{h_{nt}^2-1}{|1+h_{nt}q|^2}\frac{\left(1-|q|^2\right)^2}{|1-\bar{q}f_{h_{nt}}(q)|^2} .$$

So each c_n^q can be written by

$$c_n^q = \frac{\left(\varphi_q \circ f_{h_{(n+1)t}}\right)(q)}{\left(\varphi_q \circ f_{h_{nt}}\right)(q)} = \frac{h_{(n+1)t}^2-1}{h_{nt}^2-1}\frac{|1+h_{nt}q|^2}{\left|1+h_{(n+1)t}q\right|^2}\frac{|1-\bar{q}f_{h_{nt}}(q)|^2}{\left|1-\bar{q}f_{h_{(n+1)t}}(q)\right|^2}$$
$$= c_n\frac{|1+h_{nt}q|^2}{\left|1+h_{(n+1)t}q\right|^2}\frac{|1-\bar{q}f_{h_{nt}}(q)|^2}{\left|1-\bar{q}f_{h_{(n+1)t}}(q)\right|^2} ,$$

where $c_n = (h_{(n+1)t}^2 - 1)/(h_{nt}^2 - 1)$ as in (3.4).

We will calculate the value c_n^q in case of

$$q = \frac{-is}{2-is} \tag{3.7}$$

for some $s \in \mathbb{R}$. Let us consider

$$1-\bar{q}f_{h_{nt}}(q) = 1 - \frac{is}{2+is}\frac{q+h_{nt}}{1+h_{nt}q} = \frac{(2+is)(1+h_{nt}q)-is(q+h_{nt})}{(2+is)(1+h_{nt}q)} .$$

Applying (3.7) to the numerator only, we can get

$$1-\bar{q}f_{h_{nt}}(q) = \frac{1}{(2+is)(1+h_{nt}q)}\frac{4(1-ish_{nt})}{2-is} = \frac{4(1-ish_{nt})}{(4+s^2)(1+h_{nt}q)} .$$

Therefore, we have

$$\frac{|1-\bar{q}f_{h_{nt}}(q)|^2}{\left|1-\bar{q}f_{h_{(n+1)t}}(q)\right|^2} = \frac{1+s^2h_{nt}^2}{1+s^2h_{(n+1)t}^2}\frac{\left|1+h_{(n+1)t}q\right|^2}{|1+h_{nt}q|^2} .$$

This simplifies c_n^q as

$$c_n^q = c_n\frac{1+s^2h_{nt}^2}{1+s^2h_{(n+1)t}^2} . \tag{3.8}$$

It remains to check the value $8c_2^q - 5c_1^q c_2^q - 3$. Equation (3.8) gives

$$8c_2^q - 5c_1^q c_2^q - 3 = 8\left(c_2\frac{1+s^2h_{2t}^2}{1+s^2h_{3t}^2}\right) - 5\left(c_1\frac{1+s^2h_t^2}{1+s^2h_{2t}^2}\right)\left(c_2\frac{1+s^2h_{2t}^2}{1+s^2h_{3t}^2}\right) - 3$$
$$= \frac{8c_2 - 5c_1c_2 - 3 + s^2\left(8c_2h_{2t}^2 - 5c_1c_2h_t^2 - 3h_{3t}^2\right)}{1+s^2h_{3t}^2} .$$

From Equation (3.5) and the calculation as

$$8c_2h_{2t}^2 - 5c_1c_2h_t^2 - 3h_{3t}^2 = 8\frac{e^t(e^{2t}-1)^2}{(e^{3t}+1)^2} - 5\frac{e^{2t}(e^t-1)^2}{(e^{3t}+1)^2} - 3\frac{(e^{3t}-1)^2}{(e^{3t}+1)^2}$$
$$= \frac{-3e^{6t}+8e^{5t}-5e^{4t}-5e^{2t}+8e^t-3}{(e^{3t}+1)^2} = -\frac{(e^t-1)^4(3e^{2t}+4e^t+3)}{(e^{3t}+1)^2} ,$$

we can write the numerator by

$$8c_2 - 5c_1c_2 - 3 + s^2\left(8c_2h_{2t}^2 - 5c_1c_2h_t^2 - 3h_{3t}^2\right)$$
$$= -\frac{(e^t-1)^4(3e^{2t}+4e^t+3)}{(e^{3t}+1)^2} - s^2\frac{(e^t-1)^4(3e^{2t}+4e^t+3)}{(e^{3t}+1)^2}$$
$$= -\frac{(1+s^2)(e^t-1)^4(3e^{2t}+4e^t+3)}{(e^{3t}+1)^2} .$$

As a conclusion, the value $8c_2^q - 5c_1^q c_2^q - 3$ is always negative for any nonzero t.

Proposition 3.3. *Let f be a hyperbolic automorphism of Δ leaving 1 fixed and let*

$$c_n = \frac{\left(\varphi_0 \circ f^{(n+1)}\right)(0)}{(\varphi_0 \circ f^{(n)})(0)} .$$

Then $8c_2 - 5c_1c_2 < 3$.

Proof. Let f_α be a hyperbolic automorphism leaving 1 fixed. Another fixed point $q_\alpha = -\bar{\alpha}(1+\alpha)/\alpha(1+\bar{\alpha})$ as in (2.2) lies on $\partial\Delta \setminus \{1\}$. As mentioned in Remark 2.1, we have a suitable s such that

$$f_{p_s}(q_\alpha) = -1 .$$

Then one can easily see that the automorphism $f_{p_s} \circ f_\alpha \circ f_{p_s}^{-1}$ leaving 1 and -1 fixed. Therefore there is a nonzero t with

$$f_{h_t} = f_{p_s} \circ f_\alpha \circ f_{p_s}^{-1} = f_{p_s} \circ f_\alpha \circ f_{p_{-s}} ,$$

as showed in Section 2.2. Let

$$q = f_{p_s}(0) = \frac{1+\bar{p}_s}{1+p_s}p_s = \frac{-is}{2-is} .$$

which coincides with (3.7). Since any automorphism of Δ is isometric with respect to ds^2_Δ, we can get

$$\begin{aligned} d_{ds^2_\Delta}(0, f_\alpha(0)) &= d_{ds^2_\Delta}(f_{p_s}(0), f_{p_s}(f_\alpha(0))) \\ &= d_{ds^2_\Delta}\left(f_{p_s}(f_{p_{-s}}(q)), f_{p_s}\left(f_\alpha(f_{p_{-s}}(q))\right)\right) \\ &= d_{ds^2_\Delta}(q, f_{h_t}(q)) \end{aligned}$$

from $f_{p_{-s}}(q) = f_{p_s}^{-1}(q) = 0$. Therefore c_n is the same as c_n^q as in (3.6) since

$$c_n = \frac{\left(\varphi_0 \circ f_\alpha^{(n+1)}\right)(0)}{\left(\varphi_0 \circ f_\alpha^{(n)}\right)(0)} = \frac{\left(\varphi_q \circ f_{h_t}^{(n+1)}\right)(q)}{\left(\varphi_q \circ f_{h_t}^{(n)}\right)(q)} = c_n^q .$$

Proposition 3.2 implies $8c_2 - 5c_1c_2 < 3$. □

4. Proof of Theorem 1.2

Let S be a simple connected Riemann surface with a complete hermitian metric g of constant curvature -4. Then we can take a biholomorphism

$$F : \Delta \to S$$

from the uniformization theorem for Riemann surfaces. The Schwarz lemma for negatively curved Riemann surfaces due to L. V. Ahlfors [1] (see also [2]) implies that F is also an isometry from (Δ, ds^2_Δ) to (S, g).

Let f be an automorphism of S without fixed point and let $p \in S$. Then the automorphism

$$F^* f = F^{-1} \circ f \circ F$$

of Δ has at least one fixed point at $\partial\Delta$. Since Δ is homogeneous and rotationally symmetric, we may assume that

$$F(0) = p$$

and the extension of F^*f to $\overline{\Delta}$ leaves the point 1 fixed, that is, $(F^*f)(1) = 1$. That means that

$$F^* f = f_\alpha$$

for some $\alpha \in \Delta$ as in (2.1).

Since $F : (\Delta, ds^2_\Delta) \to (S, g)$ is isometric, it follows that

$$d_g(p, F(z)) = d_g(F(0), F(z)) = d_{ds^2_\Delta}(0, z) = \tanh^{-1}|z|$$

for any $z \in \Delta$. This implies

$$F^* \varphi_{g,p} = \varphi_{g,p} \circ F = \varphi_0$$

where $\varphi_0(z) = |z|^2 - 1$ is the test function in (3.1). Since

$$\begin{aligned} F^* f^{(n)} = F^{-1} \circ f^{(n)} \circ F &= \underbrace{(F^{-1} \circ f \circ F) \circ \cdots \circ (F^{-1} \circ f \circ F)}_{n} \\ &= (F^* f)^{(n)} = f_\alpha^{(n)} \end{aligned}$$

and $\varphi_{g,p} = \varphi_0 \circ F^{-1}$, we have that

$$c_n = \frac{\left(\varphi_{g,p} \circ f^{(n+1)}\right)(p)}{\left(\varphi_{g,p} \circ f^{(n)}\right)(p)} = \frac{\left(\varphi_0 \circ F^{-1} \circ f^{(n+1)}\right)(F(0))}{\left(\varphi_0 \circ F^{-1} \circ f^{(n)}\right)(F(0))}$$

$$= \frac{\left(\varphi_0 \circ F^{-1} \circ f^{(n+1)} \circ F\right)(0)}{\left(\varphi_0 \circ F^{-1} \circ f^{(n)} \circ F\right)(0)} = \frac{\left(\varphi_0 \circ f_\alpha^{(n+1)}\right)(0)}{\left(\varphi_0 \circ f_\alpha^{(n)}\right)(0)}$$

coinsides with c_n in Propositions 3.1, 3.2, 3.3. Therefore f_α is parabolic if and only if $8c_2 - 5c_1c_2 = 3$. This completes the proof. □

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