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PORTFOLIO SELECTION WITH HYPERBOLIC DISCOUNTING AND INFLATION RISK

Byung Hwa Lim*

Abstract. This paper investigates the time-inconsistent agent's optimal consumption and investment problem under inflation risk. The agents' discount factor is governed by hyperbolic discounting, which has a random time to change. We impose the inflation risk which plays a crucial role in long-term financial planning. We derive the semi-analytic solution to the problem of sophisticated agents when the time horizon is finite.

1. Introduction

We consider the Merton's portfolio selection problem of time-inconsistent agent who randomly changes her time preference. The agent's subjective discount factor follows a jump process which has a Poisson distribution. We call this time preference as a hyperbolic discounting, and it implies that after the preference change, the future selves behave differently from the current self. The sophisticated agent who has a hyperbolic discounting takes into account her preference change in deciding the optimal consumption and investment. So even with hyperbolic discounting, the optimal controls are time consistent. The portfolio selection problems of the sophisticated agent are well studied in [9], [4], [10], [6], [11], [12], and [7].

In this paper, we extend the model into the problem under inflation risk. The inflation risk plays an important role in long-term financial planning. To hedge the risk, there exists an inflation-linked index

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bond market in many countries(e.g., Treasury Inflation-Protected Securities(TIPS) in the US). As studied in [2] and [1], the index bond helps for long-term investment. [5] and [8] also investigate the roles of index bond in a life-insurance decision. In this paper, we incorporate the inflation risk to the sophisticated agent's portfolio selection problem. In finite horizon, the integro-differential equation for the solution is derived. Moreover, we provide the explicit solution when the time horizon is infinite.

The paper is organized as follows. Section 2 introduces the preference of the sophisticated agent and wealth dynamics in the presence of inflation risk. Section 3 provides the value function and its HJB equation. The analytic solutions are given in Section 4. Section 5 concludes.

2. Model setup

Financial Market. We consider continuous-time economy in the presence of inflation risk. The financial market contains a riskless asset, risky asset, and index bond. A risk-free asset, R_t, has nominal interest rate $R > 0$ and risky asset, S_t, follows geometric Brownian motion with constant coefficients μ_s and σ_s, which evolves according to

$$dS(t)/S(t) = \mu_s dt + \sigma_s dW(t),$$

where $W(t)$ is a standard Brownian motion under the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We suppose the agent faces an inflation risk and the price process, $P(t)$, is given by

$$dP(t)/P(t) = \mu_p dt + \sigma_p(\rho dW(t) + \sqrt{1-\rho^2} dZ(t)),$$

where μ_p and σ_p are constant drift and volatility of price process, and $Z(t)$ is another Brownian motion under $(\Omega, \mathcal{F}, \mathbb{P})$ which is independent of $W(t)$. The correlation between a price process and a risky asset is given by $\rho \in [-1, 1]$. Then the dynamics of index bond, $dI(t)$, is given by

$$dI(t)/I(t) = r + dP(t)/P(t) = (r + \mu_p)dt + \sigma_p(\rho dW(t) + \sqrt{1-\rho^2} dZ(t)),$$

where $r > 0$ is a constant real interest rate. Thanks to the index bond, the inflation risk is hedgeable in the economy and the financial market is complete.

Now, we denote by $\bar{X}(t)$ the agent's nominal wealth dynamics, $\bar{c}(t)$ the nominal consumption rate at time t. We also denote by $\pi_0(t)$, $\pi_1(t)$, and $\pi_2(t)$ the portfolio ratios for riskless asset, risky asset, and index

bond, respectively. For feasibility, we assume that $\bar{c}(t)$, $\pi_i(t), i = 0, 1, 2$, are $\mathcal{F}_t$-progressively measurable and satisfy the following conditions:

$$\int_0^T \bar{c}(t)dt < \infty \text{ a.s.} \qquad \int_0^T \pi_i^2(t)dt < \infty \text{ a.s.}$$

Then the agent's nominal liquid wealth, $\bar{X}(t)$ is unfolded by

$$d\bar{X}(t) = \pi_0(t)\bar{X}(t)dB_t/B_t + \pi_1(t)\bar{X}_t dI_t/I_t + \pi_2(t)\bar{X}_t dS_t/S_t - \bar{c}(t)dt \tag{2.1}$$

Preference. We suppose the time-inconsistent agent has a constant relative risk aversion(CRRA) utility defined by

$$u(c) = \frac{1}{1-\gamma}c^{1-\gamma}, \quad \gamma > 0,\ \gamma \neq 1,$$

where γ represents the level of risk averseness. To incorporate time-inconsistent preferences, we consider a hyperbolic discounting. As illustrated by [9], the agent behaves differently in her future life periods so future selves has different discount factors as follows.

$$D(t,s) = \begin{cases} e^{-\delta(s-t)}, & s \in (t, \tau_t), \\ \beta e^{-\delta(s-t)}, & s \in (\tau_t, \infty), \end{cases} \tag{2.2}$$

where δ is the standard subjective discount factor. β is the degree of utility loss between current self and future selves due to the time-inconsistent preference. So it represents propensity for instantaneous satisfaction which satisfies $0 < \beta \leq 1$. τ_t is a random time to switch into the future selves with different discounting, and it follows a Poisson process with a constant intensity $\lambda > 0$. Thus, for given $s > t$, the probability for preference change is defined by $\mathbb{P}(\tau_t > s) = 1 - e^{-\lambda(s-t)}$. Notice that for $\lambda = 0$ or $\beta = 1$, the time-preference is exactly same as the time consistent preference with the subjective discounting rate δ.

We will consider three kinds of agent: one time-consistent agent and two time-inconsistent agents who are naïve and sophisticated. The naïve agent believes his future selves are time-consistent but right after preference change, she recognizes her mistake and changes her optimal policies. While, the sophisticated agent takes into account the future selves' optimal decision even with preference change.

3. Problem and HJB equation

The time-inconsistent agent wants to maximize the following lifetime expected utility:

$$\mathbb{E}_t\left[\int_t^T D(t,s)u(c(s))ds + D(t,T)B(T,X(T))\right],$$

where $\mathbb{E}_t[\cdot]$ is a conditional expectation at time t and $D(t,s), s \geq t$ is a hyperbolic discounting factor given in (2.2). $B(T,X(T))$ is a bequest function given at final time T. We assume that $\tau_t < T$ so that the time-inconsistent agent's value function is given by

(3.1)

$$V(t,X(t)) = \max_{c(u),\Pi(u)} \mathbb{E}_t\left[\int_t^{t+\tau_t} e^{-\delta(s-t)}u(c(s))ds + \beta\int_{t+\tau_t}^T e^{-\delta(s-t)}u(c(s))ds\right.$$

$$\left. +\beta e^{-\delta(T-t)}B(T,X(T))\right], \tag{3.2}$$

where $\Pi(u) = (\pi_0(u), \pi_1(u), \pi_2(u))$. Notice that $X(T)$ is the wealth at final time T. If we regard the bequest function as the value function of the time consistent agent with a subjective discount factor δ and an initial wealth $X(T)$, then the value function in (3.1) is exactly same as the following problem with an infinite time horizon:

(3.3)

$$\tilde{V}(X(t)) = \max_{c(u),\Pi(u)} \mathbb{E}_t\left[\int_t^{t+\tau_t} e^{-\delta(s-t)}u(c(s))ds + \beta\int_{t+\tau_t}^\infty e^{-\delta(s-t)}u(c(s))ds\right].$$

We suppose the time T is a planning fixed time and the bequest function is defined by $B(T,X(T)) = X(T)^{1-\gamma}/M^\gamma(1-\gamma)$, where M is a positive constant.

Now, we transform the nominal wealth dynamics in (2.1) into the real terms. Let us denote by $X(t)$ the real wealth process, $\bar{X}(t)/P(t)$, and $c(t)$ by the real consumption rate, $c(t)/P(t)$. By Itô's formula, we obtain the real wealth dynamics in the following lemma.

Lemma 3.1. *The inflation-adjusted wealth process $X(t)$ has its dynamics as*

$$dX(t) = (rX(t) - \pi_0(t)\mu_0X(t) + \pi_2(t)\mu_2X(t) - c(t))dt \tag{3.4}$$

$$+ (\pi_2(t)X(t)(\sigma_s - \rho\sigma_p) - \pi_0(t)X(t)\rho\sigma_p)dW(t) \tag{3.5}$$

$$- \sqrt{1-\rho^2}\sigma_p(\pi_2(t) + \pi_0(t))X(t)dZ(t),$$

where $\mu_0 \equiv r + \mu_p - R - \sigma_p^2$ and $\mu_2 \equiv \mu_s - r - \rho\sigma_s\sigma_p - \mu_p + \sigma_p^2$.

Then, the sophisticated agent's problem is to find the value function defined in (3.1) subject to the budget constraint (3.4). For simplicity, let us rewrite the wealth dynamics as follows.

$$\begin{aligned} dX(t) = {} & f(t, c(t), \pi_0(t), \pi_2(t))dt + g_1(\pi_0(t), \pi_2(t))dW(t) \\ & + g_2(X(t), \pi_0(t), \pi_2(t))dZ(t), \end{aligned}$$

where

$$\begin{aligned} & f(t, c(t), \pi_0(t), \pi_1(t), \pi_2(t)) = rX(t) - \pi_0(t)\mu_0 X(t) + \pi_2(t)\mu_2 X(t) - c(t) \\ & g_1(\pi_0(t), \pi_2(t)) = \pi_2(t)X(t)(\sigma_s - \rho\sigma_p) - \pi_0(t)X(t)\rho\sigma_p \\ & g_2(X(t), \pi_0(t), \pi_2(t)) = -\sqrt{1-\rho^2}\sigma_p(\pi_2(t) + \pi_0(t))X(t). \end{aligned}$$

Thus, we can derive the HJB(Hamilton-Jacobi-Bellman) equations for the sophisticated agents. For better understanding, let us first consider the HJB equations for time-consistent and naïve agent first. From the standard argument of dynamic programming principle, the value function of a time consistent agent with budget constraint (3.4), $V^c(t, X(t))$, should satisfy the following HJB equation:

$$\delta V^c(t, X(t)) - V_t^c = \max_{c(t),\Pi(t)} \left\{ u(c(t)) + V_x^c f + \frac{1}{2} V_{xx}^c (g_1^2 + g_2^2) \right\}, \tag{3.6}$$

where V_t, V_x, and V_{xx} represent the partial derivatives. We can obtain the solution to the HJB equation (3.6) by using the conjectured form: $V^c(t, X(t)) = \frac{b(t)}{1-\gamma} X(t)^{1-\gamma}$.

Let us denote by $V^N(t, X(t))$ the value function of naïve agent. The naïve agent believes that her future selves are time-consistent so there would be no preference change in the future. Thus, the continuation value at the time of preference change is proportional to the time consistent agent's value function $V^c(t, X(t))$ and it becomes $\beta V^c(t, X(t))$. Since the intensity of preference change is given by λ, the HJB equation for a naïve agent is obtained from

$$\begin{aligned} & \delta V^N(t, X(t)) - V_t^N \\ & \quad = \max_{c(t),\Pi(t)} \left\{ u(c(t)) + V_x^N f + \frac{1}{2} V_{xx}^N (g_1^2 + g_2^2) + \lambda(\beta V^c - V^N) \right\}, \end{aligned}$$

with boundary condition $V^N(T, X(T)) = B(T, X(T))$.

Contrast to the case of naïve agent, the sophisticated agent takes into account the preference change of her future selves so the optimal consumption and portfolios become time-consistent. Due to the difference of the values between current and future selves, however, it is difficult

to derive the HJB equation in the standard way. We obtain the HJB equation for the sophisticated agent in the following lemma.

PROPOSITION 3.2. *The sophisticated agent's value function defined in* (3.1) *should satisfy the following HJB equation:*

(3.7)
$$\delta V^S(t,X(t)) - V_t^S + K(t,X(T)) = \max_{c(t),\Pi(t)} \left\{ u(c(t)) + V_x^S f + \frac{1}{2} V_{xx}^S (g_1^2 + g_2^2) \right\},$$

with $V^S(T,X(T)) = B(T,X(T))$. *The function* $K(t,X(T))$ *is given by*

$$K(t,X(t)) = \lambda(1-\beta)\mathbb{E}\left[\int_t^T e^{-(\lambda+\delta)(s-t)} u(c^*(s))ds\right], \tag{3.8}$$

where $c^*(s)$ *is the time-consistent optimal consumption rate.*

Proof. To derive the HJB equation of the sophisticated agent in finite horizon, we borrow the idea of [12]. All the procedures are same except for the wealth dynamics. So we briefly explain the derivation of the HJB equation in finite horizon as follows. We consider the discrete time version over $[0,T]$ with n steps and width length ϵ. Then, the wealth dynamics becomes $X(t+\epsilon) = X(t) + f(t)\epsilon + g_1(t)(W(t+\epsilon) - W(t)) + g_2(t)(Z(t+\epsilon) - Z(t))$. Let us denote $T = n\epsilon, t = j\epsilon, X(j\epsilon) = X_j, c(j\epsilon) = c_j, V(j\epsilon, w_j) = V_j$, and $(\pi_0(j\epsilon), \pi_1(j\epsilon), \pi_2(j\epsilon)) = \Pi_j$.

Now we consider a backward induction. At final time T, we have $V_n = B(T,X(T))$ and at time $(n-1)\epsilon$, the discounted value of V_n would be $\beta e^{-\delta\epsilon} V_n$ so, the value function V_{n-1} is defined by

$$V_{n-1} = \max_{c_{n-1},\Pi_{n-1}} \mathbb{E}\left[u(c_{n-1})\epsilon + \beta e^{-\delta\epsilon} V_n\right].$$

Let us denote by $\bar{u}_{n-1}$ the utility optimally chosen at $n-1$, i.e., $\bar{u}_{n-1}((n-1)\epsilon, X_{n-1}) = u_{n-1}((n-1)\epsilon, c_{n-1}^*, \Pi_{n-1}^*)$. Then the value function of the self at time $(n-2)\epsilon$ is given by

$$V_{n-2} = \max_{c_{n-2},\Pi_{n-2}} \mathbb{E}\left[u(c_{n-2})\epsilon + D(0,\epsilon)\bar{u}_{n-1}\epsilon + \beta e^{-2\delta\epsilon} V_n\right].$$

Consequently, $\bar{u}_j(j\epsilon, X_j) = u_j(j\epsilon, c_j^*, \Pi_j^*)$, $j = 0,1,2,\cdots n-1$, and the value function V_j should satisfy

$$V_j = \max_{c_j,\Pi_j} \mathbb{E}\left[u(c_j)\epsilon + \Sigma_{i=1}^{n-j-1} D(0,i\epsilon)\bar{u}_{j+i}\epsilon + \beta e^{-(n-j)\delta\epsilon} V_n\right],$$

with $V_{j+1} = \mathbb{E}\left[\Sigma_{i=1}^{n-j-2} D(0,i\epsilon)\bar{u}_{j+i+1}\epsilon + \beta e^{-(n-j-1)\delta\epsilon} V_n\right]$.

Since $e^{\delta\epsilon} \approx 1 + \delta\epsilon + o(\epsilon)$, we have

$$(1+\delta\epsilon)V_j = \max_{c_j,\Pi_j} \mathbb{E}\left[(1+\delta\epsilon)u(c_j)\epsilon + (1+\delta\epsilon)\Sigma_{i=1}^{n-j-1} D(0,i\epsilon)\bar{u}_{j+i}\epsilon \right.$$
$$\left. + \beta e^{-(n-j-1)\delta\epsilon} V_n \right].$$

By substracting V_{j+1} and divided by ϵ on both sides, we have

$$\frac{V_j - V_{j+1}}{\epsilon} + \delta V_j = \max_{c_j,\Pi_j} \mathbb{E}\left[(1+\delta\epsilon)u(c_j)\right.$$
$$\left. + \Sigma_{i=1}^{n-j-1}\left(e^{\delta\epsilon}D(0,i\epsilon) - D(0,(i-1)\epsilon)\right)\bar{u}_{j+i} + \frac{o(\epsilon)}{\epsilon}\right].$$

The second term in the right-hand side can be calculated by

$$\begin{aligned}
&\mathbb{E}\left[\Sigma_{i=1}^{n-j-1}\left(e^{\delta\epsilon}D(0,i\epsilon) - D(0,(i-1)\epsilon)\right)\bar{u}_{j+i}\right] \\
&= \Sigma_{i=1}^{n-j-1}\left(e^{\delta\epsilon}e^{-\delta i\epsilon}((1-\beta)e^{-\lambda i\epsilon} + \beta)\right. \\
&\qquad \left. - e^{-\delta(i-1)\epsilon}((1-\beta)e^{-\lambda(i-1)\epsilon} + \beta)\right) \cdot \mathbb{E}[\bar{u}_{j+i}] \\
&= \Sigma_{i=1}^{n-j-1} e^{-\delta(i-1)\epsilon}(1-\beta)e^{-\lambda i\epsilon}(1-e^{-\lambda\epsilon}) \cdot \mathbb{E}[\bar{u}_{j+i}] \\
&= -\Sigma_{i=1}^{n-j-1}\lambda(1-\beta)e^{-(\delta+\lambda)i\epsilon}\epsilon e^{\delta\epsilon} \cdot \mathbb{E}[\bar{u}_{j+i}] \\
&= -\Sigma_{i=1}^{n-j-1}\lambda(1-\beta)e^{-(\delta+\lambda)i\epsilon} \cdot \mathbb{E}[\bar{u}_{j+i}]\epsilon,
\end{aligned}$$

where the second equality holds because $\mathbb{E}[D(0,i\epsilon)] = e^{-\delta i\epsilon}(e^{-\lambda i\epsilon} \cdot 1 + (1-e^{-\lambda i\epsilon})\beta)$. Finally, since $\mathbb{E}\left[\frac{V_{j+1}-V_j}{\epsilon}\right] = V_t + V_x f + \frac{1}{2}(g_1^2 + g_2^2)V_{xx}$ and by letting $\epsilon \to 0$, we obtain the HJB equation in (3.7). □

Note that the HJB equation for the sophisticated agent can be easily extended to the problem with an infinite time horizon, where the value function is independent of time t. We summarize the HJB equation in the case of the problem with an infinite time horizon in the next corollary.

Corollary 3.3. *The HJB equation of sophisticated agent's problem in infinite-time horizon is obtained from*

(3.9)
$$\delta\tilde{V}^S(X(t)) + \tilde{K}(X(t)) = \max_{c_t,\Pi(t)}\left\{u(c(t)) + \tilde{V}_x^S f + \frac{1}{2}\tilde{V}_{xx}^S(g_1^2 + g_2^2)\right\},$$

where

$$\tilde{K}(X(t)) = \lambda(1-\beta)\mathbb{E}\left[\int_t^\infty e^{-(\lambda+\delta)(s-t)}u(c^{**}(s))ds\right],$$

and $c^{**}(s)$ is the time-consistent optimal consumption rate which satisfies (3.9).

4. Optimal policies

We derive the optimal consumption and investment when a CRRA utility function is imposed, i.e., $u(c(t)) = c(t)^{1-\gamma}/(1-\gamma)$. Then the FOCs(first order conditions) for $c^*(t), \pi_0^*(t), \pi_2^*(t)$ in the HJB equation (3.7) are determined by

$$\begin{aligned}
&c^*(t)^{-\gamma} - V_x = 0,\\
&V_x\mu_0 X(t) + V_{xx}X^2(t)(\pi_2^*\sigma_s - \rho\sigma_p(\pi_2^* + \pi_0^*)(-\rho\sigma_p)\\
&\qquad\qquad + (1-\rho^2)\sigma_p^2(\pi_2^* + \pi_0^*)) = 0,\\
&V_x\mu_2 X(t) + V_{xx}X^2(t)(\pi_2^*\sigma_s - \rho\sigma_p(\pi_2^* + \pi_0^*))(\sigma_s - \rho\sigma_p)\\
&\qquad\qquad + (1-\rho^2)\sigma_p^2(\pi_2^* + \pi_0^*) = 0.
\end{aligned}$$

To obtain the closed-form solution, we conjecture the solution as follows: $V(t, X(t)) = h(t)\frac{X(t)^{1-\gamma}}{1-\gamma}$. Then the partial derivatives and boundary condition are given by

$$V_t = h'(t)\frac{X(t)^{1-\gamma}}{1-\gamma},\ V_x = h(t)X(t)^{-\gamma},\ V_{xx} = -\gamma h(t)X(t)^{-\gamma-1},\ h(T) = 1/M^\gamma.$$

By substituting the partial derivatives into FOCs, we obtain the optimal consumption rate from $c^*(t) = h(t)^{-\frac{1}{\gamma}}X(t)$ and portfolios from

$$\begin{aligned}
\pi_0^*(t) &= \frac{1}{\gamma\sigma_s\sigma_p(1-\rho^2)}\left(\frac{\rho\sigma_s - \sigma_p}{\sigma_s}P_s - \frac{\sigma_s - \rho\sigma_p}{\sigma_p}P_I\right),\\
\pi_2^*(t) &= \frac{1}{\gamma\sigma_s(1-\rho^2)}\left(\frac{P_s}{\sigma_s} - \frac{\rho P_I}{\sigma_p}\right),
\end{aligned}$$

where $P_s \equiv \mu_s - R - \rho\sigma_s\sigma_p$ and $P_I \equiv \mu_p + r - R - \sigma_p^2$. Notice that P_s and P_I represent risk-adjusted excess premiums of risky asset and index bond respectively. Since $\pi_0^*(t) + \pi_1^*(t) + \pi_2^*(t) = 1$, we also have $\pi_1^*(t) = 1 + \frac{1}{\gamma\sigma_p(1-\rho^2)}\left(\frac{P_I}{\sigma_p} - \frac{\rho P_s}{\sigma_s}\right)$. As we can see, the time-inconsistency does not affect the optimal portfolio ratios at all.

On the other hand, to obtain the optimal consumption rate, it is necessary to calculate the function $K(t, X(t))$ in Proposition 3.2. We summarize the results in the following proposition.

PROPOSITION 4.1. *The optimal consumption rate of the sophisticated agent is given by*

$$c^*(t) = h(t)^{-\frac{1}{\gamma}} X(t),$$

where $h(t)$ satisfies the following integro-differential equation

$$\begin{aligned} h'(t) = {} & h(t)(\delta - (1-\gamma)(r+A)) - \gamma h(t)^{-\frac{1-\gamma}{\gamma}} \\ & + \lambda(1-\beta)\int_t^T e^{-\alpha(s-t)-(1-\gamma)\int_t^s h(u)du} h(s)^{-\frac{1-\gamma}{\gamma}} ds, \end{aligned}$$

with $h(T) = 1/M^\gamma$. The coefficients are determined by

$$\begin{aligned} A \equiv {} & \frac{P_s - \mu_p + \sigma_p^2}{\gamma\sigma_s(1-\rho^2)}\left(\frac{P_s}{\sigma_s} - \frac{\rho P_I}{\sigma_p}\right) \\ & - \frac{P_I}{\gamma\sigma_s\sigma_p(1-\rho^2)}\left(\frac{\rho\sigma_s - \sigma_p}{\sigma_s}P_s - \frac{\sigma_s - \rho\sigma_p}{\sigma_p}P_I\right) - \frac{1}{2}\gamma(\sigma_1^2 + \sigma_2^2), \\ \alpha = {} & \lambda + \delta - (1-\gamma)(r+A), \\ \sigma_1 = {} & \frac{1}{\gamma(1-\rho^2)}\left(\frac{P_s}{\sigma_s} - \frac{\rho P_I}{\sigma_p}\right) - \frac{\rho}{\gamma\sigma_s(1-\rho^2)}\left(\frac{\rho\sigma_s - \sigma_p}{\sigma_s}P_s - \frac{\sigma_s - \rho\sigma_p}{\sigma_p}P_I\right), \\ \sigma_2 = {} & -\frac{\sqrt{1-\rho^2}\sigma_p}{\gamma\sigma_s(1-\rho^2)}\left(\frac{\rho\sigma_s - \sigma_p}{\sigma_s}P_s - \frac{\sigma_s - \rho\sigma_p}{\sigma_p}P_I\right). \end{aligned}$$

Proof. For given A, σ_1 and σ_2 defined above, the wealth dynamics can be rewritten as

$$\begin{aligned} dX(t) = {} & X(t)\left(r - h(t)^{-\frac{1}{\gamma}} + A + \frac{1}{2}\gamma(\sigma_1^2 + \sigma_2^2)\right)dt \\ & + X(t)\sigma_1 dW(t) + X(t)\sigma_2 dZ(t), \end{aligned} \tag{4.1}$$

By substituting the partial derivatives into the HJB equation (3.7) and this wealth dynamics, we have

$$\begin{aligned} & \frac{X(t)^{1-\gamma}}{1-\gamma}(\delta h(t) - h'(t)) + K(t, X(t)) \\ & \quad = \frac{X(t)^{1-\gamma}}{1-\gamma}\left\{h(t)^{-\frac{1-\gamma}{\gamma}} + (1-\gamma)h(t)\left(r - h(t)^{-\frac{1}{\gamma}} + A\right)\right\}, \end{aligned} \tag{4.2}$$

with $h(T) = 1/M^\gamma$.

Now it is suffice to find the function $K(t, X(t))$ in (3.8). The wealth dynamics in (4.1) represents the time-consistent wealth trajectory of the sophisticated agent and it follows geometric Brownian motion. Thus, we

have

$$d(X(t)^{1-\gamma}) = (1-\gamma)X(t)^{-\gamma}dX(t) - \frac{1}{2}\gamma(1-\gamma)X(t)^{-\gamma-1}(dX(t))^2$$
$$= (1-\gamma)X(t)^{1-\gamma}\left((r-h(t)^{-\frac{1}{\gamma}}+A)dt + \sigma_1 dW(t) + \sigma_2 dZ(t)\right),$$

which implies that for $s > t$,

$$\mathbb{E}_t\left[X(s)^{1-\gamma}\right] = X(t)^{1-\gamma}\mathbb{E}_t\left[\exp\left\{(1-\gamma)\int_t^s (r-h(u)^{-\frac{1}{\gamma}}+A)du\right\}\right].$$

Therefore the function $K(t, X(t))$ is given by

$$K(t,X(t)) = \lambda(1-\beta)\mathbb{E}_t\left[\int_t^T e^{-(\lambda+\delta)(s-t)}\frac{h(s)^{-\frac{1-\gamma}{\gamma}}X(s)^{1-\gamma}}{1-\gamma}ds\right]$$
$$= \lambda(1-\beta)\int_t^T e^{-(\lambda+\delta)(s-t)}\frac{h(s)^{-\frac{1-\gamma}{\gamma}}\mathbb{E}_t[X(s)^{1-\gamma}]}{1-\gamma}ds$$
$$= \frac{\lambda(1-\beta)X(t)^{1-\gamma}}{1-\gamma}\int_t^T e^{-\alpha(s-t)-(1-\gamma)\int_t^s h(u)^{-\frac{1}{\gamma}}du}h(s)^{-\frac{1-\gamma}{\gamma}}ds,$$

where α is given in Proposition 3.2. If we plugin $K(t, X(t))$ to the HJB equation (4.2), then the fully nonlinear integro-differential equation for $h(t)$ is derived.

□

In an infinite time horizon ($T \to \infty$), we further assume that the value function in (3.3) satisfies the transversality condition as follows.

$$\lim_{T\to\infty}\mathbb{E}[e^{-\delta t}\tilde{V}^S(X(t))] = 0.$$

Then the conjectured value function is time independent and supposed to have the following form: $\tilde{V}^S(X(t)) = \frac{X(t)^{1-\gamma}}{m^\gamma(1-\gamma)}$. The optimal consumption rate is $m^*X(t)$ for a certain constant m^*, and the portfolio ratios are exactly same as those in a finite horizon case. The function $\tilde{K}(X(t))$ can also be obtained from

$$\tilde{K}(X(t)) = \frac{\lambda(1-\beta)X(t)^{1-\gamma}}{1-\gamma}\int_t^\infty e^{-((\lambda+\delta)-(1-\gamma)(r-m^*+A))(s-t)}m^{*1-\gamma}ds.$$

We summarize the optimal consumption rate in the case of infinite time horizon.

Corollary 4.2. *When $\gamma \neq 1, \lambda > 0, 0 < \beta < 1$ and $T \to \infty$, the optimal consumption rate of the sophisticated agent is given by*

$$c^*(t) = m^*X(t),$$

where m^* satisfies the following algebraic equation

$$\delta - (1-\gamma)(r+A) + \frac{\lambda(1-\beta)m^*}{\lambda+\delta-(1-r)(r-m^*+A)} = \gamma m^*.$$

5. Conclusion

We investigates the optimal consumption and investment problem of the time-inconsistent agent in the presence of inflation risk. We derive the semi-analytic solution in the sense that the solution should satisfy an integro-differential equation. We also extend the result into the problem in an infinite horizon, and obtain the closed-form solution. For the sophisticated agent, the inflation risk plays an important role in determining the consumption rate.

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Department of Economics and Finance
The University of Suwon
Hwaseong 18323, Republic of Korea
E-mail: byunghwalim@suwon.ac.kr