

SCHATTEN-HERZ CLASS TOEPLITZ OPERATORS ON WEIGHTED BERGMAN SPACES INDUCED BY DOUBLING WEIGHTS

JUNTAO DU AND SONGXIAO LI

ABSTRACT. Schatten-Herz class Toeplitz operators on weighted Bergman spaces induced by doubling weights are investigated in this paper.

1. Introduction

Let \mathbb{D} be the open unit disk in the complex plane, and $H(\mathbb{D})$ the class of all functions analytic on \mathbb{D} . For any $z \in \mathbb{D}$ and $r > 0$, let $D(z, r) = \{w \in \mathbb{D} : \beta(z, w) < r\}$ be the Bergman disk. Here $\beta(\cdot, \cdot)$ is the Bergman metric on \mathbb{D} . If $\{a_j\}_{j=1}^{\infty} \subset \mathbb{D}$ satisfying $\inf_{i \neq j} \beta(a_i, a_j) \geq s > 0$, we say that $\{a_j\}_{j=1}^{\infty} \subset \mathbb{D}$ is s -separated.

A function $\omega : \mathbb{D} \rightarrow [0, \infty)$ is called a weight if it is positive and integrable. A weight ω is radial if $\omega(z) = \omega(|z|)$ for all $z \in \mathbb{D}$. Let ω be a radial weight and $\hat{\omega}(r) = \int_r^1 \omega(s)ds$ for $r \in [0, 1]$. We say that ω is a doubling weight, denoted by $\omega \in \hat{\mathcal{D}}$, if there is a constant $C > 0$ such that $\hat{\omega}(r) < C\hat{\omega}(\frac{1+r}{2})$ when $0 \leq r < 1$. If there exist $K > 1$ and $C > 1$ such that $\hat{\omega}(r) \geq C\hat{\omega}(1 - \frac{1-r}{K})$ when $0 \leq r < 1$, we say that ω is a reverse doubling weight, denoted by $\omega \in \check{\mathcal{D}}$. We say that ω is a regular weight, denoted by $\omega \in \mathcal{R}$, if there exist $C > 1$ and $\delta \in (0, 1)$, such that

$$\frac{1}{C} < \frac{\hat{\omega}(r)}{(1-r)\omega(r)} < C, \quad \text{when } \delta < r < 1.$$

We denote by $\mathcal{D} = \hat{\mathcal{D}} \cap \check{\mathcal{D}}$. From [9], we see that $\mathcal{R} \subset \mathcal{D}$. More details about \mathcal{R} , \mathcal{D} and $\hat{\mathcal{D}}$ can be found in [7–11].

When $0 < p < \infty$ and $\omega \in \hat{\mathcal{D}}$, the weighted Bergman space A_{ω}^p is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{A_{\omega}^p}^p = \int_{\mathbb{D}} |f(z)|^p \omega(z) dA(z) < \infty,$$

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where dA is the normalized Lebesgue area measure on \mathbb{D} . When $\omega(z) = (1 - |z|^2)^\alpha$ ($\alpha > -1$), the space A_ω^p becomes the classical weighted Bergman space A_α^p . When $\alpha = 0$, we will write $A_\alpha^p = A^p$. Suppose $0 < p \leq \infty$ and μ is a positive Borel measure on \mathbb{D} . Let L_μ^p denote the Lebesgue space defined in a standard way. Meanwhile, L_τ^p denotes L_μ^p for $d\mu(z) = (1 - |z|^2)^\tau dA(z)$ for some real number τ .

Let $\omega_s = \int_0^1 r^s \omega(r) dr$ and $B_z^\omega(w) = \frac{1}{2} \sum_{k=0}^{\infty} \frac{(w\bar{z})^k}{\omega_{2k+1}}$. Then B_z^ω is the reproducing kernel for A_ω^2 , which means that for any $f \in A_\omega^2$ (see [10]),

$$f(z) = \langle f, B_z^\omega \rangle_{A_\omega^2} = \int_{\mathbb{D}} f(w) \overline{B_z^\omega(w)} \omega(w) dA(w).$$

Let μ be a positive Borel measure on \mathbb{D} . Let $f \in H(\mathbb{D})$ and $z \in \mathbb{D}$. The Toeplitz operator \mathcal{T}_μ and the Berezin transform $\widetilde{\mathcal{T}}_\mu$ of \mathcal{T}_μ are defined by

$$\mathcal{T}_\mu f(z) = \int_{\mathbb{D}} f(\xi) \overline{B_z^\omega(\xi)} d\mu(\xi), \quad \widetilde{\mathcal{T}}_\mu(z) = \frac{\langle \mathcal{T}_\mu B_z^\omega, B_z^\omega \rangle_{A_\omega^2}}{\|B_z^\omega\|_{A_\omega^2}^2},$$

respectively.

For $k = 0, 1, 2, \dots$, let $\Lambda_k = \{z \in \mathbb{D} : 1 - \frac{1}{2^k} \leq |z| < 1 - \frac{1}{2^{k+1}}\}$ and χ_k the characteristic function of Λ_k . Let $d\lambda(z) = \frac{dA(z)}{(1 - |z|^2)^2}$ be the Möbius invariant area measure on \mathbb{D} . For $0 < p \leq \infty$ and $0 \leq q \leq \infty$, let $K_q^p(\lambda)$ denote the Herz space, which consists of all measurable functions f such that

$$\|f\|_{K_q^p(\lambda)} = \|\{\|f\|_{L_{\lambda\chi_k}^p}\}_{k=0}^{\infty}\|_{l^q} < \infty.$$

Note that $K_p^p(\lambda) = L_\lambda^p$. Here, l^q ($0 \leq q \leq \infty$) consists of all complex sequences $\{a_k\}_{k=1}^{\infty}$ such that $\|\{a_k\}\|_{l^q} < \infty$, where

$$\|\{a_k\}\|_{l^q} = \begin{cases} (\sum_{k=1}^{\infty} |a_k|^q)^{\frac{1}{q}}, & 0 < q < \infty; \\ \sup_{k \geq 1} |a_k|, & q = \infty; \\ \limsup_{k \rightarrow \infty} |a_k|, & q = 0. \end{cases}$$

Let $j = 0, 1, 2, \dots$ and $\lambda_j = \inf\{\|\mathcal{T}_\mu - R\|_{A_\omega^2 \rightarrow A_\omega^2} : \text{rank}(R) \leq j\}$. If $\{\lambda_j\}_{j=0}^{\infty} \in l^p$ for some $p \in (0, \infty)$, we say that \mathcal{T}_μ belongs to the Schatten p -class, denoted by $\mathcal{T}_\mu \in \mathcal{S}_p(A_\omega^2)$. We denote by \mathcal{S}_∞ the class of all bounded linear operators on A_ω^2 . For $0 < p \leq \infty$ and $0 \leq q \leq \infty$, \mathcal{T}_μ is said to belong to the Schatten-Herz class, denoted by $\mathcal{S}_{p,q}$, if $\mathcal{T}_{\mu\chi_k} \in \mathcal{S}_p$ and

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} = \|\{\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}\}_{k=0}^{\infty}\|_{l^q} < \infty.$$

Here $\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}$ means the Schatten p -class norm of $\mathcal{T}_{\mu\chi_k}$ on A_ω^2 , see [12] for example. In [4], Loaiza, López-García and Pérez-Esteva considered Schatten-Herz class Toeplitz operators on A^2 for the first time. For more study on Herz spaces and Schatten-Herz class Toeplitz operators, see [1–6].

In [9, 11], Peláez, Rättyä and Sierra investigated the boundedness, compactness and Schatten class Toeplitz operators on Bergman spaces induced by

doubling weights. In this paper, we investigate Schatten-Herz class Toeplitz operators on A_ω^2 with $\omega \in \hat{\mathcal{D}}$. The main result of this paper is stated as follows.

Theorem 1.1. *Suppose $0 < p \leq \infty$, $0 \leq q \leq \infty$, $r > 0$, $\omega \in \hat{\mathcal{D}}$ and μ is a positive Borel measure. Let $\{a_j\} \subset \mathbb{D}$ such that $\mathbb{D} = \bigcup_{j=1}^{\infty} D(a_j, r)$ and $\{a_j\}$ are s -separated for some $0 < s < r < \infty$. Let*

$$\widehat{\mu_r}(z) = \frac{\mu(D(z, r))}{(1 - |z|)\hat{\omega}(z)} \text{ and } \|\{\widehat{\mu_r}(a_j)\}\|_{l_q^p} = \|\{\|\{(\widehat{\mu_r}\chi_k)(a_j)\}_{j=1}^{\infty}\|_{l^p}\}_{k=0}^{\infty}\|_{l^q}.$$

(i) *The following statements are equivalent.*

- (a) $\mathcal{T}_\mu \in \mathcal{S}_{p,q}$;
- (b) $\widehat{\mu_r} \in K_q^p(\lambda)$;
- (c) $\|\{\widehat{\mu_r}(a_j)\}\|_{l_q^p} < \infty$.

Moreover,

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} \approx \|\widehat{\mu_r}\|_{K_q^p(\lambda)} \approx \|\{\widehat{\mu_r}(a_j)\}\|_{l_q^p}.$$

(ii) *If $\omega \in \mathcal{D}$ such that $\frac{(1-|\cdot|)^p \hat{\omega}(\cdot)^p}{(1-|\cdot|)^2} \in \mathcal{D}$, then $\mathcal{T}_\mu \in \mathcal{S}_{p,q}$ if and only if $\widetilde{\mathcal{T}}_\mu \in K_q^p(\lambda)$.*

Moreover,

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} \approx \|\widetilde{\mathcal{T}}_\mu\|_{K_q^p(\lambda)}.$$

Throughout this paper, the letter C will denote constants and may differ from one occurrence to the other. The notation $A \lesssim B$ means that there is a positive constant C such that $A \leq CB$. The notation $A \approx B$ means $A \lesssim B$ and $B \lesssim A$.

2. The proof of main result

In this section, we prove the main result in this paper. For this purpose, let's recall some related definitions and state some lemmas.

For $j = 0, 1, 2, \dots$ and $k = 0, 1, 2, \dots, 2^j - 1$, let

$$I_{j,k} = \left\{ e^{i\theta} : \frac{2\pi k}{2^j} \leq \theta < \frac{2\pi(k+1)}{2^j} \right\}$$

and

$$R_{j,k} = \left\{ z \in \mathbb{D} : \frac{z}{|z|} \in I_{j,k}, 1 - \frac{|I_{j,k}|}{2\pi} \leq |z| < 1 - \frac{|I_{j,k}|}{4\pi} \right\}.$$

Lemma 2.1. *Suppose $0 < s < r < \infty$ are given and $\{a_n\}_{n=1}^{\infty}$ is s -separated. Then the following statements hold.*

- (i) *For any $z \in \mathbb{D}$, there exist at most $N_1 = N_1(r)$ elements of $\{R_{j,k}\}$ intersecting with $D(z, r)$;*
- (ii) *For any $R_{j,k}$, there exist at most $N_2 = N_2(r, s)$ elements of $\{D(a_n, r)\}$ intersecting with $R_{j,k}$.*

Proof. (i) Suppose $z \in \Lambda_i$ and $w \in D(z, r)$. Then we have $1 - |z| \approx 1 - |w|$. So there exists $M = M(r)$ such that $w \in \cup_{m=i-M}^{i+M} \Lambda_m$. Here, let $\Lambda_m = \emptyset$ if $m < 0$. By Proposition 4.4 in [12], $D(z, r)$ is an Euclidean disk with center $C_0 = \frac{1-(\tanh r)^2}{1-(\tanh r)^2|z|^2} z$ and radius $R_0 = \frac{1-|z|^2}{1-(\tanh r)^2|z|^2} \tanh r$. So, for all $w \in D(z, r)$, we have

$$|\operatorname{Arg}(\bar{z}w)| \leq \arcsin \frac{R_0}{C_0} \approx 1 - |z| \text{ as } |z| \rightarrow 1.$$

When $j = i - M, \dots, i + M$ and $k = 1, 2, \dots, 2^j - 1$, $|I_{j,k}| \approx \frac{1}{2^j} \approx 1 - |z|$. So, there exists $N_1 = N_1(r)$ such that there are at most N_1 elements of $\{R_{j,k}\}$ intersecting with $D(z, r)$.

(ii) Let $c_{t,j,k} = (1 - \frac{1}{2^j})e^{\frac{2\pi ik}{2^j}}$. For any $w \in R_{j,k}$ as $j \rightarrow \infty$, we have

$$\beta(c_{j,j,k}, w) \leq \beta(c_{j,j,k}, c_{j+1,j,k}) + \beta(c_{j,j,k}, c_{j,j,k+1}) \lesssim 1.$$

So, there exists $R' > 0$, for all $j = 0, 1, 2, \dots$, $k = 0, 1, \dots, 2^j - 1$ and $w \in R_{j,k}$, we have $\beta(c_{j,j,k}, w) < R'$.

Without loss of generality, assume $R_{j,k}$ intersect with $D(a_i, r)$ for $i = 1, 2, \dots, N_{j,k}$. We have

$$1 - |c_{j,j,k}| \approx 1 - |a_i|, \quad i = 1, 2, \dots, N_{j,k}$$

and

$$\cup_{i=1}^{N_{j,k}} D(a_i, s) \subset \cup_{i=1}^{N_{j,k}} D(a_i, r) \subset D(c_{j,j,k}, R' + 2r).$$

Thus

$$N_{j,k} \leq \frac{|D(c_{j,j,k}, R' + 2r)|}{\inf_{1 \leq i \leq N_{j,k}} |D(a_i, s)|} \lesssim 1,$$

which implies the desired result. The proof is complete. \square

The following lemma is a main result in [10] and plays an important role in the studying of Toeplitz operators on A_ω^2 .

Lemma 2.2. *Let $0 < p < \infty$ and $\omega \in \hat{\mathcal{D}}$. Then the following assertions hold.*

- (i) $M_p^p(r, B_z^\omega) \approx \int_0^{r|z|} \frac{1}{\hat{\omega}(t)^p(1-t)^p} dt$, $r|z| \rightarrow 1$.
- (ii) If $v \in \hat{\mathcal{D}}$, $\|B_z^\omega\|_{A_v^p}^p \approx \int_0^{|z|} \frac{\hat{v}(t)}{\hat{\omega}(t)^p(1-t)^p} dt$, $|z| \rightarrow 1$.

Here and hence forth, $M_p^p(r, B_z^\omega) = \frac{1}{2\pi} \int_0^{2\pi} |B_z^\omega(re^{i\theta})|^p d\theta$.

For any measurable function f , let

$$B_\omega f(z) = \frac{1}{\|B_z^\omega\|_{A_\omega^2}^2} \int_{\mathbb{D}} f(w) |B_z^\omega(w)|^2 \frac{\hat{\omega}(w) dA(w)}{1 - |w|}.$$

Then we have the following lemma.

Lemma 2.3. *Let $\omega \in \mathcal{D}$ and $1 \leq p < \infty$. Then there exists $\varepsilon > 0$ such that $B_\omega : L_\tau^p \rightarrow L_\tau^p$ is bounded when $\tau = -2 \pm \varepsilon$.*

Proof. Since $\omega \in \mathcal{D}$, by Lemmas A and B in [11], there are constants $0 < a < b < \infty$ such that

$$(1) \quad \frac{\hat{\omega}(t)}{(1-t)^b} \nearrow \infty \text{ and } \frac{\hat{\omega}(t)}{(1-t)^a} \searrow 0, \text{ when } 0 \leq t < 1.$$

Suppose $0 < \varepsilon < a$. Then $(1 - |\cdot|)^{-1 \pm \varepsilon} \hat{\omega}(\cdot) \in \mathcal{R}$.

We only prove the case of $\tau = -2 - \varepsilon$. The case of $\tau = -2 + \varepsilon$ can be proved in the same way.

Suppose $p = 1$. By Lemma 2.2,

$$(2) \quad B_z^\omega(z) = \|B_z^\omega\|_{A_\omega^2}^2 \approx \frac{1}{(1 - |z|)\hat{\omega}(z)}, \quad z \in \mathbb{D}.$$

For all $f \in L_\tau^p$, by Fubini's Theorem, (2) and Lemma 2.2(ii), we have

$$\begin{aligned} \|B_\omega f\|_{L_\tau^p} &= \int_{\mathbb{D}} \frac{1}{\|B_z^\omega\|_{A_\omega^2}^2} \left| \int_{\mathbb{D}} f(w) |B_z^\omega(w)|^2 \frac{\hat{\omega}(w) dA(w)}{1 - |w|} \right| (1 - |z|^2)^{-2 - \varepsilon} dA(z) \\ &\lesssim \int_{\mathbb{D}} |f(w)| \frac{\hat{\omega}(w)}{1 - |w|} \left(\int_{\mathbb{D}} |B_z^\omega(z)|^2 (1 - |z|)^{-1 - \varepsilon} \hat{\omega}(z) dA(z) \right) dA(w) \\ &\lesssim \int_{\mathbb{D}} |f(w)| \frac{\hat{\omega}(w)}{1 - |w|} \left(\int_0^{\frac{|w|+1}{2}} \frac{1}{(1 - t)^{2 + \varepsilon} \hat{\omega}(t)} dt \right) dA(w). \end{aligned}$$

Since $0 < \varepsilon < a$,

$$\begin{aligned} \int_0^{\frac{|w|+1}{2}} \frac{1}{(1 - t)^{2 + \varepsilon} \hat{\omega}(t)} dt &\lesssim \frac{(1 - |w|)^a}{\hat{\omega}(w)} \int_0^{\frac{|w|+1}{2}} \frac{1}{(1 - t)^{2 + a + \varepsilon}} dt \\ &\lesssim \frac{1}{(1 - |w|)^{1 + \varepsilon} \hat{\omega}(w)}. \end{aligned}$$

So, $\|B_\omega f\|_{L_\tau^p}^p \lesssim \|f\|_{L_\tau^p}^p$ when $p = 1$.

Suppose $1 < p < \infty$. Let $p' = \frac{p}{p-1}$, $h(z) = (1 - |z|)^s$ with $0 < s < \min\{\frac{a}{p'}, \frac{a}{p}\}$. Set

$$H(z, w) = \frac{|B_z^\omega(w)|^2 \hat{\omega}(w)}{\|B_z^\omega\|_{A_\omega^2}^2 (1 - |w|)(1 - |w|^2)^{-2 - \varepsilon}}.$$

Then $B_\omega f(z) = \int_{\mathbb{D}} f(w) H(z, w) dA_\tau(w)$. On one hand,

$$\begin{aligned} \int_{\mathbb{D}} H(z, w) h(w)^{p'} dA_\tau(w) &= \int_{\mathbb{D}} \frac{|B_z^\omega(w)|^2 \hat{\omega}(w) (1 - |w|)^{p's-1}}{\|B_z^\omega\|_{A_\omega^2}^2} dA(w) \\ &\lesssim (1 - |z|) \hat{\omega}(z) \int_0^{\frac{|z|+1}{2}} \frac{1}{(1 - t)^{2 - p's} \hat{\omega}(t)} dt \\ &\lesssim \frac{(1 - |z|)^{1+a} \hat{\omega}(z)}{\hat{\omega}(z)} \int_0^{\frac{|z|+1}{2}} \frac{1}{(1 - t)^{2+a-p's}} dt \\ &\lesssim h(z)^{p'}. \end{aligned}$$

On the other hand,

$$\begin{aligned}
\int_{\mathbb{D}} H(z, w) h(z)^p dA_\tau(z) &\approx \int_{\mathbb{D}} \frac{|B_z^\omega(w)|^2 \hat{\omega}(w) (1 - |z|)^{-2-\varepsilon+ps}}{\|B_z^\omega\|_{A_\omega^2}^2 (1 - |w|^2)^{-1-\varepsilon}} dA(z) \\
&\approx \frac{\hat{\omega}(w)}{(1 - |w|)^{-1-\varepsilon}} \int_{\mathbb{D}} |B_w^\omega(z)|^2 (1 - |z|)^{-1-\varepsilon+ps} \hat{\omega}(z) dA(z) \\
&\lesssim \frac{\hat{\omega}(w)}{(1 - |w|)^{-1-\varepsilon}} \int_0^{\frac{|w|+1}{2}} \frac{1}{(1-t)^{2+\varepsilon-ps} \hat{\omega}(t)} dt \\
&\lesssim h(w)^p.
\end{aligned}$$

By Schur's test, see [12, Theorem 3.6] for example, we have that $B_\omega : L_\tau^p \rightarrow L_\tau^p$ is bounded when $1 < p < \infty$. The proof is complete. \square

Lemma 2.4. *Suppose $\omega \in \mathcal{D}$, $1 \leq p \leq \infty$ and $0 \leq q \leq \infty$. Then B_ω is bounded on $K_q^p(\lambda)$.*

Proof. When $1 \leq p < \infty$, choose $\varepsilon > 0$ such that Lemma 2.3 holds. For $f \in K_q^p(\lambda)$, by Lemma 2.3, when $\tau = -2 \pm \varepsilon$, for $j = 0, 1, 2, \dots$, we have

$$\begin{aligned}
\|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^p}^p &= \int_{1-\frac{1}{2^k} \leq |z| < 1-\frac{1}{2^{k+1}}} |B_\omega(f\chi_j)(z)|^p \frac{dA(z)}{(1-|z|^2)^2} \\
&\approx 2^{k(\tau+2)} \int_{1-\frac{1}{2^k} \leq |z| < 1-\frac{1}{2^{k+1}}} |B_\omega(f\chi_j)(z)|^p dA_\tau(z) \\
&\lesssim 2^{k(\tau+2)} \int_{\mathbb{D}} |f(z)\chi_j(z)|^p dA_\tau(z) \approx 2^{(\tau+2)(k-j)} \|f\chi_j\|_{L_\lambda^p}^p.
\end{aligned}$$

When $j \leq k$, letting $\tau = -2 - \varepsilon$, we get

$$\|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^p}^p \lesssim 2^{-\varepsilon|j-k|} \|f\chi_j\|_{L_\lambda^p}^p.$$

When $j > k$, letting $\tau = -2 + \varepsilon$, we obtain

$$\|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^p}^p \lesssim 2^{-\varepsilon|j-k|} \|f\chi_j\|_{L_\lambda^p}^p.$$

Therefore, when $1 \leq p < \infty$,

$$\begin{aligned}
(3) \quad \|(B_\omega f)\chi_k\|_{L_\lambda^p} &= \left\| B_\omega \left(\sum_{j=0}^{\infty} f\chi_j \right) \right\|_{L_{\lambda\chi_k}^p} \\
&\leq \sum_{j=0}^{\infty} \|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^p} \lesssim \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^p}}{2^{\frac{\varepsilon|j-k|}{p}}}.
\end{aligned}$$

When $p = \infty$,

$$(4) \quad \|(B_\omega f)\chi_k\|_{L_\lambda^\infty} = \|B_\omega f\|_{L_{\lambda\chi_k}^\infty} \leq \left(\sum_{j=0}^k + \sum_{j=k+1}^{\infty} \right) \|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^\infty}.$$

After a calculation,

$$\begin{aligned}
(5) \quad & \|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^\infty} \\
& \leq \sup_{1-\frac{1}{2^k} \leq |z| < 1-\frac{1}{2^{k+1}}} \frac{1}{\|B_z^\omega\|_{A_\omega^2}^2} \int_{1-\frac{1}{2^j} \leq |w| < 1-\frac{1}{2^{j+1}}} |f(w)| |B_z^\omega(w)|^2 \frac{\hat{\omega}(w) dA(w)}{1-|w|} \\
& \lesssim \|f\|_{L_{\lambda\chi_j}^\infty} \frac{1}{2^k} \hat{\omega}\left(1 - \frac{1}{2^k}\right) \int_{1-\frac{1}{2^j}}^{1-\frac{1}{2^{j+1}}} \frac{\hat{\omega}(r) dr}{1-r} \int_0^{(1-\frac{1}{2^{j+1}})(1-\frac{1}{2^{k+1}})} \frac{1}{(1-t)^2 \hat{\omega}(t)^2} dt \\
& \approx \|f\|_{L_{\lambda\chi_j}^\infty} \frac{1}{2^k} \hat{\omega}\left(1 - \frac{1}{2^{k+1}}\right) \hat{\omega}\left(1 - \frac{1}{2^{j+1}}\right) \int_0^{x_{j,k}} \frac{1}{(1-t)^2 \hat{\omega}(t)^2} dt.
\end{aligned}$$

Here $x_{j,k} = (1 - \frac{1}{2^{j+1}})(1 - \frac{1}{2^{k+1}})$.

By (1),

$$\begin{aligned}
(6) \quad & \int_0^{x_{j,k}} \frac{1}{(1-t)^2 \hat{\omega}(t)^2} dt \lesssim \frac{(1-x_{j,k})^{2a}}{\hat{\omega}(x_{j,k})^2} \int_0^{x_{j,k}} \frac{1}{(1-t)^{2+2a}} dt \\
& \lesssim \frac{(1-x_{j,k})^{-1}}{\hat{\omega}(x_{j,k})^2} = \frac{\left(1 - (1 - \frac{1}{2^{k+1}})(1 - \frac{1}{2^{j+1}})\right)^{-1}}{\left(\hat{\omega}\left((1 - \frac{1}{2^{k+1}})(1 - \frac{1}{2^{j+1}})\right)\right)^2}.
\end{aligned}$$

Let $r_j = \frac{1}{2^{j+1}}$. Since $\frac{1-(1-\frac{1}{2^{j+1}}-\frac{1}{2^{k+1}})}{1-(1-\frac{1}{2^{k+1}})(1-\frac{1}{2^{j+1}})} \approx 1$, by (5) and (6) we have

$$\begin{aligned}
\|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^\infty} & \lesssim \|f\|_{L_{\lambda\chi_j}^\infty} \frac{\hat{\omega}\left(1 - \frac{1}{2^{k+1}}\right) \hat{\omega}\left(1 - \frac{1}{2^{j+1}}\right)}{\left(1 + 2^{k-j}\right) \left(\hat{\omega}\left(1 - \frac{1}{2^{k+1}} - \frac{1}{2^{j+1}}\right)\right)^2} \\
& = \|f\|_{L_{\lambda\chi_j}^\infty} \frac{\hat{\omega}(1-r_k) \hat{\omega}(1-r_j)}{\left(1 + 2^{k-j}\right) \hat{\omega}(1-r_k-r_j)^2}.
\end{aligned}$$

Therefore, when $j \geq k$, using the monotonicity of $\hat{\omega}$, we have

$$\begin{aligned}
(7) \quad & \|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^\infty} \lesssim \|f\|_{L_{\lambda\chi_j}^\infty} \frac{\hat{\omega}(1-r_j)}{\hat{\omega}(1-r_k-r_j)} \\
& = \|f\|_{L_{\lambda\chi_j}^\infty} \frac{\frac{\hat{\omega}(1-r_j)}{(1-(1-r_j))^a}}{\frac{\hat{\omega}(1-r_k-r_j)}{(1-(1-r_k-r_j))^a}} \frac{r_j^a}{(r_k+r_j)^a} \lesssim \frac{\|f\|_{L_{\lambda\chi_j}^\infty}}{2^{a(j-k)}}.
\end{aligned}$$

When $j < k$, it is obvious that

$$(8) \quad \|B_\omega(f\chi_j)\|_{L_{\lambda\chi_k}^\infty} \lesssim \frac{\|f\|_{L_{\lambda\chi_j}^\infty}}{2^{k-j}}.$$

From (4), (7) and (8), we have

$$(9) \quad \|(B_\omega f)\chi_k\|_{L_\lambda^\infty} \leq \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^\infty}}{2^{\min\{1,a\}|j-k|}}.$$

By (3) and (9), there exists $\varepsilon_0 > 0$ such that

$$\|(B_\omega f)\chi_k\|_{L_\lambda^p} \lesssim \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^p}}{2^{\varepsilon_0|j-k|}},$$

when $1 \leq p \leq \infty$.

When $1 \leq q \leq \infty$, we consider the sequences $X = \{x_k\}$, $Y = \{y_k\}$, where $x_k = 2^{-\varepsilon_0|k|}$ and

$$y_k = \begin{cases} \|f\chi_k\|_{L_\lambda^p}, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

Then $\|(B_\omega f)\chi_k\|_{L_\lambda^p} \lesssim X * Y(k)$. Here, $X * Y$ means the convolution of X and Y . By Young's inequality,

$$\begin{aligned} \|B_\omega f\|_{K_q^p(\lambda)} &= \|\{(B_\omega f)\chi_k\|_{L_\lambda^p}\}_{k=0}^\infty\|_{l^q} \\ &\lesssim \left\| \left\{ \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^p}}{2^{\varepsilon_0|j-k|}} \right\}_{k=0}^\infty \right\|_{l^q} = \|X * Y\|_{l^q} \lesssim \|X\|_{l^1} \|Y\|_{l^q} \approx \|f\|_{K_q^p(\lambda)}. \end{aligned}$$

When $0 < q < 1$,

$$\begin{aligned} \|B_\omega f\|_{K_q^p(\lambda)}^q &= \sum_{k=0}^{\infty} \left(\|(B_\omega f)\chi_k\|_{L_\lambda^p} \right)^q \lesssim \sum_{k=0}^{\infty} \left(\sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^p}}{2^{\varepsilon_0|j-k|}} \right)^q \\ &\leq \sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^p}^q}{2^{q\varepsilon_0|j-k|}} \leq \sum_{j=0}^{\infty} \|f\chi_j\|_{L_\lambda^p}^q \sum_{k=-\infty}^{\infty} 2^{-q\varepsilon_0|k|} \approx \|f\|_{K_q^p(\lambda)}^q. \end{aligned}$$

If $q = 0$, for any given $M \in \mathbb{N}$,

$$\begin{aligned} \|B_\omega f\|_{K_q^p(\lambda)} &= \limsup_{k \rightarrow \infty} \|(B_\omega f)\chi_k\|_{L_\lambda^p} \lesssim \limsup_{k \rightarrow \infty} \sum_{j=0}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^p}}{2^{\varepsilon_0|j-k|}} \\ &= \limsup_{k \rightarrow \infty} \left(\sum_{j=0}^M \frac{\|f\chi_j\|_{L_\lambda^p}}{2^{\varepsilon_0|j-k|}} + \sum_{j=M+1}^{\infty} \frac{\|f\chi_j\|_{L_\lambda^p}}{2^{\varepsilon_0|j-k|}} \right) \lesssim \sup_{j \geq M+1} \|f\chi_j\|_{L_\lambda^p}. \end{aligned}$$

Letting $M \rightarrow \infty$, we have $\|B_\omega f\|_{K_q^p(\lambda)} \lesssim \|f\|_{K_q^p(\lambda)}$. The proof is complete. \square

Proof of Theorem 1.1. (i) By Lemma 2.1, for any $z \in \Lambda_k$, there exists $N = N(r)$ such that $D(z, r) \subset \cup_{j=k-N}^{k+N} \Lambda_j$. Here $\Lambda_j = \emptyset$ if $j < 0$. Thus,

$$(\widehat{\mu_r} \chi_k)(z) \leq \sum_{j=k-N}^{k+N} \frac{\mu(\Lambda_j \cap D(z, r))}{(1 - |z|)\hat{\omega}(z)} = \sum_{j=k-N}^{k+N} (\widehat{\mu \chi_j})_r(z)$$

and

$$(\widehat{\mu \chi_j})_r(z) = \frac{\mu(\Lambda_j \cap D(z, r))}{(1 - |z|)\hat{\omega}(z)} \leq \frac{\mu(D(z, r))}{(1 - |z|)\hat{\omega}(z)} \sum_{k=j-N}^{j+N} \chi_k(z) = \sum_{k=j-N}^{j+N} (\widehat{\mu_r} \chi_k)(z).$$

Therefore, by Theorem 3 in [11],

$$\begin{aligned}\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} &= \|\{\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}\}_{k=0}^\infty\|_{l^q} \approx \|\{\|\widehat{(\mu\chi_k)_r}\|_{L_\lambda^p}\}_{k=0}^\infty\|_{l^q} \\ &\approx \|\{\|\widehat{\mu_r\chi_k}\|_{L_\lambda^p}\}_{k=0}^\infty\|_{l^q} = \|\widehat{\mu_r}\|_{K_q^p(\lambda)}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} &= \|\{\|\mathcal{T}_{\mu\chi_k}\|_{\mathcal{S}_p}\}_{k=0}^\infty\|_{l^q} \approx \|\{\|\{\widehat{(\mu\chi_k)_r}(a_j)\}_{j=1}^\infty\|_{l^p}\}_{k=0}^\infty\|_{l^q} \\ &\approx \|\{\|\{\widehat{\mu_r\chi_k}(a_j)\}_{j=1}^\infty\|_{l^p}\}_{k=0}^\infty\|_{l^q} = \|\{\widehat{\mu_r}(a_j)\}\|_{l_q^p}.\end{aligned}$$

(ii) Suppose $r > 0$ such that Lemma 8 in [11] holds. That is, for all $z \in \mathbb{D}$ and $w \in D(z, r)$, $|B_z^\omega(w)| \approx B_z^\omega(z)$. By Lemma 11 in [11],

$$\langle \mathcal{T}_\mu B_z^\omega, B_z^\omega \rangle_{A_\omega^2} = \langle B_z^\omega, B_z^\omega \rangle_{L_\mu^2}.$$

Using (2), we get

$$\widehat{\mu_r}(z) = \frac{\mu(D(z, r))}{(1 - |z|)\hat{\omega}(z)} \approx \frac{1}{B_z^\omega(z)} \int_{D(z, r)} |B_z^\omega(w)|^2 d\mu(w) \leq \widetilde{\mathcal{T}_\mu}(z).$$

So,

$$\|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}} \approx \|\widehat{\mu_r}\|_{K_q^p(\lambda)} \lesssim \|\widetilde{\mathcal{T}_\mu}\|_{K_q^p(\lambda)}.$$

On the other hand, since $|B_z^\omega(w)|^2$ is subharmonic and $\omega \in \mathcal{D}$, by Fubini's theorem,

$$\begin{aligned}\widetilde{\mathcal{T}_\mu}(z) &= \frac{1}{B_z^\omega(z)} \int_{\mathbb{D}} |B_z^\omega(w)|^2 d\mu(w) \lesssim \frac{1}{B_z^\omega(z)} \int_{\mathbb{D}} \int_{D(w, r)} \frac{|B_z^\omega(\eta)|^2}{(1 - |\eta|)^2} dA(\eta) d\mu(w) \\ &= \frac{1}{B_z^\omega(z)} \int_{\mathbb{D}} \widehat{\mu_r}(\eta) |B_z^\omega(\eta)|^2 \frac{\hat{\omega}(\eta) dA(\eta)}{1 - |\eta|} = B_\omega(\widehat{\mu_r})(z).\end{aligned}$$

By Lemma 2.4 we have

$$\|\widetilde{\mathcal{T}_\mu}\|_{K_q^p(\lambda)} \lesssim \|B_\omega(\widehat{\mu_r})\|_{K_q^p(\lambda)} \lesssim \|\widehat{\mu_r}\|_{K_q^p(\lambda)} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}},$$

when $1 \leq p \leq \infty$.

Next we prove the case of $p < 1$. For $z \in \mathbb{D}$,

$$\begin{aligned}\widetilde{\mathcal{T}_\mu}(z) &= \frac{1}{B_z^\omega(z)} \int_{\mathbb{D}} |B_z^\omega(w)|^2 d\mu(w) \leq \frac{1}{B_z^\omega(z)} \sum_{j=1}^\infty \int_{D(a_j, r)} |B_z^\omega(w)|^2 d\mu(w) \\ &\leq \frac{1}{B_z^\omega(z)} \sum_{j=1}^\infty \mu(D(a_j, r)) \sup_{w \in \overline{D(a_j, r)}} |B_z^\omega(w)|^2.\end{aligned}$$

Using the subharmonicity of $|B_z^\omega|^{2p}$,

$$\begin{aligned}(\widetilde{\mathcal{T}_\mu}(z))^p &\leq \frac{1}{(B_z^\omega(z))^p} \sum_{j=1}^\infty (\mu(D(a_j, r)))^p \sup_{w \in \overline{D(a_j, r)}} |B_z^\omega(w)|^{2p} \\ &\lesssim \frac{1}{(B_z^\omega(z))^p} \sum_{j=1}^\infty (\widehat{\mu_r}(a_j))^p \frac{(1 - |a_j|)^p \hat{\omega}(a_j)^p}{(1 - |a_j|)^2} \int_{D(a_j, 2r)} |B_z^\omega(w)|^{2p} dA(w).\end{aligned}$$

By Fubini's theorem and Lemma 2.2(i),

$$\begin{aligned} & \int_{\Lambda_k} \int_{D(a_j, 2r)} |B_z^\omega(w)|^{2p} dA(w) d\lambda(z) \\ & \approx 2^{2k} \int_{D(a_j, 2r)} \int_{\Lambda_k} |B_z^\omega(w)|^{2p} dA(z) dA(w) \\ & \lesssim 2^k \int_{D(a_j, 2r)} \int_0^{(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}} \frac{dt dA(w)}{\hat{\omega}(t)^{2p}(1-t)^{2p}}. \end{aligned}$$

Let $v(z) = \frac{(1-|z|)^p \hat{\omega}(z)^p}{(1-|z|)^2}$. Since $v \in \mathcal{D}$, there exist $C_1, C_2 > 1$ and $K > 1$, such that

$$\hat{v}(t) = \int_t^{\frac{1+t}{2}} v(s) ds + \hat{v}\left(\frac{1+t}{2}\right) > \int_t^{\frac{1+t}{2}} v(s) ds + \frac{1}{C_1} \hat{v}(t)$$

and

$$\hat{v}(t) = \int_t^{1-\frac{1-t}{K}} v(s) ds + \hat{v}\left(1 - \frac{1-t}{K}\right) < \int_t^{1-\frac{1-t}{K}} v(s) ds + \frac{1}{C_2} \hat{v}(t).$$

So,

$$(1-t)v(t) \approx \int_t^{\frac{1+t}{2}} v(s) ds \lesssim \hat{v}(t) \lesssim \int_t^{1-\frac{1-t}{K}} v(s) ds \approx (1-t)v(t),$$

which implies that $v \in \mathcal{R}$. Therefore,

$$\begin{aligned} & \int_0^{(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}} \frac{1}{\hat{\omega}(t)^{2p}(1-t)^{2p}} dt \\ & \approx \int_0^{(1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}} \frac{1}{\hat{v}(t)^2(1-t)^2} dt \\ & \lesssim \frac{1}{\left(\hat{v}\left((1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}\right)\right)^2 \left(1 - (1-\frac{1}{2^{k+1}})\frac{|w|+1}{2}\right)}. \end{aligned}$$

Since $w \in D(a_j, r)$, we get

$$1 - (1 - \frac{1}{2^{k+1}}) \frac{|a_j| + 1}{2} \approx 1 - (1 - \frac{1}{2^{k+1}}) \frac{|w| + 1}{2}.$$

Hence,

$$\begin{aligned} & \int_{\Lambda_k} \int_{D(a_j, 2r)} |B_z^\omega(w)|^{2p} dA(w) d\lambda(z) \\ & \lesssim \frac{2^k (1 - |a_j|)^2}{\left(\hat{v}\left((1 - \frac{1}{2^{k+1}})\frac{|a_j|+1}{2}\right)\right)^2 \left(1 - (1 - \frac{1}{2^{k+1}})\frac{|a_j|+1}{2}\right)}. \end{aligned}$$

Let $\xi_m = \left(\sum_{a_j \in \Lambda_m} (\widehat{\mu_r}(a_j))^p \right)^{\frac{1}{p}}$. Then

$$\begin{aligned} & \int_{\Lambda_k} (\widetilde{\mathcal{T}}_\mu(z))^p d\lambda(z) \\ & \lesssim \frac{1}{2^{(k+1)p}} \hat{\omega} \left(1 - \frac{1}{2^{k+1}}\right)^p \\ & \quad \times \sum_{m=0}^{\infty} \xi_m^p \frac{\frac{1}{2^{(m+1)p}} \hat{\omega} \left(1 - \frac{1}{2^{m+1}}\right)^p}{\frac{1}{2^{2(m+1)}}} \frac{\frac{2^k}{2^{2(m+1)}}}{\left(\hat{v}\left(1 - \frac{1}{2^{k+1}} - \frac{1}{2^{m+1}}\right)\right)^2 \left(\frac{1}{2^{k+1}} + \frac{1}{2^{m+1}}\right)} \\ & \approx \sum_{m=0}^{\infty} \xi_m^p \frac{\hat{v}\left(1 - \frac{1}{2^{m+1}}\right) \hat{v}\left(1 - \frac{1}{2^{k+1}}\right)}{\left(\hat{v}\left(1 - \frac{1}{2^{m+1}} - \frac{1}{2^{k+1}}\right)\right)^2} \frac{1}{1 + 2^{m-k}}. \end{aligned}$$

Similarly to get (7) and (8), there exists $\varepsilon > 0$ such that

$$\int_{\Lambda_k} (\widetilde{\mathcal{T}}_\mu(z))^p d\lambda(z) \lesssim \sum_{m=0}^{\infty} \xi_m^p \cdot \frac{1}{2^{\varepsilon|k-m|}}.$$

Note that

$$\|\{\xi_m\}\|_{l^q} = \left\| \left\{ \left(\sum_{j=1}^{\infty} ((\widehat{\mu_r}\chi_m)(a_j))^p \right)^{\frac{1}{p}} \right\}_{m=0}^{\infty} \right\|_{l^q} = \|\{\widehat{\mu_r}(a_j)\}\|_{l_q^p} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}.$$

If $0 < p < q = \infty$, then

$$\|\widetilde{\mathcal{T}}_\mu\|_{K_\infty^p(\lambda)} = \sup_{k \geq 0} \left(\int_{\Lambda_k} (\widetilde{\mathcal{T}}_\mu(z))^p d\lambda(z) \right)^{\frac{1}{p}} \lesssim \|\{\xi_m\}\|_{l^\infty} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,\infty}}.$$

When $0 < p < q < \infty$, let $X = \{x_k\}$ and $Y = \{y_k\}$, where $x_k = \frac{1}{2^{\varepsilon|k|}}$ and

$$y_k = \begin{cases} \xi_k^p, & k \geq 0, \\ 0, & k < 0. \end{cases}$$

Then,

$$\begin{aligned} \|\widetilde{\mathcal{T}}_\mu\|_{K_q^p(\lambda)} &= \left(\sum_{k=0}^{\infty} \left(\int_{\Lambda_k} (\widetilde{\mu}(z))^p d\lambda(z) \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \lesssim \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} \xi_m^p \cdot \frac{1}{2^{\varepsilon|k-m|}} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\ &= \|X * Y\|_{l_p^{\frac{q}{p}}}^{\frac{1}{p}} \leq \left(\|X\|_{l^1} \|Y\|_{l_p^{\frac{q}{p}}} \right)^{\frac{1}{p}} \lesssim \|\xi_m\|_{l^q} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}. \end{aligned}$$

If $0 < q \leq p < 1$, then

$$\|\widetilde{\mathcal{T}}_\mu\|_{K_q^p(\lambda)} = \left(\sum_{k=0}^{\infty} \left(\int_{\Lambda_k} (\widetilde{\mathcal{T}}_\mu(z))^p d\lambda(z) \right)^{\frac{q}{p}} \right)^{\frac{1}{q}}$$

$$\begin{aligned}
&\lesssim \left(\sum_{k=0}^{\infty} \left(\sum_{m=0}^{\infty} \xi_m^p \cdot \frac{1}{2^{\varepsilon|k-m|}} \right)^{\frac{q}{p}} \right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{k=0}^{\infty} \sum_{m=0}^{\infty} \xi_m^q \cdot \frac{1}{2^{\frac{q\varepsilon|k-m|}{p}}} \right)^{\frac{1}{q}} \\
&\lesssim \left(\sum_{m=0}^{\infty} \xi_m^q \sum_{k=-\infty}^{\infty} \frac{1}{2^{\frac{q\varepsilon|k|}{p}}} \right)^{\frac{1}{q}} \lesssim \|\xi_m\|_{l^q} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}.
\end{aligned}$$

When $0 = q < p < 1$, for any given $M \in \mathbb{N}$,

$$\limsup_{k \rightarrow \infty} \int_{\Lambda_k} (\widetilde{\mathcal{T}}_\mu(z))^p d\lambda(z) \lesssim \limsup_{k \rightarrow \infty} \left(\sum_{m=0}^M + \sum_{m=M+1}^{\infty} \right) \xi_m^p \cdot \frac{1}{2^{\varepsilon|k-m|}} \leq \sup_{m>M} \xi_m^p.$$

Letting $M \rightarrow \infty$, we obtain

$$\limsup_{k \rightarrow \infty} \left(\int_{\Lambda_k} (\widetilde{\mathcal{T}}_\mu(z))^p d\lambda(z) \right)^{\frac{1}{p}} \lesssim \limsup_{m \rightarrow \infty} \xi_m.$$

So,

$$\|\widetilde{\mathcal{T}}_\mu\|_{K_0^p(\lambda)} \lesssim \|\xi_m\|_{l^0} \approx \|\mathcal{T}_\mu\|_{\mathcal{S}_{p,q}}.$$

The proof is complete. \square

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JUNTAO DU

DEPARTMENT OF MATHEMATICS

SHANTOU UNIVERSITY

SHANTOU, GUANGDONG 515063, P. R. CHINA

Email address: jtdu007@163.com

SONGXIAO LI

INSTITUTE OF FUNDAMENTAL AND FRONTIER SCIENCES

UNIVERSITY OF ELECTRONIC SCIENCE AND TECHNOLOGY OF CHINA

610054, CHENGDU, SICHUAN, P. R. CHINA

Email address: jyulsx@163.com