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FOURIER COEFFICIENTS OF WEIGHT 2 MODULAR FORMS OF PRIME LEVEL

Yan-Bin Li

Abstract. We use a kind of modified Hurwitz class numbers to express the number of representing a positive integer by some ternary quadratic forms such as the sum of three squares, and the Fourier coefficients of weight 2 modular forms of prime level with trivial character.

1. Introduction

Gross [4] studied how to express the newforms (weight 2 and 3/2) corresponding to an isogeny class of elliptic curves over the field $\mathbb{Q}$ of rational numbers with a prime conductor by certain theta series coming from the definite quaternion algebra ramified precisely at the prime number and infinity. In the process, he proved Eichler's trace formula on Brandt matrices of the maximal orders of the definite quaternion algebra [4, p. 120, Prop. 1.9], and the identity on the numbers of certain elements of the maximal orders in [4, p. 123], and obtained a conditional formula for the number of representing a positive integer by the sum of three squares (see [4, p. 177]).

The first two results have been generalized by Boylan, Skoruppa, and Zhou to definite quaternion algebras ramified precisely at odd number of prime numbers and infinity using the theory of Jacobi forms (see [2, Cor. 1.1, Eq. (5)]).

We notice that the identity in [4, p. 123] can complement the conditional formula in [4, p. 177], and [4, p. 120, Prop. 1.9] deduces a simple trace formula for Hecke operators acting on the subspace of weight 2 cusp forms of prime level with trivial character. The conclusion of the former is our formula (7) (Theorem 3.1) on the sum of three squares, and the latter one can give general formulae for the Fourier coefficients of all weight 2 modular forms of prime level with trivial character (see Theorem 4.1, Theorem 4.2 and Theorem 4.3).

2. Preliminary

This section introduces the notation and basics of the paper.

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For any negative discriminant $-D$ (i.e., positive integer $D \equiv 0, 3 \bmod 4$), recall the definition formula of the Hurwitz class number:

$$H(D) = \sum_{df^2=-D} \frac{h(d)}{u(d)},$$

where the sum runs through all negative discriminants d dividing $-D$ (positive integer f depends on d, satisfying identity $df^2 = -D$), and $h(d)$ (resp. $u(d)$) equals the class number of ideals (resp. a half of the number of units) of an order of discriminant d of an image quadratic field. For convenience, set also $H(0) = -1/12$ and $H(D) = 0$ if $D \equiv 1, 2 \bmod 4$.

Given a prime number p, the modified Hurwitz class number in [2, Eq. (2)] defined on any negative discriminant $-D$ is

$$H^{(p)}(D) := H(D/f_p^2)\left(1 - \left(\frac{-D/f_p^2}{p}\right)\right) \tag{1}$$

(double the modified invariant $H_p(D)$ in [4, p. 120, Eq. (1.8)]), where f_p is the largest power of p such that $-D/f_p^2$ is still a negative discriminant, and the Kronecker symbol on integer m is

$$\left(\frac{m}{2}\right) = \begin{cases} 1 & \text{if } m \equiv \pm 1 \bmod 8, \\ -1 & \text{if } m \equiv \pm 3 \bmod 8, \\ 0 & \text{otherwise,} \end{cases}$$

or Legendre symbol $\left(\frac{m}{p}\right)$ if $p > 2$.

There are also

$$H^{(p)}(0) := -\frac{1}{12}(1-p),$$

and $H^{(p)}(D) := 0$ for positive integer $D \equiv 1, 2 \bmod 4$.

On the other hand, up to isomorphism, there is only one definite quaternion algebra over $\mathbb{Q}$ ramified exactly at prime number p and ∞. Let Q_p stand for it and $\mathbb{Z}$ be the ring of integers in $\mathbb{Q}$. Call a finitely generated $\mathbb{Z}$-submodule of Q_p an *ideal of* Q_p if it contains a $\mathbb{Q}$-basis of Q_p. An *order of* Q_p is an ideal and a subring of Q_p containing $\mathbb{Z}$. Every element in Q_p has a characteristic polynomial $(x^2 - tx + n)^2$, where rational numbers t, n are defined to be its *reduced trace* and *reduced norm* respectively. In particular, the reduced trace (resp. reduced norm) of any element in an order of definite quaternion algebra Q_p is an *integer* (resp. a *nonnegative integer*). A *maximal order* is an order not properly contained in another order.

For every prime number q, use $\mathbb{Q}_q$ for the field of q-adic numbers, and $\mathbb{Z}_q$ for the ring of q-adic integers in $\mathbb{Q}_q$. Let $\mathcal{O} = \mathcal{O}_1$ be a maximal order of Q_p. A *left $\mathcal{O}$-ideal* $\mathfrak{a}$ is an ideal of Q_p which is locally (left) principal, that is

$$\mathfrak{a}_q := \mathbb{Z}_q \otimes_{\mathbb{Z}} \mathfrak{a} = \mathcal{O}_q x_q,$$

where $\mathcal{O}_q := \mathbb{Z}_q \otimes_{\mathbb{Z}} \mathcal{O}$ and x_q an unit in $\mathbb{Q}_q \otimes_{\mathbb{Q}} Q_p$. Say two left $\mathcal{O}$-ideals $\mathfrak{a}$ and $\mathfrak{b}$ are in same class if $\mathfrak{a} = \mathfrak{b}x$ for some unit $x \in Q_p$.

Select a system of left $\mathcal{O}$-ideals $\mathfrak{a}_1 = \mathcal{O}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_{h_p}$ representing all finite number of left $\mathcal{O}$-ideal classes, where h_p is the number of left $\mathcal{O}$-ideal classes. For any $1 \leq i \leq h_p$, let $\mathcal{O}_i$ be *the right order of* $\mathfrak{a}_i$, and $\operatorname{card}\left(\mathcal{O}_i^\times\right)$ be the number of the elements of the unit group of $\mathcal{O}_i$. Here the right order of $\mathfrak{a}_i$ (also a maximal order of Q_p) is defined as

$$\mathcal{O}_r(\mathfrak{a}_i) := \{x \in Q_p \mid \mathfrak{a}_i x \subseteq \mathfrak{a}_i\}.$$

Similarly, the definition of its *left order* is

$$\mathcal{O}_l(\mathfrak{a}_i) := \{x \in Q_p \mid x\mathfrak{a}_i \subseteq \mathfrak{a}_i\} = \mathcal{O}.$$

The measure of the class number

$$\sum_{i=1}^{h_p} \frac{1}{\operatorname{card}\left(\mathcal{O}_i^\times\right)} = \frac{p-1}{24}, \tag{2}$$

which is Eichler's mass formula.

To introduce the Brandt matrices of the maximal orders of Q_p, we need the inverse (an ideal)

$$\mathfrak{a}^{-1} := \{x \in Q_p \mid \mathfrak{a}x\mathfrak{a} \subseteq \mathfrak{a}\}$$

of a left $\mathcal{O}$-ideal $\mathfrak{a}$. The relation

$$\mathcal{O}_r(\mathfrak{a}_j^{-1}) = \mathcal{O} = \mathcal{O}_l(\mathfrak{a}_i)$$

guarantees one can define the product ideal

$$\mathfrak{a}_j^{-1}\mathfrak{a}_i := \{\sum_{k=1}^{t} x_k y_k \mid t \in \mathbb{Z}, t \geq 1, x_k \in \mathfrak{a}_j^{-1}, y_k \in \mathfrak{a}_i\},$$

where $1 \leq i, j \leq h_p$. In particular

$$\mathfrak{a}_i^{-1}\mathfrak{a}_i = \mathcal{O}_l(\mathfrak{a}_i^{-1}) = \mathcal{O}_i = \mathcal{O}_r(\mathfrak{a}_i)$$

and $\mathcal{O} = \mathfrak{a}_i\mathfrak{a}_i^{-1}$.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals of Q_p. The positive rational number generating the fractional ideal of $\mathbb{Q}$ generated by set $\{\mathrm{n}(x) \mid x \in \mathfrak{a}\}$ is said to be *the norm* $\mathrm{n}(\mathfrak{a})$ *of* $\mathfrak{a}$, where $\mathrm{n}(x)$ is the reduced norm of element x. One has $\mathrm{n}(\mathfrak{a}^{-1}) = \mathrm{n}(\mathfrak{a})^{-1}$ and $\mathrm{n}(\mathfrak{a}\mathfrak{b}) = \mathrm{n}(\mathfrak{a})\,\mathrm{n}(\mathfrak{b})$ (if $\mathcal{O}_r(\mathfrak{a}) = \mathcal{O}_l(\mathfrak{b})$).

On the modular group

$$\mathrm{SL}(2, \mathbb{Z}) := \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mid a, b, c, d \in \mathbb{Z}, ad - bc = 1\},$$

define the congruence subgroup

$$\Gamma_0(p) := \{\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in \mathrm{SL}(2, \mathbb{Z}) \mid c \equiv 0 \bmod p\}$$

of level p. Denote $q(\tau) = e^{2\pi i\tau}$ be the function on the upper half complex plane $\mathbb{H}$. All theta series

$$f_{ij}(\tau) := \frac{1}{\operatorname{card}\left(\mathcal{O}_j^\times\right)} \sum_{b \in \mathfrak{a}_j^{-1}\mathfrak{a}_i} q^{\frac{\mathrm{n}(b)}{\mathrm{n}(\mathfrak{a}_j^{-1}\mathfrak{a}_i)}} = \sum_{n \geq 0} B_{ij}(n) q^n,$$

where $1 \le i, j \le h_p$, span the space $\mathcal{M}_2(p)$ of modular forms of weight 2 on $\Gamma_0(p)$ with the trivial character.

For integer $n \ge 0$, define *the* n^{th} *Brandt matrix* $B(n)$ as the matrix $(B_{ij}(n))$ where $1 \le i, j \le h_p$. By [2, Cor. 1.1] or [4, p. 120, Prop. 1.9], the trace of $B(n)$ is

$$\operatorname{tr}(B(n)) = \sum_{i=1}^{h_p} B_{ii}(n) = \frac{1}{2} \sum_{\substack{r \in \mathbb{Z} \\ r^2 \le 4n}} H^{(p)}(4n - r^2), \tag{3}$$

and the class number

$$h_p = \frac{p-1}{12} + \frac{1}{4}\left(1 - \left(\frac{-4}{p}\right)\right) + \frac{1}{3}\left(1 - \left(\frac{-3}{p}\right)\right) \tag{4}$$

since $B_{ii}(1) = 1$. In particular $h_p = 1$ if and only if $p = 2, 3, 5, 7$ and 13.

Let $\rho_{\mathcal{O}_i}(n, r)$ be the number of the elements in $\mathcal{O}_i$ with reduced norm n and reduced trace r. At the end of the section, we cite the following formula from [2, Eq. (5)] (also see [4, p. 123]):

$$\sum_{i=1}^{h_p} \frac{\rho_{\mathcal{O}_i}(n, r)}{\operatorname{card}\left(\mathcal{O}_i^{\times}\right)} = \frac{H^{(p)}(4n - r^2)}{2}. \tag{5}$$

3. The sum of three squares

First by the equation (5) and Eichler's mass formula, one can get

$$\rho_{\mathcal{O}_1}(n, r) = \frac{12 H^{(p)}(4n - r^2)}{p - 1} \tag{6}$$

when $h_p = 1$.

Write $Q_p = \left(\frac{a,b}{\mathbb{Q}}\right)$, whose a $\mathbb{Q}$-basis $\{1, i, j, k\}$ satisfies

$$i^2 = a, \quad j^2 = b, \quad ij = k = -ji.$$

On this $\mathbb{Q}$-basis, any element

$$x = x_0 + x_1 i + x_2 j + x_3 k \in Q_p$$

has reduced trace $\operatorname{tr}(x) = 2x_0$ and reduced norm

$$\operatorname{n}(x) = x_0^2 - a x_1^2 - b x_2^2 + ab x_3^2.$$

Consider the Hurwitz order

$$\mathcal{O}_1 = \mathbb{Z}[\frac{1 + i + j + k}{2}, i, j, k]$$

of the quaternion algebra $Q_2 = \left(\frac{-1,-1}{\mathbb{Q}}\right)$, i.e., the $\mathbb{Z}$-module generated by $(1 + i + j + k)/2, i, j$ and k. It is a maximal order. For any element

$$x_0 \frac{1 + i + j + k}{2} + x_1 i + x_2 j + x_3 k \in \mathcal{O}_1$$

with $x_0, x_1, x_2, x_3 \in \mathbb{Z}$, its reduced trace $r = x_0$, reduced norm

$$n = x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_0(x_1 + x_2 + x_3).$$

Hence the equation (6) deduces

$$\#\{(x_1, x_2, x_3) \in \mathbb{Z}^3 \mid (x_1^2+x_2^2+x_3^2)+r(x_1+x_2+x_3) = n-r^2\} = 12H^{(2)}(4n-r^2).$$

This kind of equations are not quadratic forms unless $r = 0$. The latter case gives a formula for the number of representing a positive integer n by the sum of three squares:

Theorem 3.1.

$$|\{x_1^2 + x_2^2 + x_3^2 = n\}| = 12H^{(2)}(4n), \tag{7}$$

where $(x_1, x_2, x_3) \in \mathbb{Z}^3$.

It improves the conditional formula in [4, p. 177]:

$$|\{y_1^2 + y_2^2 + y_3^2 = D, y_1 \equiv y_2 \equiv y_3 \bmod 2\}| = 12H^{(2)}(D),$$

where $(y_1, y_2, y_3) \in \mathbb{Z}^3$.

For the cases $p = 3, 5, 7, 13$, from [5, Prop. 5.1, Prop. 5.2] we know $\left(\frac{-1,-p}{\mathbb{Q}}\right)$ (resp. $\left(\frac{-2,-p}{\mathbb{Q}}\right)$) is the unique quaternion algebra (up to isomorphism) exactly ramified at infinity and p if $p \equiv 3 \bmod 4$ (resp. $p \equiv 5 \bmod 8$). Therein Pizer gave a maximal order

$$\mathcal{O}_1 = \mathbb{Z}[\frac{1+j}{2}, \frac{i+k}{2}, j, k]$$

(resp. $\mathcal{O}_1 = \mathbb{Z}[\frac{1+j+k}{2}, \frac{i+2j+k}{4}, j, k]$).

It is similar to the above argument that we have

$$|\{(x_1^2 + 3x_2^2 + 3x_3^2) + 3x_1x_3 = n\}| = 6H^{(3)}(4n),$$
$$|\{(2x_1^2 + 7x_2^2 + 7x_3^2) + 7x_1x_3 = n\}| = 2H^{(7)}(4n),$$

and

$$|\{(2x_1^2 + 5x_2^2 + 10x_3^2) + 5(x_1x_3 + x_1x_2) = n\}| = 3H^{(5)}(4n),$$
$$|\{(5x_1^2 + 13x_2^2 + 26x_3^2) + 13(x_1x_3 + x_1x_2) = n\}| = H^{(13)}(4n).$$

If following the method in [4, pp. 172–177], one can also get

$$|\{(3y_1^2 + 4y_2^2 + 12y_3^2) + 12y_2y_3 = D\}| = 6H^{(3)}(D),$$
$$|\{(7y_1^2 + 8y_2^2 + 28y_3^2) + 28y_2y_3 = D\}| = 2H^{(7)}(D),$$

and

$$|\{(15y_1^2 + 8y_2^2 + 20y_3^2) + 20(y_2y_3 + y_1y_3 + y_1y_2) = D\}| = 3H^{(5)}(D),$$
$$|\{(39y_1^2 + 20y_2^2 + 52y_3^2) + 52(y_2y_3 + y_1y_3 + y_1y_2) = D\}| = H^{(13)}(D).$$

4. Fourier coefficients of modular forms of weight 2 with prime level

In the section, we will study the space $\mathcal{M}_2(p)$ and its subspace of cusp forms $\mathcal{S}_2(p)$. As the corresponding quotient space, the Eisenstein subspace $\mathcal{E}_2(p) := \mathcal{M}_2(p)/\mathcal{S}_2(p)$ is one-dimensional. Let $\mathcal{S}_2^{\mathrm{new}}(p)$ be the subspace of newforms of $\mathcal{S}_2(p)$, and $\mathcal{M}_2(1)$ be the space of modular forms of weight 2 on $\mathrm{SL}(2,\mathbb{Z})$ with the trivial character. Since $\dim \mathcal{M}_2(1) = 0$, we have $\mathcal{S}_2^{\mathrm{new}}(p) = \mathcal{S}_2(p)$ and $h_p - 1 = \dim \mathcal{S}_2^{\mathrm{new}}(p)$ (the genus of the modular curve $X_0(p)$, see [4, p. 144]).

For positive integer n, let $\mathrm{T}(n)$ be the Hecke operator of index n on $\mathcal{M}_2(p)$ and

$$\sigma(n)_p := \sum_{\substack{p \nmid d | n \\ d>0}} d,$$

where the sum runs through positive divisors of n without the multiples of p. Gross [4, p. 144, Eq. (5.7)] proved that

$$E_p = \sum_{j=1}^{h_p} f_{ij} = \frac{p-1}{24} + \sum_{n \geq 1} \sigma(n)_p q^n$$

is the unique normalized Eisenstein series for any $1 \leq i \leq h_p$. It satisfies

$$\mathrm{T}(n) E_p = \sigma(n)_p E_p.$$

He also cited and proved Eichler's identity: the trace of Brandt matrix $B(n)$ equals the trace of Hecke operator $\mathrm{T}(n)$ acting on all modular forms in $\mathcal{M}_2(p)$ (see [4, p. 142, Eq. (5.4), Prop. 5.5]).

The above conclusions and equation (3) imply that the trace of $\mathrm{T}(n)$ on $\mathcal{S}_2(p)$ is

$$\mathrm{Tr}_{\mathcal{S}_2(p)}(\mathrm{T}(n)) = \mathrm{tr}(B(n)) - \sigma(n)_p = 2^{-1} \sum_{r^2 \leq 4n} H^{(p)}(4n - r^2) - \sigma(n)_p. \tag{8}$$

If $h_p = 1$, theta series

$$f_{11}(\tau) = \frac{1}{\mathrm{card}\left(\mathcal{O}_1^{\times}\right)} \sum_{b \in \mathcal{O}_1} q^{\mathrm{n}(b)} = \sum_{n \geq 0} B_{11}(n) q^n$$

is a basis of $\mathcal{M}_2(p)$, namely:

Theorem 4.1. *For $p = 2, 3, 5, 7$ and 13 we have*

$$\sigma(n)_p = B_{11}(n) = \mathrm{tr}(B(n)) = 2^{-1} \sum_{r^2 \leq 4n} H^{(p)}(4n - r^2), \tag{9}$$

where $n \geq 1$.

On the structure of $\mathcal{S}_2(p)$, we know from [1, Lem. 18, Lem. 19, Thm. 3, Thm. 5] that there is a direct sum decomposition

$$\mathcal{S}_2(p) = \mathcal{S}_2^{\mathrm{new}}(p) = \bigoplus V_i, \tag{10}$$

where one-dimensional subspaces V_i generated by a normalized newform are simultaneous eigenspaces for all Hecke operators $\mathrm{T}(q)$ with prime number $q \neq p$, Atkin-Lehner involution W_p and $\mathrm{T}(p) = U(p)$. Here for each normalized newform $F = \sum_{n\geq 1} A(n)q^n$, one has

$$F|U(p) = A(p)F = -F|W_p,$$

where

$$F|W_p = p(pc\tau + pd)^{-2}F(\frac{pa\tau + b}{pc\tau + pd})$$

for $W_p = \begin{bmatrix} pa & b \\ pc & pd \end{bmatrix}$ with $a, b, c, d \in \mathbb{Z}$ and $p^2ad - pbc = p$.

Let $\mathfrak{S}_2^{\mathrm{new}}(p, 1)$ and $\mathfrak{S}_2^{\mathrm{new}}(1, p)$ be the subspaces spanned by all common eigenforms F with respect to Atkin-Lehner involution W_p in $\mathcal{S}_2^{\mathrm{new}}(p)$ such that the Atkin-Lehner sign (eigenvalue) of F is -1 and $+1$ respectively. The direct sum decomposition (10) can be rewritten as

$$\mathcal{S}_2^{\mathrm{new}}(p) = \mathfrak{S}_2^{\mathrm{new}}(p, 1) \bigoplus \mathfrak{S}_2^{\mathrm{new}}(1, p). \tag{11}$$

Moreover, the eigenvalue of Hecke operator $\mathrm{T}(n)$ on a normalized newform of $\mathcal{S}_2^{\mathrm{new}}(p)$ is the n^{th} Fourier coefficient of the newform [3, Thm. 13.3.5]. Hence the sum of canonical basis of newforms in $\mathcal{S}_2(p) = \mathcal{S}_2^{\mathrm{new}}(p)$ gives trace form

$$\mathcal{T}_2(p) := \sum_{n\geq 1} \mathrm{Tr}_{\mathcal{S}_2(p)}(\mathrm{T}(n))q^n. \tag{12}$$

Consider prime p satisfying $h_p = 2$. The calculation formula for h_p deduces $h_p = 2$ if and only if $p = 11, 17, 19$. For them, we have:

Theorem 4.2. *The unique normalized newform in $\mathcal{M}_2(p)$ is the trace form $\mathcal{T}_2(p)$, its n^{th} Fourier coefficient $(n \geq 1)$ is*

$$2^{-1} \sum_{r^2 \leq 4n} H^{(p)}(4n - r^2) - \sigma(n)_p.$$

For $p \geq 23$, let $s_p = \lfloor \frac{p+1}{6} \rfloor + 1$, where $\lfloor \frac{p+1}{6} \rfloor$ equals the largest integer less than or equal to $\frac{p+1}{6}$. By [3, Thm. 12.4.16], we have:

Theorem 4.3. *Set*

$$\{\mathrm{T}(k)\mathcal{T}_2(p) \,|\, 1 \leq k \leq s_p\}$$

generates subspace $\mathcal{S}_2(p)$. There are at most $h_p - 1$ in them linear independent. Here the n^{th} Fourier coefficient of $\mathrm{T}(k)\mathcal{T}_2(p)$ is

$$\sum_{\substack{d|\gcd(n,k) \\ d>0}} \chi(d)d\,\mathrm{Tr}_{\mathcal{S}_2(p)}(\mathrm{T}(\frac{nk}{d^2})),$$

where $\chi(d) = 1$ if $p \nmid d$, otherwise $\chi(d) = 0$, and

$$\mathrm{Tr}_{\mathcal{S}_2(p)}(\mathrm{T}(\frac{nk}{d^2})) = 2^{-1} \sum_{r^2 \leq \frac{4nk}{d^2}} H^{(p)}(\frac{4nk}{d^2} - r^2) - \sigma(\frac{nk}{d^2})_p.$$

Based on Theorem 4.3 and the equation (11), we can give an algorithm for general formulae of the Fourier coefficients of all weight 2 newforms of prime level p with trivial character. For such a newform F, the algorithm consists of the following steps.

Step 1. Calculate the first $s_p = \lfloor \frac{p+1}{6} \rfloor + 1$ Fourier coefficients of F, and the dimension

$$\dim \mathcal{S}_2(p) = \dim \mathcal{S}_2^{\mathrm{new}}(p) = h_p - 1 \leq \lfloor \frac{p+1}{12} \rfloor$$

using the equation (4).

Step 2. Calculate the Atkin-Lehner signs (eigenvalues) of all elements (normalized eigenforms) of the canonical basis of $\mathcal{S}_2^{\mathrm{new}}(p)$ on W_p in Step 1. Note that $\mathcal{T}_2(p)$ is the sum of these eigenforms and

$$\mathrm{T}(p)\mathcal{T}_2(p) = \mathcal{T}_2(p)|U(p) = -\mathcal{T}_2(p)|W_p.$$

One can get

$$F = \frac{\mathcal{T}_2(p) + \mathrm{T}(p)\mathcal{T}_2(p)}{2} \quad \text{or} \quad F = \frac{\mathcal{T}_2(p) - \mathrm{T}(p)\mathcal{T}_2(p)}{2}$$

by the equation (11), when ($\mathbb{C}$ denote the field of complex numbers)

$$\mathfrak{S}_2^{\mathrm{new}}(p,1) = \mathbb{C}F \quad \text{or} \quad \mathfrak{S}_2^{\mathrm{new}}(1,p) = \mathbb{C}F.$$

Step 3. If F is not a basis of $\mathfrak{S}_2^{\mathrm{new}}(p,1)$ or $\mathfrak{S}_2^{\mathrm{new}}(1,p)$, compute the first s_p Fourier coefficients of all cusp forms in $\{\mathrm{T}(k)\mathcal{T}_2(p) \,|\, 1 \leq k \leq s_p\}$. These coefficients form a square matrix A of order s_p (write the s_p coefficients of a cusp form as a row or column). Theorem 4.3 implies that there exists a nonsingular submatrix of A which is a square matrix of order $h_p - 1$. It corresponds to a maximal linearly independent subset of $\{\mathrm{T}(k)\mathcal{T}_2(p) \,|\, 1 \leq k \leq s_p\}$. To locate the positions of the entries of such a nonsingular submatrix in A, one only need to do elementary row and column transformations on A. Using matrix multiplication to the inverse of such a submatrix and the corresponding Fourier coefficients of F, one will obtain the coefficients in linear combination expressing F by a maximal linearly independent subset (a basis of $\mathcal{S}_2^{\mathrm{new}}(p)$).

Remark 4.4. Reviewing the algorithm, we point out:

(1) Step 3 always works for any newform of prime level (of weight 2 with trivial character).

(2) In general, if F satisfies the condition in Step 2, one can get two general formulae for its Fourier coefficients by using Step 2 and Step 3 to it.

The data needed by the algorithm also can be found in The L-functions and Modular Forms Database (LMFDB, available at https://www.lmfdb.org).

With the assistance of LMFDB, applying the algorithm, we make the short Table 1 to give several examples for the newforms of weight 2 corresponding to the isogeny classes of elliptic curves over $\mathbb{Q}$ with prime conductor, where the LMFDB label stands for the corresponding elliptic curve isogeny class. One

TABLE 1. Newforms corresponding to elliptic curves with prime conductor $p \leq 101$

p	LMFDB label	Newform
11	11.a	$\mathcal{T}_2(11)$
17	17.a	$\mathcal{T}_2(17)$
19	19.a	$\mathcal{T}_2(19)$
37	37.a	$\frac{\mathcal{T}_2(37)-\mathrm{T}(37)\mathcal{T}_2(37)}{2}$
37	37.b	$\frac{\mathcal{T}_2(37)+\mathrm{T}(37)\mathcal{T}_2(37)}{2}$
43	43.a	$\frac{\mathcal{T}_2(43)-\mathrm{T}(43)\mathcal{T}_2(43)}{2}$
53	53.a	$\frac{\mathcal{T}_2(53)-\mathrm{T}(53)\mathcal{T}_2(53)}{2}$
61	61.a	$\frac{\mathcal{T}_2(61)-\mathrm{T}(61)\mathcal{T}_2(61)}{2}$
67	67.a	$\frac{6\mathcal{T}_2(67)+9\,\mathrm{T}(2)\mathcal{T}_2(67)-3\,\mathrm{T}(3)\mathcal{T}_2(67)+4\,\mathrm{T}(4)\mathcal{T}_2(67)+\mathrm{T}(5)\mathcal{T}_2(67)}{40}$
73	73.a	$\frac{-\,\mathrm{T}(3)\mathcal{T}_2(73)+\mathrm{T}(4)\mathcal{T}_2(73)+2\,\mathrm{T}(5)\mathcal{T}_2(73)}{3}$
79	79.a	$\frac{\mathcal{T}_2(79)-\mathrm{T}(79)\mathcal{T}_2(79)}{2}$
83	83.a	$\frac{\mathcal{T}_2(83)-\mathrm{T}(83)\mathcal{T}_2(83)}{2}$
89	89.a	$\frac{\mathcal{T}_2(89)-\mathrm{T}(89)\mathcal{T}_2(89)}{2}$
89	89.b	$\frac{8\mathcal{T}_2(89)+4\,\mathrm{T}(2)\mathcal{T}_2(89)+6\,\mathrm{T}(3)\mathcal{T}_2(89)-3\,\mathrm{T}(4)\mathcal{T}_2(89)-5\,\mathrm{T}(5)\mathcal{T}_2(89)+2\,\mathrm{T}(6)\mathcal{T}_2(89)+2\,\mathrm{T}(7)\mathcal{T}_2(89)}{45}$
101	101.a	$\frac{\mathcal{T}_2(101)-\mathrm{T}(101)\mathcal{T}_2(101)}{2}$

can use Theorem 4.3 to give general formulae for the Fourier coefficients of these newforms. In this process, we get relations

$$-\frac{\mathrm{T}(2)\mathcal{T}_2(37)}{2} = \frac{\mathcal{T}_2(37) - \mathrm{T}(37)\mathcal{T}_2(37)}{2}$$

and

$$-\frac{\mathrm{T}(2)\mathcal{T}_2(43)}{4} - \frac{\mathrm{T}(3)\mathcal{T}_2(43)}{4} = \frac{\mathcal{T}_2(43) - \mathrm{T}(43)\mathcal{T}_2(43)}{2}$$

when using Step 2 and Step 3 of the algorithm to the newforms corresponding to the elliptic curve isogeny classes (LMFDB label) 37.a and 43.a in Table 1.

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Yan-Bin Li
School of Mathematical Sciences
Tongji University
No.1239, Siping Rd, Yangpu District, Shanghai, P. R. China
Email address: liyanbin0454@163.com