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ON ARTINIANNESS OF GENERAL LOCAL COHOMOLOGY MODULES

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Abstract. In this paper, we show some results on the artinianness of local cohomology modules with respect to a system of ideals. If M is a Φ-minimax ZD-module, then $H_{\Phi}^{\dim M}(M)/\mathfrak{a}H_{\Phi}^{\dim M}(M)$ is artinian for all $\mathfrak{a} \in \Phi$. Moreover, if M is a Φ-minimax ZD-module, t is a non-negative integer and $H_{\Phi}^{i}(M)$ is minimax for all $i > t$, then $H_{\Phi}^{i}(M)$ is artinian for all $i > t$.

1. Introduction

Throughout this paper, R is a noetherian commutative (with non-zero identity) ring and Φ is a system of ideals of R. It is well-known that the local cohomology theory of Grothendieck is an important tool in commutative algebra and algebraic geometry. In [3], a non-empty set of ideals Φ of R is called to be a system of ideals if whenever $\mathfrak{a}, \mathfrak{b} \in \Phi$, then there is an ideal $\mathfrak{c} \in \Phi$ such that $\mathfrak{c} \subseteq \mathfrak{a}\mathfrak{b}$. For an R-module M, the Φ-torsion submodule of M is $\Gamma_{\Phi}(M) = \{x \in M \mid \mathfrak{a}x = 0 \text{ for some } \mathfrak{a} \in \Phi\}$. The authors denoted by H_{Φ}^{i} the i-th right derived functor of the functor Γ_{Φ}. It is clear that when $\Phi = \{\mathfrak{a}^n \mid n \in \mathbb{N}\}$, the functor H_{Φ}^{i} coincides with the usual local cohomology functor $H_{\mathfrak{a}}^{i}$. In [3, Proposition 2.3], Bijan-Zadeh showed that

$$H_{\Phi}^{i}(M) \cong \varinjlim_{\mathfrak{a}\in\Phi} \operatorname{Ext}_R^i(R/\mathfrak{a}, M)$$

for all $i \geq 0$. Moreover, [4, Lemma 2.1] gave us the isomorphism

$$H_{\Phi}^{i}(M) \cong \varinjlim_{\mathfrak{a}\in\Phi} H_{\mathfrak{a}}^{i}(M)$$

for all $i \geq 0$.

We recall that an R-module M is minimax if there is a finitely generated submodule N of M such that M/N is artinian. The minimax modules were first introduced in [12] and then developed in [9, 13]. It is clear that if M is a minimax R-module and $\operatorname{Supp}_R M \subseteq \operatorname{Max} R$, then M is artinian. We see that M

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is minimax if and only if M/N has finite Goldie dimension for each submodule N of M. Note that, an R-module N is said to have finite Goldie dimension (written $\mathrm{Gdim} N < \infty$) if N does not contain an infinite direct sum of non-zero submodules, or equivalently the injective hull $E(N)$ of N decomposes as a finite direct sum of indecomposable (injective) submodules.

For a prime ideal $\mathfrak{p}$, let $\mu^0(\mathfrak{p}, N)$ denote the 0-th Bass number of N with respect to the prime ideal $\mathfrak{p}$. It is known that $\mu^0(\mathfrak{p}, N) > 0$ if and only if $\mathfrak{p} \in \mathrm{Ass}_R N$. It is clear by the definition of the Goldie dimension that

$$\mathrm{Gdim} N = \sum_{\mathfrak{p} \in \mathrm{Ass}_R N} \mu^0(\mathfrak{p}, N).$$

In [6], the authors introduced and studied the concept $\mathfrak{a}$-relative Goldie dimension which is a generalization of Goldie dimension. They used the $\mathfrak{a}$-relative Goldie dimension to investigate the artinianness of local cohomology modules with respect to an ideal. Let $\mathfrak{a}$ be an ideal of R. The $\mathfrak{a}$-relative Goldie dimension of N, denoted by $\mathrm{Gdim}_{\mathfrak{a}} N$, is defined as

$$\mathrm{Gdim}_{\mathfrak{a}} N = \sum_{\mathfrak{p} \in V(\mathfrak{a}) \cap \mathrm{Ass}_R N} \mu^0(\mathfrak{p}, N).$$

Since $\mathrm{Ass}_R N \cap V(\mathfrak{a}) = \mathrm{Ass}_R \Gamma_{\mathfrak{a}}(N)$, one obtains

$$\mathrm{Gdim}_{\mathfrak{a}} N = \mathrm{Gdim} \Gamma_{\mathfrak{a}}(N).$$

By using $\mathfrak{a}$-relative Goldie dimension and ZD-modules, many results on the artinianness of local cohomology modules were provided in [6].

In [5], the author showed that $H_{\Phi}^{\dim R}(R)$ is artinian provided that R is a local ring. In generally, if M is a finitely generated R-module, then $H_{\Phi}^{\dim M}(M)$ is not artinian.

The purpose of this paper is to investigate the artinianness of local cohomology modules with respect to a system of ideals $H_{\Phi}^i(M)$. First, we introduce the concept Φ-relative Goldie dimension of a module which is an extension of $\mathfrak{a}$-relative Goldie dimension in [6]. Next, an R-module M is called Φ-minimax if the Φ-relative Goldie dimension of any quotient module of M is finite. The first main result is Theorem 3.1 which says that if M is a Φ-minimax ZD-module, then $H_{\Phi}^{\dim M}(M)/\mathfrak{a} H_{\Phi}^{\dim M}(M)$ is artinian for all $\mathfrak{a} \in \Phi$. Next, we will see in Theorem 3.2 that if $(R, \mathfrak{m})$ is a local ring and M is a Φ-minimax ZD-module such that $\mathrm{Supp}_R H_{\Phi}^i(M) \subseteq \{\mathfrak{m}\}$ for all $i < t$, then $H_{\Phi}^i(M)$ is artinian for all $i < t$. Theorem 3.3 is devoted to the study the relationship on the vanishing and the finiteness of $H_{\Phi}^i(M)$. The paper is closed by Theorem 3.8 which shows that if M is a Φ-minimax ZD-module and $H_{\Phi}^i(M)$ is minimax for all $i > t$, then $H_{\Phi}^i(M)$ is artinian for all $i > t$.

2. Φ-minimax modules

Let Φ be a system of ideals of R. An R-module M is called Φ-torsion-free if $\Gamma_\Phi(M) = 0$. It is worth noting that

$$\Gamma_\Phi(M) = \bigcup_{\mathfrak{a}\in\Phi}(0 :_M \mathfrak{a}) \cong \varinjlim_{\mathfrak{a}\in\Phi} \Gamma_{\mathfrak{a}}(M).$$

Therefore, if M is Φ-torsion-free, then it is $\mathfrak{a}$-torsion-free for all $\mathfrak{a} \in \Phi$. Let $\Omega = \cup_{\mathfrak{a}\in\Phi} V(\mathfrak{a})$, we introduce a new notion which is motivated by $\mathfrak{a}$-relative Goldie dimension.

Definition 2.1. Let Φ be a system of ideals of R and M an R-module. The Φ-relative Goldie dimension of M, denoted by $\mathrm{Gdim}_\Phi M$, is defined as

$$\mathrm{Gdim}_\Phi M = \sum_{\mathfrak{p}\in\Omega\cap\mathrm{Ass}_R M} \mu^0(\mathfrak{p}, M).$$

Lemma 2.2. *Let Φ be a system of ideals of R and M an R-module. Then*

$$\mathrm{Ass}_R\Gamma_\Phi(M) = \mathrm{Ass}_R M \cap \Omega.$$

Proof. Let $\mathfrak{p} \in \mathrm{Ass}_R\Gamma_\Phi(M)$. There exists a non-zero element $x \in \Gamma_\Phi(M)$ such that $\mathfrak{p} = \mathrm{Ann}_R\, x$. Since $x \in \Gamma_\Phi(M)$, we have an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a}x = 0$. This implies that $\mathfrak{a} \subseteq \mathfrak{p}$ and then $\mathrm{Ass}_R\Gamma_\Phi(M) \subseteq \mathrm{Ass}_R M \cap \Omega$.

Let $\mathfrak{p} \in \mathrm{Ass}_R M \cap \Omega$. Then we have a non-zero element $x \in M$ and $\mathfrak{a} \in \Phi$ such that $\mathfrak{a} \subseteq \mathfrak{p} = \mathrm{Ann}_R\, x$. Therefore, one obtains $\mathfrak{a}x = 0$. Hence, $x \in \Gamma_\Phi(M)$ and $\mathfrak{p} \in \mathrm{Ass}_R\Gamma_\Phi(M)$. □

Corollary 2.3. *Let Φ be a system of ideals of R and M an R-module. Then* $\mathrm{Gdim}_\Phi M = \mathrm{Gdim}\Gamma_\Phi(M)$.

In [7], an R-module M is said to be a ZD-module if for any submodule N of M, the set $Z_R(M/N)$ is a finite union of prime ideals in $\mathrm{Ass}_R(M/N)$. Let S be a multiplicatively closed subset of R, we denote

$$S^{-1}\Phi = \{S^{-1}\mathfrak{a} \mid \mathfrak{a} \in \Phi\}.$$

If $\mathfrak{p} \in \mathrm{Spec}R$ and $S = R \setminus \mathfrak{p}$, we rewrite $S^{-1}\Phi$ by $\Phi_{\mathfrak{p}}$.

Proposition 2.4. *Let M be a ZD-module. The following statements are equivalent*:

(i) $\mathrm{Gdim}_\Phi M$ *is finite*;
(ii) $\mathrm{Gdim}_{\Phi_{\mathfrak{p}}} M_{\mathfrak{p}}$ *is finite for all prime ideals* $\mathfrak{p}$ *of* R;
(iii) $\mathrm{Gdim}_{\Phi_{\mathfrak{p}}} M_{\mathfrak{p}}$ *is finite for all prime ideals* $\mathfrak{p}$, *which is maximal in* $\mathrm{Ass}_R M$.

Proof. (i)$\Rightarrow$ (ii) It follows from Corollary 2.3 that $\mathrm{Gdim}_{\Phi_{\mathfrak{p}}} M_{\mathfrak{p}} = \mathrm{Gdim}\Gamma_{\Phi_{\mathfrak{p}}}(M_{\mathfrak{p}})$. Let $E(\Gamma_\Phi(M))$ be the injective hull of $\Gamma_\Phi(M)$. Note that, if S is a multiplicatively closed subset of R, then

$$S^{-1}E_R(R/\mathfrak{p}) \cong \begin{cases} 0, & S\cap\mathfrak{p} \neq \emptyset, \\ E_R(R/\mathfrak{p}), & S\cap\mathfrak{p} = \emptyset. \end{cases}$$

On the other hand, if $S \cap \mathfrak{p} = \emptyset$, then

$$S^{-1}E_R(R/\mathfrak{p}) \cong E_{S^{-1}R}(S^{-1}(R/\mathfrak{p})).$$

Hence, we have

$$\begin{aligned} E(\Gamma_\Phi(M))_\mathfrak{p} &= \bigoplus_{\mathfrak{q}\subseteq\mathfrak{p},\mathfrak{q}\in \mathrm{Ass}_R\Gamma_\Phi(M)} \mu^0(\mathfrak{q}, M)E_R(R/\mathfrak{q}) \\ &\cong \bigoplus_{\mathfrak{q}\subseteq\mathfrak{p},\mathfrak{q}\in \mathrm{Ass}_R\Gamma_\Phi(M)} \mu^0(\mathfrak{q}, M)E_{R_\mathfrak{p}}(R_\mathfrak{p}/\mathfrak{q}R_\mathfrak{p}) \\ &= E(\Gamma_{\Phi_\mathfrak{p}}(M_\mathfrak{p})). \end{aligned}$$

Therefore, we can claim that $\mathrm{Gdim}_{\Phi_\mathfrak{p}} M_\mathfrak{p} \leq \mathrm{Gdim}_\Phi M$.

(ii)⇒(iii) It is clear.

(iii)⇒(i) Since M is a ZD-module, by [6, Lemma 2.3] we may assume that $\{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ is the set of maximal elements in $\mathrm{Ass}_R M$. For each $\mathfrak{p}_i$, we have by Corollary 2.3 that

$$\begin{aligned} \mathrm{Gdim}_{\Phi_{\mathfrak{p}_i}} M_{\mathfrak{p}_i} &= \sum_{\mathfrak{p}R_{\mathfrak{p}_i}\in \mathrm{Ass}_{R_{\mathfrak{p}_i}}\Gamma_{\Phi_{\mathfrak{p}_i}}(M_{\mathfrak{p}_i})} \mu^0(\mathfrak{p}R_{\mathfrak{p}_i}, M_{\mathfrak{p}_i}) \\ &= \sum_{\mathfrak{p}\in \mathrm{Ass}_R\Gamma_\Phi(M),\mathfrak{p}\subseteq\mathfrak{p}_i} \mu^0(\mathfrak{p}, M). \end{aligned}$$

This implies that

$$\begin{aligned} \mathrm{Gdim}_\Phi M &\leq \sum_{i=1}^{k} \left(\sum_{\mathfrak{p}\in \mathrm{Ass}_R\Gamma_\Phi(M),\mathfrak{p}\subseteq\mathfrak{p}_i} \mu^0(\mathfrak{p}, M) \right) \\ &= \sum_{i=1}^{k} \mathrm{Gdim}_{\Phi_{\mathfrak{p}_i}} M_{\mathfrak{p}_i}, \end{aligned}$$

and which completes the proof. □

Azami, Naghipour and Vakili [2] defined that an R-module N is $\mathfrak{a}$-minimax if the $\mathfrak{a}$-relative Goldie dimension of any quotient module of N is finite. The concept of $\mathfrak{a}$-minimax modules is a generalization of the one of miminax modules. The following definition is an extension of $\mathfrak{a}$-minimax modules.

Definition 2.5. An R-module M is Φ-minimax if the Φ-relative Goldie dimension of any quotient module of M is finite.

We have some primary properties on Φ-minimax modules.

Proposition 2.6. *Let M be an R-module. The following statements hold*:

(i) *If M is a Φ-minimax R-module, then $\mathrm{Ass}_R M \cap \Omega$ is a finite set.*
(ii) *Let $0 \to A \to B \to C \to 0$ be a short exact sequence of R-modules. Then B is Φ-minimax if and only if A and C are both Φ-minimax. Thus any submodule and quotient of a Φ-minimax module as well as any finite direct sum of Φ-minimax modules are Φ-minimax.*

(iii) *Let N be a finitely generated R-module and M a Φ-minimax R-module. Then $\mathrm{Ext}_R^i(N,M)$ and $\mathrm{Tor}_i^R(N,M)$ are Φ-minimax for all $i \geq 0$.*

Proof. (i) It follows from Definition 2.5.

(ii) We can assume that A is a submodule of B and $C \cong B/A$.

($\Rightarrow$) Let B be a Φ-minimax R-module and A' a submodule of A. The short exact sequence

$$0 \to \frac{A}{A'} \to \frac{B}{A'} \to \frac{B}{A} \to 0$$

induces the following exact sequence

$$0 \to \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), \frac{A_\mathfrak{p}}{A'_\mathfrak{p}}) \to \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), \frac{B_\mathfrak{p}}{A'_\mathfrak{p}}),$$

where $\mathfrak{p} \in \mathrm{Spec}R$ and $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Moreover, we have

$$\mathrm{Ass}_R(A/A') \cap \Omega \subseteq \mathrm{Ass}_R(B/A') \cap \Omega.$$

Consequently, we can conclude that A is Φ-minimax. Next, let C' be a submodule of C. There exists a submodule D of B containing A such that $C' \cong D/A$. It follows that $C/C' \cong B/D$ and then C is Φ-minimax.

($\Leftarrow$) Assume that A, C are both Φ-minimax and B' a submodule of B. The short exact sequence

$$0 \to \frac{A}{A \cap B'} \to \frac{B}{B'} \to \frac{B}{A+B'} \to 0$$

induces the following exact sequence

$$0 \to \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), \frac{A_\mathfrak{p}+B'_\mathfrak{p}}{B'_\mathfrak{p}}) \to \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), \frac{B_\mathfrak{p}}{B'_\mathfrak{p}}) \to \mathrm{Hom}_{R_\mathfrak{p}}(k(\mathfrak{p}), \frac{B_\mathfrak{p}}{A_\mathfrak{p}+B'_\mathfrak{p}}),$$

where $\mathfrak{p} \in \mathrm{Spec}R$ and $k(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$. Moreover, there is an isomorphism

$$\frac{B}{A+B'} \cong \frac{B/A}{(A+B')/A}$$

and an inclusion

$$\mathrm{Ass}_R(B/B') \cap \Omega \subseteq (\mathrm{Ass}_R(A/A \cap B') \cap \Omega) \cup (\mathrm{Ass}_R(B/(A+B')) \cap \Omega).$$

Therefore, one can claim that B is Φ-minimax.

(iii) We will prove the assertion for the Ext modules, and it is similar to the case of the Tor modules. Since N is a finitely generated R-module over a noetherian ring, N has a free resolution

$$\mathbf{F}: \quad \cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to 0,$$

where F_i is finitely generated free for all $i \geq 0$. Consequently, for each non-negative integer i, there is a positive integer t such that $\mathrm{Hom}_R(F_i, M) = \oplus^t M$. We know that $\mathrm{Ext}_R^i(N,M) = H^i(\mathrm{Hom}_R(\mathbf{F}, M))$ which is a subquotient of the Φ-minimax module $\mathrm{Hom}_R(F_i, M)$. Hence, the assertion follows from (ii). $\square$

Lemma 2.7. *Let M be an R-module such that* $\mathrm{Ass}_R M \subseteq \Omega$. *Then* $M = \Gamma_\Phi(M)$.

Proof. Let $x \in M$ a non-zero element. Then $\mathrm{Ass}_R(Rx) = \{\mathfrak{p}_1, \dots, \mathfrak{p}_k\}$ is a finite set. There exist $\mathfrak{a}_1, \dots, \mathfrak{a}_k \in \Phi$ such that $\mathfrak{p}_i \in V(\mathfrak{a}_i)$ for each $i = 1, 2, \dots, k$. Note that

$$\sqrt{\mathrm{Ann}_R(Rx)} = \cap_{i=1}^k \mathfrak{p}_i \supseteq \cap_{i=1}^k \mathfrak{a}_i \supseteq \Pi_{i=1}^k \mathfrak{a}_i.$$

Hence, there is an ideal $\mathfrak{a} \in \Phi$ such that $\mathfrak{a}x = 0$. We assert that $M = \Gamma_\Phi(M)$, as required. □

The following result is an extension of [2, Proposition 2.6].

Proposition 2.8. *Let M be a Φ-minimax R-module and* $\mathrm{Ass}_R M \subseteq \Omega$. *Then $H^i_\Phi(M)$ is Φ-minimax for all $i \geq 0$.*

Proof. Since $\Gamma_\Phi(M)$ is a submodule of M, it follows from Proposition 2.6(ii) that $\Gamma_\Phi(M)$ is Φ-minimax.

By Lemma 2.7, we have $M = \Gamma_\Phi(M)$. It follows from [10, 1.4] that $H^i_\Phi(M) = 0$ for all $i > 0$, and the proof is complete. □

3. On the artinianness of local cohomology modules

In this section, we will consider the artinianness of general local cohomology module $H^i_\Phi(M)$ under condition that M is a Φ-minimax ZD-module. In [6, Corollary 3.3], if M is a $\mathfrak{a}$-minimax ZD-module of dimension n, then $H^n_\mathfrak{a}(M)$ is artinian. We have the first main result of this paper which is an extension of [6, Corollary 3.3].

Theorem 3.1. *Let M be a ZD-module of dimension n. Assume that M is Φ-minimax. Then $H^n_\Phi(M)/\mathfrak{a}H^n_\Phi(M)$ is artinian for all $\mathfrak{a} \in \Phi$.*

Proof. The proof is by induction on n. Let $n = 0$. By Proposition 2.6(ii), $H^0_\Phi(M)$ is Φ-minimax. Hence, the set $\mathrm{Ass}_R H^0_\Phi(M)$ is finite. Moreover, we have $\mathrm{Ass}_R H^0_\Phi(M) \subseteq \mathrm{Ass}_R M \subseteq \mathrm{Max} R$ since $\dim M = 0$. This implies that the injective hull $E(H^0_\Phi(M))$ is artinian. Hence, so is $H^0_\Phi(M)$.

Let $n > 0$. It follows from [10, 1.4] that $H^n_\Phi(M) \cong H^n_\Phi(M/\Gamma_\Phi(M))$. We may assume that M is Φ-torsion-free. Therefore, M is $\mathfrak{a}$-torsion-free for all $\mathfrak{a} \in \Phi$. Let $\mathfrak{a} \in \Phi$, there exists an element $x \in \mathfrak{a}$ which is M-regular. The short exact sequence

$$0 \to M \xrightarrow{x} M \to M/xM \to 0$$

yields the following exact sequence

$$H^{n-1}_\Phi(M/xM) \to H^n_\Phi(M) \xrightarrow{x} H^n_\Phi(M) \to H^n_\Phi(M/xM).$$

By [7, Proposition 4] and Proposition 2.6, M/xM is a Φ-minimax ZD-module and $\dim M/xM \leq n-1$. In view of [3, 2.7], one obtains $H^n_\Phi(M/xM) = 0$.

Applying the functor $R/\mathfrak{a}\otimes_R -$ to the above exact sequence, we have a following exact sequence

$$H_\Phi^{n-1}(M/xM)/\mathfrak{a}H_\Phi^{n-1}(M/xM) \to H_\Phi^n(M)/\mathfrak{a}H_\Phi^n(M) \xrightarrow{x} H_\Phi^n(M)/\mathfrak{a}H_\Phi^n(M) \to 0.$$

The inductive hypothesis shows that $H_\Phi^{n-1}(M/xM)/\mathfrak{a}H_\Phi^{n-1}(M/xM)$ is artinian. This leads the artinianness of $0 :_{H_\Phi^n(M)/\mathfrak{a}H_\Phi^n(M)} x$. Since $x \in \mathfrak{a}$, it follows from [8, Theorem 1.3] that $H_\Phi^n(M)/\mathfrak{a}H_\Phi^n(M)$ is artinian. □

Theorem 3.2. *Let $(R,\mathfrak{m})$ be a local ring, M a ZD-module and t a non-negative integer. Assume that M is Φ-minimax and $\mathrm{Supp}_R H_\Phi^i(M) \subseteq \{\mathfrak{m}\}$ for all $i < t$. Then $H_\Phi^i(M)$ is artinian for all $i < t$.*

Proof. The proof is by induction on i. It is similar to the argument of the proof of Theorem 3.1, we see that $H_\Phi^0(M)$ is artinian. Let $i > 0$ and assume that $H_\Phi^{t-2}(M)$ is artinian. Let $\overline{M} = M/\Gamma_\Phi(M)$, it follows from [10, 1.4] that $H_\Phi^i(M) \cong H_\Phi^i(\overline{M})$. Since $\overline{M}$ is Φ-torsion-free, it is $\mathfrak{a}$-torsion-free for any $\mathfrak{a} \in \Phi$. Let $\mathfrak{a} \in \Phi$, then $\mathfrak{a}$ contains an element x which is $\overline{M}$-regular. The short exact sequence

$$0 \to \overline{M} \xrightarrow{x} \overline{M} \to \overline{M}/x\overline{M} \to 0$$

leads the following exact sequence

$$H_\Phi^{i-1}(\overline{M}/x\overline{M}) \to H_\Phi^i(\overline{M}) \xrightarrow{x} H_\Phi^i(\overline{M}) \to \cdots .$$

By the assumption, $\mathrm{Supp}_R H_\Phi^i(\overline{M}/x\overline{M}) \subseteq \{\mathfrak{m}\}$ for all $i < t-1$. Combining Proposition 2.6 with [7, Proposition 1.4] we see that $\overline{M}/x\overline{M}$ is a ZD-module as well as Φ-minimax module. The inductive hypothesis deduces that $H_\Phi^i(\overline{M}/x\overline{M})$ is artinian for all $i < t-1$. Hence, one gets that $0 :_{H_\Phi^{t-1}(M)} x$ is artinian. Since $\mathrm{Supp}_R H_\Phi^{t-1}(M) \subseteq \{\mathfrak{m}\} \subseteq V(xR)$, the artinianness of $H_\Phi^{t-1}(M)$ is followed from [8, Theorem 1.3]. □

Next, we have a connection on the finiteness and the vanishing of local cohomology modules with resect to a system of ideals. This is also an improvement of [11, Proposition 3.1].

Theorem 3.3. *Let $(R,\mathfrak{m})$ be a local ring, M a Φ-minimax ZD-module and t a positive integer. The following statements are equivalent*:

(i) $H_\Phi^i(M) = 0$ *for all* $i \geq t$;
(ii) $H_\Phi^i(M)$ *is finitely generated for all* $i \geq t$.

Proof. (i) ⇒ (ii) Trivial.

(ii) ⇒ (i) The proof is by induction on $\dim M$. Let $n = \dim M$. If $n = 0$, then $H_\Phi^i(M) = 0$ for all $i > 0$.

Let $n > 0$, it follows from [10, 1.4] that

$$H_\Phi^i(M) \cong H_\Phi^i(M/\Gamma_\Phi(M))$$

for all $i > 0$. Let $\overline{M} = M/\Gamma_\Phi(M)$, it is clear that $\overline{M}$ is Φ-torsion-free. This implies that $\overline{M}$ is $\mathfrak{a}$-torsion-free for all $\mathfrak{a} \in \Phi$. In particular, there is an element $x \in \mathfrak{m}$ which is regular on $\overline{M}$. Now, the short exact sequence

$$0 \to \overline{M} \xrightarrow{\cdot x} \overline{M} \to \overline{M}/x\overline{M} \to 0$$

induces a long exact sequence

$$\cdots \to H^i_\Phi(\overline{M}) \xrightarrow{\cdot x} H^i_\Phi(\overline{M}) \to H^i_\Phi(\overline{M}/x\overline{M}) \to \cdots .$$

By the assumption, $H^i_\Phi(\overline{M}/x\overline{M})$ is finitely generated for all $i \geq t$. Since $\dim(\overline{M}/x\overline{M}) < \dim(\overline{M}) \leq n$ and $\overline{M}/x\overline{M}$ is a ZD-module, it follows from the inductive hypothesis that $H^i_\Phi(\overline{M}/x\overline{M}) = 0$ for all $i \geq t$. Now the long exact sequence yields

$$H^i_\Phi(\overline{M}) = xH^i_\Phi(\overline{M})$$

for all $i \geq t$. By Nakayama's Lemma, we can conclude that $H^i_\Phi(\overline{M}) = 0$ for all $i \geq t$, and the proof is complete. □

It is clear that finitely generated R-modules are Φ-minimax ZD-modules. Hence, the following consequence is deduced immediately from Theorem 3.3.

Corollary 3.4. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module and t a positive integer. The following statements are equivalent*:

(i) $H^i_\Phi(M) = 0$ *for all* $i \geq t$;
(ii) $H^i_\Phi(M)$ *is finitely generated for all* $i \geq t$.

Corollary 3.5. *Let $(R, \mathfrak{m})$ be a local ring, M a minimax R-module and $t > 1$ a positive integer. The following statements are equivalent*:

(i) $H^i_\Phi(M) = 0$ *for all* $i \geq t$;
(ii) $H^i_\Phi(M)$ *is finitely generated for all* $i \geq t$.

Proof. (i) $\Rightarrow$ (ii) Trivial. We now prove (ii) $\Rightarrow$ (i). Since M is a minimax R-module, there is a short exact sequence

$$0 \to N \to M \to A \to 0,$$

where N is finitely generated and A is artinian. By applying the functor $\Gamma_\Phi(-)$ to the above exact sequence, we get a long exact sequence

$$0 \to H^0_\Phi(N) \to H^0_\Phi(M) \to H^0_\Phi(A) \to H^1_\Phi(N) \to H^1_\Phi(M) \to 0$$

and

$$H^i_\Phi(N) \cong H^i_\Phi(M)$$

for all $i \geq 2$. By the hypothesis, $H^i_\Phi(N)$ is finitely generated for all $i \geq t$. It follows from Corollary 3.4 that $H^i_\Phi(N) = 0$ for all $i \geq t$ and which completes the proof. □

We recall the cohomological dimension of M with respect to a system of ideals

$$\operatorname{cd}(\Phi, M) = \sup\{i \mid H^i_\Phi(M) \neq 0\}.$$

Corollary 3.6. *Let $(R, \mathfrak{m})$ be a local ring and M a finitely generated R-module with* $\operatorname{cd}(\Phi, M) > 0$. *Then $H_\Phi^{cd(\Phi,M)}(M)$ is not finitely generated.*

Proof. The assertion follows easily from Corollary 3.4. □

Corollary 3.7. *Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module with finite dimension and $t > 1$ a positive integer such that $H_\Phi^i(M) = 0$ for all $i \geq t$. Then $H_\Phi^{t-1}(M)/\mathfrak{a}H_\Phi^{t-1}(M) = 0$ for all $\mathfrak{a} \in \Phi$.*

Proof. Let $\mathfrak{a} \in \Phi$, in the proof of Theorem 3.3, there is a long exact sequence

$$\cdots \to H_\Phi^{t-1}(\overline{M}) \xrightarrow{\cdot x} H_\Phi^{t-1}(\overline{M}) \to H_\Phi^{t-1}(\overline{M}/x\overline{M}) \to 0,$$

where $x \in \mathfrak{a}$. By the inductive hypothesis of the dimension of M, one asserts that $H_\Phi^{t-1}(\overline{M}/x\overline{M})/\mathfrak{a}H_\Phi^{t-1}(\overline{M}/x\overline{M}) = 0$. Moreover, there is an isomorphism

$$H_\Phi^{t-1}(\overline{M})/\mathfrak{a}H_\Phi^{t-1}(\overline{M}) \cong H_\Phi^{t-1}(\overline{M}/x\overline{M})/\mathfrak{a}H_\Phi^{t-1}(\overline{M}/x\overline{M}),$$

and which complete the proof. □

By using Theorem 3.3 and a fact of minimax modules, we have the following result.

Theorem 3.8. *Let M be a Φ-minimax ZD-module and t a positive integer. Assume that $H_\Phi^i(M)$ is a minimax R-module for all $i \geq t$. Then $H_\Phi^i(M)$ is an artinian R-module for all $i \geq t$.*

Proof. Let $i \geq t$. Since $H_\Phi^i(M)$ is minimax, we see that $H_\Phi^i(M)_\mathfrak{p} \cong H_{\Phi_\mathfrak{p}}^i(M_\mathfrak{p})$ is a finitely generated $R_\mathfrak{p}$-module for all prime ideals $\mathfrak{p}$ which is not maximal. By [7, Proposition 3(2)], $M_\mathfrak{p}$ is a ZD $R_\mathfrak{p}$-module. By using Proposition 2.6, we can check that $M_\mathfrak{p}$ is a $\Phi_\mathfrak{p}$-minimax $R_\mathfrak{p}$-module. It follows from Theorem 3.3 that $H_\Phi^i(M)_\mathfrak{p} = 0$. This implies that $\operatorname{Supp}_R H_\Phi^i(M) \subseteq \operatorname{Max} R$ and then we have the artinianness of $H_\Phi^i(M)$. □

The following consequence is an extension of [1, Theorem 2.3].

Corollary 3.9. *Let M be an $\mathfrak{a}$-minimax ZD-module and t a positive integer. Assume that $H_\mathfrak{a}^i(M)$ is a minimax R-module for all $i \geq t$. Then $H_\mathfrak{a}^i(M)$ is an artinian R-module for all $i \geq t$.*

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References

[1] M. Aghapournahr and L. Melkersson, *Finiteness properties of minimax and coatomic local cohomology modules*, Arch. Math. (Basel) **94** (2010), no. 6, 519–528. `https://doi.org/10.1007/s00013-010-0127-z`

[2] J. Azami, R. Naghipour, and B. Vakili, *Finiteness properties of local cohomology modules for $\mathfrak{a}$-minimax modules*, Proc. Amer. Math. Soc. **137** (2009), no. 2, 439–448. `https://doi.org/10.1090/S0002-9939-08-09530-0`

[3] M. H. Bijan-Zadeh, *Torsion theories and local cohomology over commutative and Noetherian rings*, J. London Math. Soc. (2) **19** (1979), no. 3, 402–410. https://doi.org/10.1112/jlms/s2-19.3.402

[4] ———, *A common generalization of local cohomology theories*, Glasgow Math. J. **21** (1980), no. 2, 173–181. https://doi.org/10.1017/S0017089500004158

[5] ———, *On the Artinian property of certain general local cohomology modules*, J. London Math. Soc. (2) **32** (1985), no. 3, 399–403. https://doi.org/10.1112/jlms/s2-32.3.399

[6] K. Divaani-Aazar and M. A. Esmkhani, *Artinianness of local cohomology modules of ZD-modules*, Comm. Algebra **33** (2005), no. 8, 2857–2863. https://doi.org/10.1081/AGB-200063983

[7] E. G. Evans, Jr., *Zero divisors in Noetherian-like rings*, Trans. Amer. Math. Soc. **155** (1971), 505–512. https://doi.org/10.2307/1995700

[8] L. Melkersson, *On asymptotic stability for sets of prime ideals connected with the powers of an ideal*, Math. Proc. Cambridge Philos. Soc. **107** (1990), no. 2, 267–271. https://doi.org/10.1017/S0305004100068535

[9] P. Rudlof, *On minimax and related modules*, Canad. J. Math. **44** (1992), no. 1, 154–166. https://doi.org/10.4153/CJM-1992-009-7

[10] R. Y. Sharp and P. Schenzel, *Cousin complexes and generalized Hughes complexes*, Proc. London Math. Soc. (3) **68** (1994), no. 3, 499–517. https://doi.org/10.1112/plms/s3-68.3.499

[11] K.-I. Yoshida, *Cofiniteness of local cohomology modules for ideals of dimension one*, Nagoya Math. J. **147** (1997), 179–191. https://doi.org/10.1017/S0027763000006371

[12] H. Zöschinger, *Minimax-moduln*, J. Algebra **102** (1986), no. 1, 1–32. https://doi.org/10.1016/0021-8693(86)90125-0

[13] ———, *Über die Maximalbedingung für radikalvolle Untermoduln*, Hokkaido Math. J. **17** (1988), no. 1, 101–116. https://doi.org/10.14492/hokmj/1381517790

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