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THE CONVEX HULL OF THREE BOUNDARY POINTS IN COMPLEX HYPERBOLIC SPACE

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Abstract. The convex hull of a generic triple of boundary points has non-zero finite volume in complex hyperbolic 2-space.

1. Introduction

In hyperbolic space, the convex hull $\mathcal{C}(p_1, \ldots, p_n)$ of boundary points $p_1, \ldots,$ p_n is the smallest geodesically convex set whose closure contains $p_1, \ldots, p_n$. For example, the convex hull $\mathcal{C}(p_1, p_2)$ of two boundary points p_1 and p_2 is the geodesic connecting p_1 and p_2. For three boundary points p_1, p_2, p_3 of real hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$ $(n \geq 2)$, the convex hull $\mathcal{C}(p_1, p_2, p_3)$ is the ideal triangle whose vertices are p_1, p_2, p_3. It is embedded in a 2-dimensional totally geodesic subspace. Since the isometry group acts triply transitively on the boundary at infinity of $\mathbb{H}^n_{\mathbb{R}}$, convex hulls of three boundary points are all isometric to each other. However, in complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$, the shape of the convex hull of three boundary points becomes very mysterious. Here, we investigate the convex hull of three boundary points in complex hyperbolic 2-space $\mathbb{H}^2_{\mathbb{C}}$.

A triple $P = (p_1, p_2, p_3)$ of distinct boundary points in $\partial\mathbb{H}^2_{\mathbb{C}}$ is parameterized by the Cartan angular invariant $\mathbb{A}(P)$ which has the following properties [1].

- $-\frac{\pi}{2} \leq \mathbb{A}(P) \leq \frac{\pi}{2}$.
- $\mathbb{A}(P) = 0 \Leftrightarrow p_1, p_2, p_3$ lie on the boundary of a Lagrangian plane.
- $\mathbb{A}(P) = \pm\frac{\pi}{2} \Leftrightarrow p_1, p_2, p_3$ lie on the boundary of a complex line.

For two triples of boundary points P and Q,

- $\mathbb{A}(P) = \mathbb{A}(Q) \Leftrightarrow g(P) = Q$ for a holomorphic isometry g of $\mathbb{H}^2_{\mathbb{C}}$.
- $\mathbb{A}(P) = -\mathbb{A}(Q) \Leftrightarrow h(P) = Q$ for an anti-holomorphic isometry h of $\mathbb{H}^2_{\mathbb{C}}$.

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Hence, up to the action of the group of holomorphic isometries $\mathrm{PU}(2,1)$, there is a 1-dimensional parameter space $[-\frac{\pi}{2}, \frac{\pi}{2}]$ of the set of all triples of distinct boundary points in $\partial\mathbb{H}^2_{\mathbb{C}}$. In what follows, we will prove that the convex hull $\mathcal{C}(p_1, p_2, p_3)$ of a generic triple (p_1, p_2, p_3) has non-zero finite volume.

2. Complex hyperbolic space and the boundary at infinity

We refer to [2] and [4] for the basics of complex hyperbolic geometry.

2.1. Complex hyperbolic space and the boundary at infinity

Let $\mathbb{C}^{2,1}$ be the three dimensional complex vector space $\mathbb{C}^3$ with the Hermitian form of signature $(2,1)$ given by

$$\langle \mathbf{z}, \mathbf{w} \rangle = \mathbf{w}^* J \mathbf{z} = z_1 \overline{w_1} + z_2 \overline{w_2} - z_3 \overline{w_3}, \tag{1}$$

where the Hermitian matrix J is $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$. Let V_- and V_0 be the set of negative vectors and null vectors respectively:

$$\begin{aligned} V_- &= \{\mathbf{z} \in \mathbb{C}^{2,1} : \langle \mathbf{z}, \mathbf{z} \rangle < 0\}, \\ V_0 &= \{\mathbf{z} \in \mathbb{C}^{2,1} \setminus \{0\} : \langle \mathbf{z}, \mathbf{z} \rangle = 0\}. \end{aligned} \tag{2}$$

Let $\mathbb{P} : \mathbb{C}^{2,1} \setminus \{0\} \to \mathbb{CP}^2$ be the canonical projection onto complex projective space. Then the complex hyperbolic 2-space $\mathbb{H}^2_{\mathbb{C}}$ is defined to be $\mathbb{P}V_-$ and the boundary at infinity $\partial\mathbb{H}^2_{\mathbb{C}}$ to be $\mathbb{P}V_0$. Considering the section defined by $z_3 = 1$, we obtain the ball model of complex hyperbolic 2-space. For any $z = (z_1, z_2) \in \mathbb{C}^2$, we lift the point z to $\mathbf{z} = (z_1, z_2, 1) \in \mathbb{C}^{2,1}$, called the *standard lift of* z. Then $\langle \mathbf{z}, \mathbf{z} \rangle = |z_1|^2 + |z_2|^2 - 1$. Hence the ball model of complex hyperbolic 2-space is

$$\mathbb{H}^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 < 1\}$$

and its boundary at infinity is

$$\partial\mathbb{H}^2_{\mathbb{C}} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}.$$

The *Bergman metric* ρ on $\mathbb{H}^2_{\mathbb{C}}$ is defined by

$$\cosh^2\left(\frac{\rho(z,w)}{2}\right) = \frac{\langle \mathbf{z}, \mathbf{w} \rangle \langle \mathbf{w}, \mathbf{z} \rangle}{\langle \mathbf{z}, \mathbf{z} \rangle \langle \mathbf{w}, \mathbf{w} \rangle}, \tag{3}$$

where $\mathbf{z}$ and $\mathbf{w}$ are the standard lifts of z and $w \in \mathbb{H}^2_{\mathbb{C}}$. Let $\mathrm{SU}(2,1)$ be the group of unitary matrices which preserve the given Hermitian form with the determinant 1. Then the group of holomorphic isometries of $\mathbb{H}^2_{\mathbb{C}}$ is $\mathrm{PU}(2,1) = \mathrm{SU}(2,1)/\{I, \omega I, \omega^2 I\}$, where $\omega = (-1 + i\sqrt{3})/2$ is a cube root of unity.

Let p and q be distinct boundary points of $\mathbb{H}^2_{\mathbb{C}}$ and $\widetilde{p}$, $\widetilde{q} \in V_0$ the lifts of p and q, respectively, with $\langle \widetilde{p}, \widetilde{q} \rangle = -1$. Then $\mathbb{P}(e^{\frac{t}{2}}\widetilde{p} + e^{-\frac{t}{2}}\widetilde{q}) \in \mathbb{H}^2_{\mathbb{C}}$ $(t \in \mathbb{R})$ is the geodesic connecting p and q.

2.2. Cartan angular invariant

Consider a triple of distinct boundary points (p_1, p_2, p_3) in $\partial\mathbb{H}^2_{\mathbb{C}}$. Then the Cartan angular invariant of (p_1, p_2, p_3) is defined as

$$\mathbb{A}(p_1, p_2, p_3) = \arg(-\langle\widetilde{p_1}, \widetilde{p_2}\rangle\langle\widetilde{p_2}, \widetilde{p_3}\rangle\langle\widetilde{p_3}, \widetilde{p_1}\rangle),$$

where $\widetilde{p_i} \in \mathbb{C}^{2,1}$ is a lift of p_i $(i = 1, 2, 3)$. The argument of the Hermitian triple product does not depend on the chosen lifts because

$$\langle\lambda_1\widetilde{p_1}, \lambda_2\widetilde{p_2}\rangle\langle\lambda_2\widetilde{p_2}, \lambda_3\widetilde{p_3}\rangle\langle\lambda_3\widetilde{p_3}, \lambda_1\widetilde{p_1}\rangle = |\lambda_1|^2|\lambda_2|^2|\lambda_3|^2\langle\widetilde{p_1}, \widetilde{p_2}\rangle\langle\widetilde{p_2}, \widetilde{p_3}\rangle\langle\widetilde{p_3}, \widetilde{p_1}\rangle$$

for $\lambda_i \in \mathbb{C}^*$ $(i = 1, 2, 3)$.

Geometrically, the Cartan angular invariant can be seen as follows: Let L be the unique complex line spanned by p_1, p_2 and $\Pi : \mathbb{H}^2_{\mathbb{C}} \to L$ the orthogonal projection onto L. Then the Cartan angular invariant is the half of the signed area of the geodesic triangle whose vertices are p_1, p_2 and $\Pi(p_3)$:

$$\mathbb{A}(p_1, p_2, p_3) = \frac{1}{2}\mathrm{Area}(\triangle(p_1, p_2, \Pi(p_3))).$$

If (p_1, p_2, p_3) lies on the boundary of a Lagrangian plane R, the projection $\Pi(p_3)$ belongs to the geodesic connecting p_1 and p_2. Thus the area of $\triangle(p_1, p_2, \Pi(p_3))$ is 0 and $\mathbb{A}(p_1, p_2, p_3) = 0$.

If (p_1, p_2, p_3) lies on the boundary of a complex line, the projection $\Pi(p_3) = p_3$ is a boundary point of the complex line L. Thus $\triangle(p_1, p_2, \Pi(p_3))$ is an ideal triangle whose area is $\pm\pi$ where the sign depends on the orientation of p_1, p_2, p_3 along the boundary of the complex line L. Thus $\mathbb{A}(p_1, p_2, p_3) = \pm\frac{\pi}{2}$.

Complex hyperbolic 2-space $\mathbb{H}^2_{\mathbb{C}}$ has no co-dimension 1 totally geodesic subspaces. Lagrangian planes and complex lines are the only 2-dimensional totally geodesic subspaces of $\mathbb{H}^2_{\mathbb{C}}$. If a triple (p_1, p_2, p_3) belongs to the boundary of a Lagrangian plane or a complex line, i.e., $\mathbb{A}(p_1, p_2, p_3) = 0$ or $\pm\frac{\pi}{2}$ respectively, the convex hull $\mathcal{C}(p_1, p_2, p_3)$ is an ideal triangle embedded in the Lagrangian plane or the complex line respectively in $\mathbb{H}^2_{\mathbb{C}}$. Thus the volume of $\mathcal{C}(p_1, p_2, p_3)$ is 0 in complex hyperbolic 2-space $\mathbb{H}^2_{\mathbb{C}}$. We will call a triple (p_1, p_2, p_3) with the Cartan angular invariant $\mathbb{A}(p_1, p_2, p_3) \neq 0, \pm\frac{\pi}{2}$ generic. In the next section, we will investigate the convex hull of a generic triple.

3. Convex hull

Let (p_1, p_2, p_3) be a generic triple of boundary points in $\partial\mathbb{H}^2_{\mathbb{C}}$, i.e., $\mathbb{A}(p_1, p_2, p_3) \neq 0, \pm\frac{\pi}{2}$. Applying isometries if necessary, we may assume that $p_1 = (1, 0)$, $p_2 = (-1, 0)$, $p_3 = \left(\frac{vi}{-2+vi}, \frac{2}{-2+vi}\right)$ for $v = \tan\mathbb{A}(p_1, p_2, p_3) \in \mathbb{R} - \{0\}$. Let $\mathcal{C} = \mathcal{C}(p_1, p_2, p_3)$ be the convex hull of the triple (p_1, p_2, p_3). Let γ_i $(i = 1, 2, 3)$ be a geodesic between the boundary points as in Figure 1.

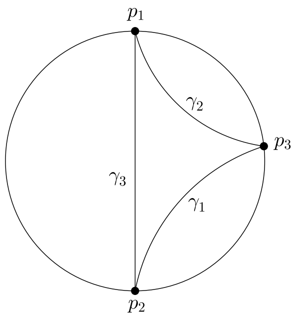

FIGURE 1. Three geodesic in the convex hull $\mathcal{C}(p_1, p_2, p_3)$

For the geodesic $\gamma_1(t)$ between p_2 and p_3, let $\widetilde{p_2} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$ and $\widetilde{p_3} = \frac{2-vi}{2-2vi}\mathbf{p}_3 = \begin{pmatrix} \frac{-vi}{2-2vi} \\ \frac{-2}{2-2vi} \\ \frac{2-vi}{2-2vi} \end{pmatrix}$ be the lifts of p_2 and p_3, respectively. Then $\langle \widetilde{p_2}, \widetilde{p_3} \rangle = -1$ and

$$e^{\frac{t}{2}}\widetilde{p_2} + e^{-\frac{t}{2}}\widetilde{p_3} = \begin{pmatrix} -e^{\frac{t}{2}} - e^{-\frac{t}{2}}\frac{vi}{2-2vi} \\ -e^{-\frac{t}{2}}\frac{2}{2-2vi} \\ e^{\frac{t}{2}} + e^{-\frac{t}{2}}\frac{2-vi}{2-2vi} \end{pmatrix}.$$

We normalized it such that the third coordinate becomes 1, to obtain the geodesic

$$\gamma_1(t) = \left(\frac{-e^t(2-2vi) - vi}{e^t(2-2vi) + 2 - vi}, \ \frac{-2}{e^t(2-2vi) + 2 - vi} \right). \tag{4}$$

Similarly, for the geodesic $\gamma_2(t)$ between p_1 and p_3, we choose the lifts $\widetilde{p_1} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ and $\widetilde{\widetilde{p_3}} = \begin{pmatrix} \frac{-vi}{2} \\ -1 \\ \frac{2-vi}{2} \end{pmatrix}$ of p_1 and p_3, respectively. Then

$$e^{\frac{t}{2}}\widetilde{p_1} + e^{-\frac{t}{2}}\widetilde{\widetilde{p_3}} = \begin{pmatrix} e^{\frac{t}{2}} - e^{-\frac{t}{2}}\frac{vi}{2} \\ -e^{-\frac{t}{2}} \\ e^{\frac{t}{2}} + e^{-\frac{t}{2}}\frac{2-vi}{2} \end{pmatrix}$$

gives us the geodesic

$$\gamma_2(t) = \left(\frac{2e^t - vi}{2e^t + 2 - vi}, \ \frac{-2}{2e^t + 2 - vi} \right). \tag{5}$$

Lastly, for the geodesic $\gamma_3(t)$ between p_1 and p_2, we choose the lifts $\widetilde{p_1} = \left(\begin{smallmatrix}1\\0\\1\end{smallmatrix}\right)$ and $\widetilde{\widetilde{p_2}} = \left(\begin{smallmatrix}-\frac{1}{2}\\0\\\frac{1}{2}\end{smallmatrix}\right)$ of p_1 and p_2, respectively. Then

$$e^{\frac{t}{2}}\widetilde{p_1} + e^{-\frac{t}{2}}\widetilde{\widetilde{p_2}} = \begin{pmatrix} e^{\frac{t}{2}} - \frac{e^{-\frac{t}{2}}}{2} \\ 0 \\ e^{\frac{t}{2}} + \frac{e^{-\frac{t}{2}}}{2} \end{pmatrix}$$

projects to the geodesic

$$\gamma_3(x) = (x,\ 0) \tag{6}$$

for $-1 \le x \le 1$.

Theorem 3.1. *The convex hull $\mathcal{C}$ of a generic triple has non-zero volume in complex hyperbolic space.*

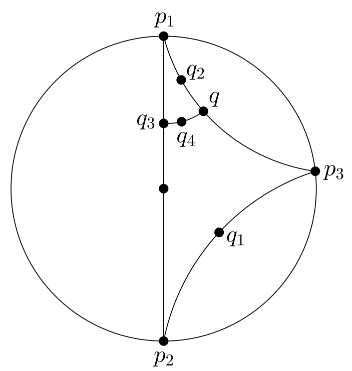

FIGURE 2. Points in the convex hull $\mathcal{C}$

Proof. The geodesic $\gamma_3(x)$ passes $(0,0) \in \mathbb{C}^2$ when $x = 0$. Thus the convex hull $\mathcal{C}$ contains $(0,0)$. Now we will find four points $q_1, q_2, q_3, q_4 \in \mathcal{C}$ and show that the four points have real rank 4 in $\mathbb{C}^2$.

First, we choose q_1, q_2 and q_3 as follows (see Figure 2)

$$q_1 = \gamma_1(0) = \left(\frac{-2+vi}{4-3vi},\ \frac{-2}{4-3vi}\right),$$
$$q_2 = \gamma_2(\ln 2) = \left(\frac{4-vi}{6-vi},\ \frac{-2}{6-vi}\right),$$
$$q_3 = \gamma_3\left(\frac{1}{2}\right) = \left(\frac{1}{2},\ 0\right),$$

where $\gamma_1, \gamma_2, \gamma_3$ are the geodesic (4), (5) and (6). For q_4, let $q = \gamma_2(0) = \left(\frac{2-vi}{4-vi}, \frac{-2}{4-vi}\right)$ and $\gamma_4(t)$ be the geodesic segment connecting q and q_3. Let $\widetilde{q} = \frac{1}{6-vi}\begin{pmatrix} 2-vi \\ -2 \\ 4-vi \end{pmatrix}$ and $\widetilde{q_3} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$ be the lifts of q and q_3 respectively with $\langle \widetilde{q}, \widetilde{q_3} \rangle = -1$. Then the lift of $\gamma_4(t)$ is of the following form

$$\widetilde{q} + t\widetilde{q_3} = \begin{pmatrix} \frac{2-vi}{6-vi} + t \\ \frac{-2}{6-vi} \\ \frac{4-vi}{6-vi} + 2t \end{pmatrix}$$

for a real number t satisfying $\langle \widetilde{q} + t\widetilde{q_3}, \widetilde{q} + t\widetilde{q_3} \rangle < 0$. In particular, we choose $t = 1$ for q_4, i.e.,

$$q_4 = \mathbb{P}(\widetilde{q} + \widetilde{q_3}) = \left(\frac{8-2vi}{16-3vi}, \frac{-2}{16-3vi} \right).$$

We identify $\mathbb{C}^2$ with $\mathbb{R}^4$ via for $(z_1, z_2) \in \mathbb{C}^2$, $(z_1, z_2) \mapsto (\mathrm{Re} z_1, \mathrm{Im} z_1, \mathrm{Re} z_2, \mathrm{Im} z_2) \in \mathbb{R}^4$. Let M be the 4×4 matrix whose entries are the real coordinates of $q_1, q_2, q_3, q_4 \in \mathbb{C}^2$. Equivalently, consider the 4×4 matrix whose entries are the real coordinates of $|4-3vi|^2 q_1$, $|6-vi|^2 q_2$, $2q_3$, $|16-3vi|^2 q_4$,

$$M' = \begin{pmatrix} -8-3v^2 & -2v & -8 & -6v \\ 24+v^2 & -2v & -12 & -2v \\ 1 & 0 & 0 & 0 \\ 128+6v^2 & -8v & -32 & -6v \end{pmatrix}.$$

Since v is non-zero, M' has rank 4. Therefore, the convex hull $\mathcal{C}$ has non-zero volume. $\square$

Using the fact that a closed connected subset S of a CAT(k)-space ($k \leq 0$) is convex if the set S is locally convex [5], we will prove that the convex hull $\mathcal{C}$ has finite volume.

Lemma 3.2. *The convex hull $\mathcal{C}$ has finite volume in complex hyperbolic 2-space $\mathbb{H}^2_{\mathbb{C}}$.*

Proof. For $i = 1, 2, 3$, let $r_i(t)$ be a geodesic ray from $(0,0) \in \mathcal{C}$ to the boundary point p_i, which is parameterized by arc length $t \geq 0$. For a positive number A, we consider a convex neighborhood N_i of the geodesic ray r_i ($i = 1, 2, 3$) (see Figure 3),

$$N_i = \{p \in \mathbb{H}^2_{\mathbb{C}} : \rho(p, r_i(t)) \leq Ae^{-t} \text{ for some } t \geq 0\}.$$

Then N_i has finite volume and so does $N = N_1 \cup N_2 \cup N_3$. For a large A, N is a locally convex set containing p_1, p_2 and p_3. Since $\mathbb{H}^2_{\mathbb{C}}$ is a complete CAT(-1)-space, N itself is convex. From the fact that the convex hull $\mathcal{C}$ is the smallest convex set containing p_1, p_2 and p_3, $\mathcal{C}$ is contained in N. Therefore, $\mathcal{C}$ has finite volume. $\square$

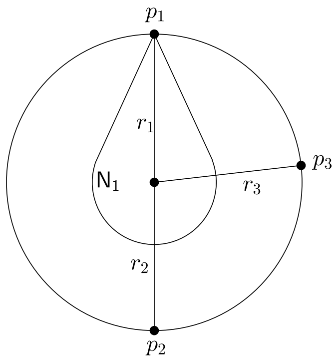

FIGURE 3. Three geodesic rays r_1, r_2, r_3 and a convex neighborhood N_1

Lemma 3.2 implies that the convex hull $\mathcal{C}(p_1, p_2, p_3)$ has exactly three boundary points p_1, p_2 and p_3. In [3], they prove that if a set S is a union of finitely many convex sets of a complete CAT(-1)-space, then S and the convex hull of S have the same boundary at infinity.

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