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A FINITE DIFFERENCE/FINITE VOLUME METHOD FOR SOLVING THE FRACTIONAL DIFFUSION WAVE EQUATION

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Abstract. In this paper, we present and analyze a fully discrete numerical method for solving the time-fractional diffusion wave equation: $\partial_t^\beta u - \mathrm{div}(a\nabla u) = f$, $1 < \beta < 2$. We first construct a difference formula to approximate $\partial_t^\beta u$ by using an interpolation of derivative type. The truncation error of this formula is of $O(\triangle t^{2+\delta-\beta})$-order if function $u(t) \in C^{2,\delta}[0,T]$ where $0 \le \delta \le 1$ is the Hölder continuity index. This error order can come up to $O(\triangle t^{3-\beta})$ if $u(t) \in C^3[0,T]$. Then, in combinination with the linear finite volume discretization on spatial domain, we give a fully discrete scheme for the fractional wave equation. We prove that the fully discrete scheme is unconditionally stable and the discrete solution admits the optimal error estimates in the H^1-norm and L_2-norm, respectively. Numerical examples are provided to verify the effectiveness of the proposed numerical method.

1. Introduction

Fractional partial differential equations provide a nature framework for the study of a variety physics models related to nonlocality and spatial heterogeneity, see e.g., [1, 2, 8, 16] and the references therein. At present, many numerical methods have been proposed for solving time-fractional diffusion and diffusion wave equations. These numerical methods are basically to combine finite difference discretization for time fractional derivative with various types of spatial discretization methods, for example, the finite difference method [4,13,15,19–22], finite element method [5–7,12,17,23,26], finite volume method [25] and spectral method [10], collocation method [9], wavelet method [14,18], and so on. However, few finite volume methods are presented for the fractional diffusion wave equations.

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In this paper, we present and analize a finite difference/finite volume method for solving the fractional diffusion wave equation:

$$\partial_t^\beta u - \mathrm{div}(a(x)\nabla u) = f(t,x),\ 1 < \beta < 2. \tag{1.1}$$

We first construct a difference formula to discretize the time-fractional derivative $\partial_t^\beta u(t)$ with $1 < \beta < 2$. This difference formula is established by using an interpolation of derivative type to approximate the integrand $u''(t)$. We show that the truncation error of this formula is of $O(\triangle t^{2+\delta-\beta})$-order if function $u(t) \in C^{2,\delta}[0,T]$ where $0 \le \delta \le 1$ is the Hölder continuity index. It is well known that for the difference formula discretizing $\partial_t^\beta u(t)$, the truncation error is of $O(\triangle t^{3-\beta})$-order if $u \in C^3[0,T]$. Noting that when $u \in C^3[0,T] \subset C^{2,1}[0,T]$, our error order also reaches $O(\triangle t^{3-\beta})$. So our difference formula has a more delicate error boundness for function $u(t)$ with lower smoothness. Then, we further consider the spatial discretization by using the linear finite volume method on space domain. Thus, a fully discrete numerical scheme is presented to solve the fractional wave equation (1.1). We prove that this fully discrete scheme is unconditionally stable and the discrete solution admits the optimal error estimates in the H^1-norm and L_2-norm, respectively.

This paper is organized as follows. In Section 2, we establish the difference formula and give its truncation error bound. In Section 3, we propose the fully discrete finite difference/finite volume scheme and prove the unconditional stability. Section 4 is contributed to the error analysis. In Section 5, numerical experiments are provided to test the effictiveness of the proposed difference formula and fully discrete method.

Throughout this paper, for a non-negative integer m, we adopt the notation $H^m(\Omega)$ to denote the usual Sobolev space on domain Ω equipped with the norm $\|\cdot\|_m$. The notations $(\cdot,\cdot)$ and $\|\cdot\|$ denote the inner product and norm in the L_2 space, respectively. We use the letter C to represent a generic positive constant, independent of the mesh sizes $\triangle t$ and h.

2. The difference formula and its error bound

In this section, we establish the difference formula to approximate the fractional derivative $\partial_t^\beta u$ and give the rigorous error bound for function $u(t)$ with limited smoothness.

For $1 < \beta < 2$, the Caputo type fractional derivative of order β with respect to t is as follows

$$\partial_t^\beta u(t) = \frac{1}{\Gamma(2-\beta)} \int_0^t (t-\tau)^{1-\beta} u''(\tau)d\tau,\ 0 < t \le T, \tag{2.1}$$

where $\Gamma(\cdot)$ denotes the Gamma function.

Let us consider the discretization of $\partial_t^\beta u(t)$. Let $0 = t_0 < t_1 < \cdots < t_N = T$ be an equidistant partition of time interval $[0,T]$ with step size $\triangle t = T/N$ for some positive integer N. At node t_n, we have from (2.1) that

$$\partial_t^\beta u(t_n) = \frac{1}{\Gamma(2-\beta)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - \tau)^{1-\beta} u''(\tau) d\tau. \tag{2.2}$$

For a mesh function w^n on node set $\{t_n\}$, we introduce the notations:

$$\delta_t w^n = \frac{1}{\triangle t}(w^n - w^{n-1}), \quad w^{n-\frac{1}{2}} = \frac{1}{2}(w^n + w^{n-1}),$$

and set $w^n = w(t_n)$ if $w(t)$ is a continuous function on $[0, T]$. Also we introduce the piecewise quadratic polynomial function which is a special approximation to $u(t)$:

$$H_{2,k}u(t) = \frac{(t-t_{k-1})^2}{2\triangle t} u'(t_k) - \frac{(t_k - t)^2}{2\triangle t} u'(t_{k-1}),\ t \in (t_{k-1}, t_k),\ 1 \le k \le N. \tag{2.3}$$

Obviously,

$$H'_{2,k}u(t) = \frac{t - t_{k-1}}{\triangle t} u'(t_k) + \frac{t_k - t}{\triangle t} u'(t_{k-1}),\ t \in (t_{k-1}, t_k),\ k = 1, \ldots, N, \tag{2.4}$$

$$H''_{2,k}u(t) = \delta_t^k u'(t_k) = \frac{u'(t_k) - u'(t_{k-1})}{\triangle t},\ t \in (t_{k-1}, t_k),\ k = 1, \ldots, N. \tag{2.5}$$

Replacing $u(\tau)$ by $H_{2,k}u(\tau)$ in (2.2), we obtain from (2.5) that

$$\partial_t^\beta u(t_n) = \frac{1}{\Gamma(2-\beta)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - \tau)^{1-\beta} \delta_t u'(t_k) d\tau + R_1^n(u), \tag{2.6}$$

where the error function

$$R_1^n(u) = \frac{1}{\Gamma(2-\beta)} \sum_{k=1}^{n} \int_{t_{k-1}}^{t_k} (t_n - \tau)^{1-\beta} (u''(\tau) - \delta_t u'(t_k)) d\tau. \tag{2.7}$$

Set

$$b_k = (k+1)^{2-\beta} - k^{2-\beta},\ k = 0, 1, \ldots, \quad \Gamma_\triangle^\beta = \Gamma(3-\beta)\triangle t^{\beta-1}. \tag{2.8}$$

Since

$$\int_{t_{k-1}}^{t_k} (t_n - \tau)^{1-\beta} d\tau = \frac{1}{2-\beta}\big((t_n - t_{k-1})^{2-\beta} - (t_n - t_k)^{2-\beta}\big) = \frac{\triangle t^{2-\beta}}{2-\beta}\, b_{n-k},$$

then, it follows from (2.6) that

$$\partial_t^\beta u(t_n) = \frac{1}{\Gamma_\triangle^\beta} \sum_{k=1}^{n} b_{n-k}\big(u'(t_k) - u'(t_{k-1})\big) + R_1^n(u). \tag{2.9}$$

We need to further discretize the derivative in (2.9). Using the summation by parts formula:

$$\sum_{k=1}^{n} v_k (w_k - w_{k-1}) = \sum_{k=1}^{n-1} (v_k - v_{k+1}) w_k + v_n w_n - v_1 w_0, \tag{2.10}$$

we obtain from (2.9)

$$\partial_t^\beta u(t_n) = \frac{1}{\Gamma_\triangle^\beta}\Big[\sum_{k=1}^{n-1}(b_{n-k} - b_{n-k-1})u'(t_k) + b_0 u'(t_n) - b_{n-1}u'(t_0)\Big] + R_1^n(u). \tag{2.11}$$

Here and afterwards, we consider the sum to be equal to zero if the upper summation index is less than the lower one. Now, let $w^n = u'(t_n)$, it follows from (2.11) that

$$\begin{aligned}\partial_t^\beta u^{n-\frac{1}{2}} &= \frac{\partial_t^\beta u(t_n) + \partial_t^\beta u(t_{n-1})}{2}\\ &= \frac{1}{\Gamma_\triangle^\beta}\Big[\frac{w^n + w^{n-1}}{2} + \sum_{k=1}^{n-1}\big(b_{n-k} - b_{n-k-1}\big)\frac{w^k + w^{k-1}}{2} - b_{n-1}w^0\Big]\\ &\quad + \frac{R_1^n(u) + R_1^{n-1}(u)}{2},\ w^n = u'(t_n),\ n = 1, 2, \ldots.\end{aligned} \tag{2.12}$$

Using the Taylor expansion:

$$u(t_k) = u(t_{k-1}) + \triangle t u'(t_{k-1}) + \int_{t_{k-1}}^{t_k} u''(s)(t_k - s)ds,$$

$$u(t_{k-1}) = u(t_k) - \triangle t u'(t_k) + \int_{t_k}^{t_{k-1}} u''(s)(t_{k-1} - s)ds,$$

we obtain

$$\frac{u'(t_k) + u'(t_{k-1})}{2} = \frac{u(t_k) - u(t_{k-1})}{\triangle t} + R_2^k(u), \tag{2.13}$$

where the error remainder

$$R_2^k(u) = \frac{1}{2\triangle t}\Big[\int_{t_{k-1}}^{t_k} u''(s)(s - t_{k-1})ds - \int_{t_{k-1}}^{t_k} u''(s)(t_k - s)ds\Big]. \tag{2.14}$$

Substituting (2.13) into (2.12), we derive the approximation to the fractional derivative $(\partial_t^\beta u(t_n) + \partial_t^\beta u(t_{n-1}))/2$ as follows

$$\partial_t^\beta u^{n-\frac{1}{2}} = \triangle_n^\beta u^{n-\frac{1}{2}} + r_n(u), \tag{2.15}$$

where the difference formula

$$\triangle_n^\beta u^{n-\frac{1}{2}} = \frac{1}{\Gamma_\triangle^\beta}\Big[\delta_t u^n + \sum_{k=1}^{n-1}\big(b_{n-k} - b_{n-k-1}\big)\delta_t u^k - b_{n-1}u'(0)\Big], \tag{2.16}$$

with $\Gamma_\triangle^\beta = \Gamma(3-\beta)\triangle t^{\beta-1}$ and the truncation error

$$r_n(u) = \frac{1}{\Gamma_\triangle^\beta}\Big[R_2^n(u) + \sum_{k=1}^{n-1}\big(b_{n-k} - b_{n-k-1}\big)R_2^k(u)\Big] + \frac{R_1^n(u) + R_1^{n-1}(u)}{2}. \tag{2.17}$$

Below we estimate the truncation error $r_n(u)$.

Let $H_{2,k}u$ be the piecewise quadratic polynomial given in (2.3) and error function $R_{H,k}(t) = u(t) - H_{2,k}u(t)$, $1 \le k \le N$. From (2.4) we see that $H'_{2,k}u$ is the linear interpolation of derivative function $u'(t)$ on $[t_{k-1}, t_k]$. Then, we have from the interpolation error formula of Newton type,

$$\begin{aligned} u'(t) &= H'_{2,k}u(t) + R'_{H,k}(t), \\ R'_{H,k}(t) &= (t - t_{k-1})(t - t_k)u'[t_{k-1}, t_k, t],\ t \in (t_{k-1}, t_k), \end{aligned} \tag{2.18}$$

where the two-order difference quotient

$$\begin{aligned} u'[t_{k-1}, t_k, t] &= \big(u'[t_k, t] - u'[t_{k-1}, t_k]\big)/(t - t_{k-1}), \\ u'[t_i, t_j] &= \big(u'(t_j) - u'(t_i)\big)/(t_j - t_i). \end{aligned}$$

We introduce the Hölder continuous function space with indexes $m \ge 0$ and $0 \le \delta \le 1$,

$$\begin{aligned} C^{0,\delta}[a,b] &= \{u(t) \in C^{(0)}(a,b) : |u|_{C^{0,\delta}[a,b]} < \infty\}, \\ C^{m,\delta}[a,b] &= \{u(t) \in C^{(m-1)}[a,b] \bigcap C^{(m)}(a,b) : |u|_{C^{m,\delta}[a,b]} < \infty\},\ m \ge 1, \end{aligned}$$

where the semi-norm

$$|u|_{C^{m,\delta}[a,b]} = \sup_{t_1,\, t_2 \in (a,b),\, t_1 \ne t_2} \frac{|u^{(m)}(t_1) - u^{(m)}(t_2)|}{|t_1 - t_2|^{\delta}}.$$

Lemma 2.1. *Let $u \in C^{2,\delta}[0,T]$ and error function $R_{H,k}(t) = u(t) - H_{2,k}u(t)$, $1 \le k \le N$. Then, it holds*

$$|R'_{H,k}(t)| \le \triangle t^{1+\delta}|u|_{C^{2,\delta}[t_{k-1},t_k]},\quad |R''_{H,k}(t)| \le \triangle t^{\delta}|u|_{C^{2,\delta}[t_{k-1},t_k]},\ t \in (t_{k-1}, t_k).$$

Proof. First, from (2.18) we have

$$\begin{aligned} |R'_{H,k}(t)| &= |(t - t_k)(u'[t_k, t] - u'[t_{k-1}, t_k])| \\ &\le \triangle t|u''(\xi_k) - u''(\eta_k)| \\ &\le \triangle t^{1+\delta}|u''(\xi_k) - u''(\eta_k)|/|\xi_k - \eta_k|^{\delta} \\ &\le \triangle t^{1+\delta}|u|_{C^{2,\delta}[t_{k-1},t_k]},\ t \in (t_{k-1}, t_k). \end{aligned}$$

Next, it follows from (2.6)

$$\begin{aligned} |R''_{H,k}(t)| &= \left| u''(t) - \frac{u(t_k) - u(t_{k-1})}{\triangle t} \right| \\ &= |u''(t) - u''(\xi_k)| \le \triangle t^{\delta}|u|_{C^{2,\delta}[t_{k-1},t_k]},\ t \in (t_{k-1}, t_k). \end{aligned}$$

The proof is completed. □

Lemma 2.2. *For $1 < \beta < 2$, series $b_k = (k+1)^{2-\beta} - k^{2-\beta}$ has the properties*:

$$1 = b_0 > b_1 > \cdots > b_{k-1} > b_k > (2-\beta)(k+1)^{1-\beta},\ k = 1, 2, \ldots. \tag{2.19}$$

Proof. Since

$$b_k = (k+1)^{2-\beta} - k^{2-\beta} = (2-\beta)\int_k^{k+1} t^{1-\beta}dt,$$

we have $b_0 = 1$,

$$\frac{2-\beta}{(k+1)^{\beta-1}} < b_k < \frac{2-\beta}{k^{\beta-1}}, \quad k = 1, 2, \ldots.$$

This implies the conclusion of Lemma 2.2. □

Now, we can give the error bound of the difference formula $\triangle_n^\beta u^{n-\frac{1}{2}}$.

Theorem 2.1. *For function $u(t) \in C^{2,\delta}[0, t_n]$, $0 \le \delta \le 1$, it holds*

$$|r_n(u)| = |\partial_t^\beta u^{n-\frac{1}{2}} - \triangle_n^\beta u^{n-\frac{1}{2}}| \le \frac{3\triangle t^{2+\delta-\beta}}{\Gamma(3-\beta)}|u|_{C^{2,\delta}[0,\, t_n]},\ 1 \le n \le N. \tag{2.20}$$

Proof. Let $r_n(u)$ be the truncation error shown in (2.17) in which R_1^n and R_2^n are given by (2.7) and (2.14), respectively. We first estimate R_1^n. From (2.7) and integration by parts, we obtain (noting that $R_k'(t_{k-1}) = R_k'(t_k) = 0$)

$$\begin{aligned}
&R_1^n(u)\\
&= \frac{1}{\Gamma(2-\beta)}\sum_{k=1}^{n}\int_{t_{k-1}}^{t_k}(t_n-\tau)^{1-\beta}R''_{H,k}(\tau)d\tau\\
&= \frac{1}{\Gamma(2-\beta)}\Big(\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}(t_n-\tau)^{1-\beta}R''_{H,k}(\tau)d\tau + \int_{t_{n-1}}^{t_n}(t_n-\tau)^{1-\beta}R''_{H,n}(\tau)d\tau\Big)\\
&= \frac{1}{\Gamma(2-\beta)}\Big(\sum_{k=1}^{n-1}\int_{t_{k-1}}^{t_k}(1-\beta)(t_n-\tau)^{-\beta}R'_{H,k}(\tau)d\tau + \int_{t_{n-1}}^{t_n}(t_n-\tau)^{1-\beta}R''_{H,n}(\tau)d\tau\Big).
\end{aligned}$$

Hence, it follows from Lemma 2.1 that

$$\begin{aligned}
|R_1^n| &\le \frac{1}{\Gamma(2-\beta)}\Big(\int_0^{t_{n-1}}(\beta-1)\triangle t^{1+\delta}(t_n-\tau)^{-\beta}d\tau\\
&\qquad + \triangle t^\delta\int_{t_{n-1}}^{t_n}(t_n-\tau)^{1-\beta}d\tau\Big)|u|_{C^{2,\delta}[0,\, t_n]}\\
&= \frac{\triangle t^\delta}{\Gamma(2-\beta)}\Big(\triangle t(\triangle t^{1-\beta} - t_n^{1-\beta}) + \frac{1}{2-\beta}\triangle t^{2-\beta}\Big)|u|_{C^{2,\delta}[0,\, t_n]}\\
&\le \frac{\triangle t^{2+\delta-\beta}}{\Gamma(2-\beta)}\left(1 + \frac{1}{2-\beta}\right)|u|_{C^{2,\delta}[0,\, t_n]} = \frac{3-\beta}{\Gamma(3-\beta)}\triangle t^{2+\delta-\beta}|u|_{C^{1,\delta}[0,\, t_n]}\,.
\end{aligned}$$

Next, we estimate the first term in (2.17). From (2.14) and the mean value theorem, we have

$$R_2^k(u) = \frac{1}{2\triangle t}\big[u''(\xi_k)\frac{\triangle t^2}{2} - u''(\eta_k)\frac{\triangle t^2}{2}\big] \le \frac{\triangle t^{1+\delta}}{4}|u|_{C^{2,\delta}[t_{k-1},\, t_k]},\ 1 \le k \le N.$$

Hence, by using Lemma 2.2, we obtain

$$\begin{aligned}&\frac{1}{\Gamma_{\triangle}^{\beta}}\left[R_2^n(u)+\sum_{k=1}^{n-1}\big(b_{n-k}-b_{n-k-1}\big)R_2^k(u)\right]\\ &\le \frac{1}{\Gamma_{\triangle}^{\beta}}\big(1+b_0-b_{n-1}\big)\frac{\triangle t^{1+\delta}}{4}|u|_{C^{2,\delta}[0,\,t_n]}\le \frac{\triangle t^{2+\delta-\beta}}{2\Gamma(3-\beta)}|u|_{C^{2,\delta}[0,\,t_n]}.\end{aligned}$$

Substituting this estimate into (2.17) and combining the estimate of R_1^n, the proof is completed. □

Note that space $C^2[0,T]\subset C^{2,0}[0,T]$ and $C^3[0,T]\subset C^{2,1}[0,T]$, then from Theorem 2.1, we also obtain the following result

$$|r_n(u)|\le \frac{3\triangle t^{m-\beta}}{\Gamma(3-\beta)}\|u\|_{C^m[0,T]},\ m=2,3\,.$$

We can further reduce the regularity requirement for function $u(t)$ in Theorem 2.1. In fact, from the proof of Theorem 2.1, we see that if $u(t)$ is piecewise smooth on $(0,t^*)\bigcup(t^*,T)$ and t^* is a mesh point, then the argument in Theorem 2.1 maintains to hold. Therefore, we have the following conclusion.

Corollary 2.1 *If function* $u(t)\in C^{2,\delta}[0,t^*]\bigcap C^{2,\delta}[t^*,T]$ *and* $t^*=t_k$ *is a mesh point of the difference formula, then it holds*

$$|r_n(u)|\le \frac{3\triangle t^{2+\delta-\beta}}{\Gamma(3-\beta)}\big(|u|_{C^{2,\delta}[0,t^*]}+|u|_{C^{2,\delta}[t^*,T]}\big),\ 1\le n\le N.$$

Remark 2.1. This difference formula presented in this paper is completely similar to that given in [20]. However, we give here a new error bound with respect to function $u(t)$ with lower smoothness.

3. The fully discrete method for the fractional wave problem

In this section, based on the difference formula given in Section 2, we present a fully discrete finite difference/finite volume method for solving the fractional diffusion wave equation and carry out the stability analysis.

Consider the initial-boundary value problem of fractional diffusion wave equation:

$$\text{(3.1)}\qquad \begin{cases}\partial_t^{\beta}u-\mathrm{div}(a(x)\nabla u)=f(t,x)\ \text{in}\ \ \Omega,\ 0<t\le T,\ 1<\beta<2,\\ u=0\ \ \text{on}\ \ \partial\Omega,\ 0<t\le T,\\ u(0,x)=\phi(x),\ u_t(0,x)=\psi(x),\ x\in\Omega,\end{cases}$$

where $\Omega\subset R^d$ $(2\le d\le 3)$ is a bounded domain with boundary $\partial\Omega$. As usual, we assume that there exist positive constants a_0 and a_1 such that $a_0\le a(x)\le a_1,\ x\in\Omega$.

We first introduce the finite volume discretization on spatial domain, see [25] for details. Let $T_h=\bigcup\{K\}$ be a regular triangulation of domain Ω and $T_h^*=\bigcup\{K_p^*\}$ be the accompanying dual partition. On triangulations T_h and

T_h^*, we introduce the piecewise linear trial function space S_h and the piecewise constant test function space S_h^*, respectively. Then we introduce the interpolation operator $\gamma_h : u_h \in S_h \to \gamma_h u_h \in S_h^*$ such that

$$\gamma_h u_h = \sum_{P \in N_h} u_h(P)\chi_p, \ \forall\, u_h \in S_h, \tag{3.2}$$

where χ_p is the characteristic function of the dual element K_P^* and N_h is the set of all mesh points of T_h.

Now we define the semi-discrete finite volume approximation of problem (3.1) by finding $u_h(t) : (0, T] \to S_h$ such that

$$\begin{cases} (\partial_t^\beta u_h, \gamma_h v_h) + a_h(u_h, \gamma_h v_h) = (f, \gamma_h v_h), \ \forall\, v_h \in S_h, \ 0 < t \le T, \\ u_h(0) \in S_h, \end{cases} \tag{3.3}$$

where the bilinear form

$$a_h(u, \gamma_h v_h) = -\sum_{K_P^* \in T_h^*} \int_{\partial K_P^*} n \cdot (a\nabla u)\gamma v_h ds, \ u \in H^1(\Omega), \ v_h \in S_h. \tag{3.4}$$

It is well known (see [24], for example) that for h small, there exist positive constants C_1 and C_2 such that for $u_h, \ v_h \in S_h$,

$$C_1\|\nabla u_h\|^2 \le a_h(u_h, \gamma_h u_h), \ |a_h(u_h, \gamma_h v_h)| \le C_2\|\nabla u_h\| \, \|\nabla v_h\| \,. \tag{3.5}$$

Based on the semi-discrete scheme (3.3), we define the fully discrete finite difference/finite volume approximation of the problem (3.1) by finding $u_h^n \in S_h$ such that

$$\begin{cases} (\triangle_n^\beta u_h^{n-\frac{1}{2}}, \gamma_h v_h) + a_h(u_h^{n-\frac{1}{2}}, \gamma_h v_h) = (f^{n-\frac{1}{2}}, \gamma_h v_h), \\ u_h^0 \in S_h, \forall\, v_h \in S_h, \ 0 < t \le T, \end{cases} \tag{3.6}$$

where $u_h^{n-\frac{1}{2}} = (u_h^n + u_h^{n-1})/2$ and the difference formula (see (2.16))

$$\triangle_n^\beta u^{n-\frac{1}{2}} = \frac{1}{\Gamma_\triangle^\beta}\left[\delta_t u^n + \sum_{k=1}^{n-1}\big(b_{n-k} - b_{n-k-1}\big)\delta_t u^k - b_{n-1}\psi\right], \tag{3.7}$$

where $\delta_t u^n = (u^n - u^{n-1})/\triangle t$ and $\Gamma_\triangle^\beta = \Gamma(3-\beta)\triangle t^{\beta-1}$. Using (3.7), discrete scheme (3.6) also can be written as

$$\begin{cases} \quad (\delta_t u_h^n, \gamma_h v_h) + \Gamma_\triangle^\beta a_h(u_h^{n-\frac{1}{2}}, \gamma_h v_h) \\ = \displaystyle\sum_{k=1}^{n-1}\big(b_{n-k-1} - b_{n-k}\big)(\delta_t u_h^k, \gamma_h v_h) + b_{n-1}(\psi, \gamma_h v_h) \\ \quad +\Gamma_\triangle^\beta(f^{n-\frac{1}{2}}, \gamma_h v_h), \ v_h \in S_h, \\ u_h^0 \in S_h, \ n = 1, 2, \ldots, N. \end{cases} \tag{3.8}$$

Below we discuss the stability. First we give a useful lemma.

Lemma 3.1 ([25]). *For $u_h, v_h \in S_h$, it holds*

$$(u_h, \gamma_h v_h) = (\gamma_h u_h, v_h), \tag{3.9}$$

$$\frac{5}{12}\|u_h\|^2 \le (u_h, \gamma_h u_h) \le 4\|u_h\|^2, \tag{3.10}$$

$$\|u_h\|_*^2 \le \|\gamma_h u_h\|^2 \le \frac{12}{5}\|u_h\|_*^2. \tag{3.11}$$

According to Lemma 3.1, we can define the inner-product and norm on space S_h:

$$(u_h, v_h)_* = (u_h, \gamma_h v_h), \ \ \|u_h\|_*^2 = (u_h, \gamma_h u_h), \ \ u_h, v_h \in S_h. \tag{3.12}$$

Furthermore, introduce the energy norm (see (3.5)):

$$\|u_h\|_A^2 = a_h(u_h, \gamma_h u_h), \ \|u_h\|_A \ge C\|\nabla u_h\|, \ u_h \in S_h. \tag{3.13}$$

Theorem 3.1. *The solution u_h^n of the discrete equation* (3.6) *uniquely exists and satisfies the following stability estimate.*

$$\|u_h^n\|_A^2 \le \|u_h^0\|_A^2 + \frac{12}{5}\frac{t_n^{2-\beta}}{\Gamma(3-\beta)}\|\psi\|^2 + \frac{12}{5}t_n^{\beta}\Gamma(2-\beta)\max_{1\le k\le n}\|f^{k-\frac{1}{2}}\|^2. \tag{3.14}$$

Proof. Taking $v_h = \delta_t u_h^n$ in (3.8) and using the Cauchy inequality and (3.11), we have

$$\begin{aligned}
&\|\delta_t u_h^n\|_*^2 + \frac{\Gamma_\triangle^\beta}{2\triangle t}(\|u_h^n\|_A^2 - \|u_h^{n-1}\|_A^2)\\
&\le \sum_{k=1}^{n-1}\big(b_{n-k-1} - b_{n-k}\big)\frac{1}{2}(\|\delta_t u_h^k\|_*^2 + \|\delta_t u_h^n\|_*^2)\\
&\quad + \frac{b_{n-1}}{2}\left(\frac{12}{5}\|\psi\|^2 + \|\delta_t u_h^n\|_*^2\right) + \Gamma_\triangle^\beta|(f^{n-\frac{1}{2}}, \gamma_h\delta_t u_h^n)|.
\end{aligned}$$

Hence, noting that $\sum_{k=1}^{n-1}\big(b_{n-k-1} - b_{n-k}\big) = 1 - b_{n-1}$, it yields

$$\begin{aligned}
\|\delta_t u_h^n\|_*^2 + \frac{\Gamma_\triangle^\beta}{\triangle t}(\|u_h^n\|_A^2 - \|u_h^{n-1}\|_A^2) &\le \sum_{k=1}^{n-1}\big(b_{n-k-1} - b_{n-k}\big)\|\delta_t u_h^k\|_*^2\\
&\quad + \frac{12}{5}b_{n-1}\|\psi\|^2 + 2\Gamma_\triangle^\beta|(f^{n-\frac{1}{2}}, \gamma_h\delta_t u_h^n)|,
\end{aligned}$$

or

$$\begin{aligned}
\frac{\Gamma_\triangle^\beta}{\triangle t}\|u_h^n\|_A^2 + \sum_{k=1}^{n} b_{n-k}\|\delta_t u_h^k\|_*^2 &\le \frac{\Gamma_\triangle^\beta}{\triangle t}\|u_h^{n-1}\|_A^2 + \sum_{k=1}^{n-1} b_{n-k-1}\|\delta_t u_h^k\|_*^2\\
&\quad + \frac{12}{5}b_{n-1}\|\psi\|^2 + 2\Gamma_\triangle^\beta|(f^{n-\frac{1}{2}}, \gamma_h\delta_t u_h^n)|.
\end{aligned} \tag{3.15}$$

Set

$$F^0 = \frac{\Gamma_\triangle^\beta}{\triangle t}\|u_h^0\|_A^2, \quad F^n = \frac{\Gamma_\triangle^\beta}{\triangle t}\|u_h^n\|_A^2 + \sum_{k=1}^{n} b_{n-k}\|\delta_t u_h^k\|_*^2.$$

Then, it follows from (3.15) that

$$F^n \le F^{n-1} + \frac{12}{5}b_{n-1}\|\psi\|^2 + 2\Gamma_\triangle^\beta|(f^{n-\frac{1}{2}}, \gamma_h\delta_t u_h^n)|, \ n = 1, 2, \ldots, N.$$

Summing, it yields

$$\frac{\Gamma_\triangle^\beta}{\triangle t}\|u_h^n\|_A^2+\sum_{k=1}^{n}b_{n-k}\|\delta_t u_h^k\|_*^2$$
$$\le \frac{\Gamma_\triangle^\beta}{\triangle t}\|u_h^0\|_A^2+\frac{12}{5}\sum_{k=1}^{n}b_{k-1}\|\psi\|^2+2\Gamma_\triangle^\beta\Big|\sum_{k=1}^{n}(f^{k-\frac{1}{2}},\gamma_h\delta_t u_h^k)\Big|.$$

Hence, again using (3.11) and Cauchy inequality, we obtain

$$\|u_h^n\|_A^2\le\|u_h^0\|_A^2+\frac{\triangle t}{\Gamma_\triangle^\beta}\frac{12}{5}\sum_{k=1}^{n}b_{k-1}\|\psi\|^2+\frac{\triangle t}{\Gamma_\triangle^\beta}\sum_{k=1}^{n}\frac{12(\Gamma_\triangle^\beta)^2}{5b_{n-k}}\|f^{k-\frac{1}{2}}\|^2. \quad (3.16)$$

From (2.8) and Lemma 2.2, we know that

$$\sum_{k=1}^{n}b_{k-1}=n^{2-\beta},\quad b_{n-k}\ge(2-\beta)(n-k+1)^{1-\beta}\ge(2-\beta)n^{1-\beta},\ 1<\beta<2.$$

Hence, it follows from (3.16) that

$$\|u_h^n\|_A^2\le\|u_h^0\|_A^2+\frac{12t_n^{2-\beta}}{5\Gamma(3-\beta)}\|\psi\|^2+\frac{12}{5}t_n^\beta\Gamma(2-\beta)\max_{1\le k\le n}\|f^{k-\frac{1}{2}}\|^2.$$

The proof is completed. □

4. Error analysis

Let $u(t)$ be the solution of problem (3.1). From (3.1), we see that $u^n=u(t_n)$ satisfies (also see [25])

$$(\triangle_n^\beta u^{n-\frac{1}{2}},\gamma_h v_h)+a_h(u^{n-\frac{1}{2}},\gamma_h v_h)=(f^{n-\frac{1}{2}}-r_n(u),\gamma_h v_h),\ \forall\, v_h\in S_h, \quad (4.1)$$

where $r_n(u)=\partial_t^\beta u^{n-1/2}-\triangle_n^\beta u^{n-1/2}$ is the truncation error. In order to do the error analysis, we introduce the finite volume projection $V_h: u\in H_0^1(\Omega)\to V_h u\in S_h$ such that

$$a_h(u-V_h u,\gamma_h v_h)=0,\ \forall\, v_h\in S_h. \quad (4.2)$$

It is easy to see that $V_h u$ just is the solution of the finite volume method for the elliptic problem: $-\text{div}(a(x)\nabla u)=f$. Then, from the known result (see [3, 11], for example), we have the error estimates:

$$\|u-V_h u\|\le Ch^2\|u\|_3,\ \ \|u-V_h u\|_1\le Ch\|u\|_2. \quad (4.3)$$

Let $u(t)$ and u_h^n be the solutions of problems (3.1) and (3.6), respectively. We decompose the error:

$$u(t_n)-u_h^n=u(t_n)-V_h u(t_n)+V_h u(t_n)-u_h^n=\eta^n+\theta^n. \quad (4.4)$$

From (4.2)-(4.3), we know that $\eta(t)$ satisfies the error estimate:

$$\|\partial_t^s\eta(t)\|\le Ch^2\|\partial_t^s u(t)\|_3,\ \ \|\partial_t^s\eta(t)\|_1\le Ch\|\partial_t^s u(t)\|_2,\ s=0,1,2. \quad (4.5)$$

Moreover, it follows from equations (4.1) and (4.2) that for $v_h \in S_h$,

$$(V_h \triangle_n^\beta u^{n-\frac{1}{2}}, \gamma_h v_h) + a_h(V_h u^{n-\frac{1}{2}}, \gamma_h v_h) = (f^{n-\frac{1}{2}} + S_n - r_n(u), \gamma_h v_h), \tag{4.6}$$

where

$$S_n = V_h \triangle_n^\beta u^{n-\frac{1}{2}} - \triangle_n^\beta u^{n-\frac{1}{2}}. \tag{4.7}$$

Lemma 4.1. *Assume that $u(t)$ and u_h^n are the solutions of problems* (3.1) *and* (3.6), *respectively,* $u(0), \psi \in H_0^1(\Omega)$, $u_{tt} \in H_0^1(\Omega)$. *Then, we have for* $\theta^n = V_h u^n - u_h^n$,

$$\|\theta^n\|_A \le \|\theta^0\|_A + C\Big(t_n^{2-\beta}\|\eta_t(0)\| + t_n^2 \max_{0\le\tau\le t_n}\|\eta_{tt}(\tau)\| + t_n^\beta \max_{1\le k\le n}\|r_k(u)\|\Big). \tag{4.8}$$

In particular, if choosing the initial value $u_h^0 = V_h u(0)$, *we have*

$$\|\theta^n\|_A \le C\Big(t_n^{2-\beta}\|\eta_t(0)\| + t_n^2 \max_{0\le\tau\le t_n}\|\eta_{tt}(\tau)\| + t_n^\beta \max_{1\le k\le n}\|r_k(u)\|\Big). \tag{4.9}$$

Proof. From equations (3.6) and (4.6), we have

$$(V_h \triangle_n^\beta u^{n-\frac{1}{2}} - \triangle_n^\beta u_h^{n-\frac{1}{2}}, \gamma_h v_h) + a_h(\theta^{n-\frac{1}{2}}, \gamma_h v_h) = (S_n - r_n(u), \gamma_h v_h).$$

Hence, from (3.7), we obtain the equation satisfied by θ^n:

$$\begin{aligned}
&(\delta_t \theta^n, \gamma_h v_h) + \Gamma_\triangle^\beta a_h(\theta^{n-\frac{1}{2}}, \gamma_h v_h) \\
&= \sum_{k=1}^{n-1} \big(b_{n-k-1} - b_{n-k}\big)(\delta_t \theta^k, \gamma_h v_h) + b_{n-1}(V_h\psi - \psi, \gamma_h v_h) \\
&\quad + \Gamma_\triangle^\beta (S_n - r_n(u), \gamma_h v_h),\ v_h \in S_h.
\end{aligned} \tag{4.10}$$

Comparing (4.10) with (3.8), similar to the stability argument in Theorem 3.1, we can derive

$$\begin{aligned}
\|\theta^n\|_A^2 &\le \|\theta^0\|_A^2 + \frac{12 t_n^{2-\beta}}{5\Gamma(3-\beta)}\|V_h\psi - \psi\|^2 \\
&\quad + \frac{12}{5} t_n^\beta \Gamma(2-\beta) \max_{1\le k\le n}\|S_k - r_k(u)\|^2 \\
&\le \|\theta^0\|_A^2 + C t_n^{2-\beta}\|\eta_t(0)\|^2 + C t_n^\beta \max_{1\le k\le n}\|S_k - r_k(u)\|^2.
\end{aligned} \tag{4.11}$$

Hence, we only need to estimate $\|S_k\| = \|\triangle_k^\beta u^{k-\frac{1}{2}} - V_h \triangle_k^\beta u^{k-\frac{1}{2}}\|$. According to the definition of $\triangle_n^\beta u^{n-\frac{1}{2}}$, we have

$$\begin{aligned}
S_n &= \frac{1}{\Gamma_\triangle^\beta}\left[\delta_t \eta^n + \sum_{k=1}^{n-1}(b_{n-k} - b_{n-k-1})\delta_t \eta^k - b_{n-1}\eta_t(0)\right] \\
&= \frac{1}{\Gamma_\triangle^\beta}\left[\sum_{k=1}^{n} b_{n-k}\delta_t \eta^k - \sum_{k=1}^{n-1} b_{n-k-1}\delta_t \eta^k - b_{n-1}\eta_t(0)\right]
\end{aligned} \tag{4.12}$$

$$= \frac{1}{\Gamma_{\triangle}^{\beta}}\left[\sum_{k=1}^{n-1} b_{n-k-1}(\delta_t\eta^{k+1} - \delta_t\eta^k) + b_{n-1}\delta_t\eta^1 - b_{n-1}\eta_t(0)\right].$$

By using the mean value theorem, we obtain

$$\delta_t\eta^{k+1} - \delta_t\eta^k = \eta_t(\xi') - \eta_t(\xi'') \leq \triangle t \max_{0\leq\tau\leq t_n} |\eta_{tt}(\tau)|,\ \xi',\xi'' \in (t_{k-1}, t_{k+1}),$$
$$\delta_t\eta^1 - \eta_t(0) = \eta_t(\xi) - \eta_t(0) \leq \triangle t \max_{0\leq\tau\leq t_1} |\eta_{tt}(\tau)|,\ \xi \in (0, t_1).$$

Hence, it yields from (4.12) that

$$\begin{aligned}|S_n| &\leq \frac{1}{\Gamma_{\triangle}^{\beta}}\triangle t \max_{0\leq\tau\leq t_n}|\eta_{tt}(\tau)|\sum_{k=0}^{n-1} b_{n-k-1}\\ &= \frac{1}{\Gamma_{\triangle}^{\beta}}\triangle t \max_{0\leq\tau\leq t_n}|\eta_{tt}(\tau)|n^{2-\beta} = \frac{t_n^{2-\beta}}{\Gamma(3-\beta)}\max_{0\leq\tau\leq t_n}|\eta_{tt}(\tau)|.\end{aligned}$$

Substituting this into (4.11), estimate (4.8) is derived. When $u_h^0 = V_h u(0)$, it holds that $\theta^0 = V_h u(0) - u_h^0 = 0$. Then, estimate (4.9) follows from (4.8). $\square$

Let X be a linear normed space and $0 < t^* \leq T$. For the X-value function $u(t) : [0, t^*] \to X$, we define the space

$$L_\infty(0, t^*; X) = \{u(t) \in X : \|u(t)\|_{L_\infty(0,t^*;X)} = \sup_{0\leq t\leq t^*}\|u(t)\|_X < \infty\}.$$

Similarly, we can define the space $C^{2,\delta}([0, t^*]; X)$ with the corresponding norm $\|\cdot\|_{C^{2,\delta}([0,t^*];X)}$.

Theorem 4.1. *Assume that $u(t)$ and u_h^n are the solutions of problems* (3.1) *and* (3.6)*, respectively, $u(0), \psi \in H_0^1(\Omega)\bigcap H^2(\Omega)$, $u \in C^{2,\delta}([0, T]; L_2(\Omega))$, $u_{tt} \in L_\infty(0, T;\ H^2(\Omega))$, and the initial approximation: $\|u_h^0 - u(0)\|_A \leq Ch\|u(0)\|_2$. Then, the following optimal H^1-error estimate holds for $n \geq 1$,*

$$\begin{aligned}\|u^n - u_h^n\|_A &\leq C\triangle t^{2+\delta-\beta}\|u\|_{C^{2,\delta}([0,t_n];L_2)}\\ &\quad + Ch\Big(\|u(0)\|_2 + \|\psi\|_2 + \|u_{tt}\|_{L_\infty(0,\,t_n;H^2)}\Big).\end{aligned} \tag{4.13}$$

Moreover, if $u_h(0) = V_h u(0)$ and $\psi \in H^3(\Omega)$, $u_{tt} \in L_\infty(0, T;\ H^3(\Omega))$, then the following optimal L_2-error estimate holds.

$$\|u^n - u_h^n\| \leq C\triangle t^{2+\delta-\beta}\|u\|_{C^{2,\delta}([0,t_n];L_2)} + Ch^2\Big(\|\psi\|_3 + \|u_{tt}\|_{L_\infty(0,\,t_n;H^3)}\Big).$$

Proof. From Theorem 2.1 and (4.5), we obtain

$$\begin{aligned}&\max_{1\leq k\leq n}\|r_k(u)\| \leq C\triangle t^{2+\delta-\beta}\|u\|_{C^{2,\delta}([0,t_n];L_2)},\\ &\|\eta(t)\|_1 + \|\eta_t(0)\|_1 + \|\eta_{tt}(t)\|_1 \leq Ch(\|\psi\|_2 + \|u_{tt}(t)\|_2),\\ &\|\eta(t)\| + \|\eta_t(0)\| + \|\eta_{tt}(t)\| \leq Ch^2(\|\psi\|_3 + \|u_{tt}(t)\|_3).\end{aligned}$$

Combining these estimates with Lemma 4.1 and using the triangle inequality: $\|u^n - u_h^n\|_A \leq \|\theta^n\|_A + \|\eta^n\|_A$, we can derive the conclusions of Theorem 4.1, noting that $\|\theta^n\| \leq C\|\nabla\theta^n\| \leq C\|\theta^n\|_A$. □

For the finite volume method, to obtain the optimal order L_2-error estimate, the H^3-regularity requirement in (4.3) and Theorem 4.1 is necessary, see [3,11].

In Theorem 4.1, if $u \in C^3([0,T]; L_2(\Omega))$, then we have that $\|u^n - u_h^n\| \leq C(\triangle t^{3-\beta} + h^2)$.

5. Numerical experiment

In this section, we use numerical examples to verify the convergence rates given by Theorem 2.1, Corollary 2.1 and Theorem 4.1 for the difference formula (2.16) and the fully discrete method (3.6), respectively.

To estimate the $C^{2,\delta}$-regularity, we first give a lemma.

Lemma 5.1. *Let $u(t) = t^\sigma$, $0 < \sigma < 1$. Then, $u(t) \in C^\delta[a,b]$, $\forall\ 0 \leq \delta < \sigma$.*

Proof. Using the Hölder inequality with indexes $p = 1/(1-\delta), q = 1/\delta$, it is easy to see that for $0 \leq \delta < \sigma$ and $t_1, t_2 \in (0,T)$,

$$|u(t_2) - u(t_1)| = |t_2^\sigma - t_1^\sigma| = |\sigma \int_{t_1}^{t_2} t^{\sigma-1} dt| \leq \sigma\Big(\int_{t_1}^{t_2} t^{(\sigma-1)p} dt\Big)^{\frac{1}{p}} |t_2 - t_1|^{\frac{1}{q}}$$
$$= \sigma\lambda^{\delta-1}|t_2^\lambda - t_1^\lambda|^{1-\delta}|t_2 - t_1|^\delta,\ \lambda = (\sigma-\delta)/(1-\delta).$$

Therefore, we can conclude that $u(t) \in C^\delta[0,T]$. □

Example 1. In this example, we test the convergence rate given in Theorem 2.1 for function $u(t) \in C^{2,\delta}[0,T]$ $(0 < \delta < 1)$ by computing the error

$$E(N) = \max_{1\leq n\leq N} |r_n(u)| = \max_{1\leq n\leq N} |\partial_t^\beta u^{n-\frac{1}{2}} - \triangle_n^\beta u^{n-\frac{1}{2}}|,\ t_n = n\triangle t,\ 1 < \beta < 2.$$

We take the test function

$$u_\delta(t) = t^{2+\delta},\ \partial_t^\beta u_\delta(t) = \frac{\Gamma(3+\delta)}{\Gamma(3+\delta-\beta)} t^{2+\delta-\beta},\ 0 < \delta < 1,\ t \in [0,T].$$

Since $u_\delta''(t) = (2+\delta)(1+\delta)t^\delta$, according to Lemma 5.1, we can conclude that $u_\delta(t) \in C^{2,\delta_-}[0,T]$ where number δ_- is such that $\delta - \varepsilon < \delta_- < \delta, \forall \varepsilon > 0$. In this example, we set $T = 1, \triangle t = 1/N$. For $N = 2^j, j = 2, 3, \ldots$, the numerical convergence rate r^c is computed by the formula $r^c = \ln[E(N)/E(2N)]/\ln 2$. Table 5.1 gives the numerical results for different parameters β and δ, and the theoretical convergence rate $r^* = 2 + \delta_- - \beta$ (see Theorem 2.1) also is listed in the last column in Table 5.1. From the numerical results we observe that the convergence rates r^c and r^* are almost uniform.

Example 2. In this example, we test the convergence rate given in Corollary 2.1 for piecewise smooth function. Take the test function:

$$u_\gamma(t) = \begin{cases} 1, & 0 \leq t \leq 1/2, \\ (t-1/2)^{2+\gamma} + 1, & 1/2 \leq t \leq 1,\ 0 < \gamma < 1 \end{cases}$$

TABLE 5.1. Error and convergence rate for $u_\delta(t), N = 128$.

β	δ	error	Numer. rate	$2+\delta_- - \beta$
$\beta = 1.2$	0.3	0.0010	1.100	1.1_-
	0.5	0.0006	1.300	1.3_-
	0.7	0.0003	1.500	1.5_-
	0.9	0.0002	1.700	1.7_-
$\beta = 1.4$	0.3	0.0035	0.900	0.9_-
	0.5	0.0023	1.100	1.1_-
	0.7	0.0012	1.300	1.3_-
	0.9	0.0006	1.500	1.5_-
$\beta = 1.6$	0.3	0.0117	0.700	0.7_-
	0.5	0.0076	0.900	0.9_-
	0.7	0.0042	1.100	1.1_-
	0.9	0.0024	1.300	1.3_-
$\beta = 1.8$	0.3	0.0376	0.500	0.5_-
	0.5	0.0248	0.700	0.7_-
	0.7	0.0146	0.900	0.9_-
	0.9	0.0089	1.100	1.1_-

and

$$\partial_t^\beta u_\gamma(t) = \begin{cases} 0, \quad 0 \le t \le 1/2, \\ \dfrac{\Gamma(3+\gamma)}{\Gamma(3+\gamma-\beta)}(t-1/2)^{2+\gamma-\beta}, \ 1/2 \le t \le 1, \ 1 < \beta < 2. \end{cases}$$

According to Lemma 5.1, it is easy to see that the piecewise smooth function $u_\gamma(t) \in C^1[0,1] \bigcap C^{2,\gamma_-}[0,1/2] \bigcap C^{2,\gamma_-}[1/2,1]$ where γ_- is such that $\gamma - \varepsilon < \gamma_- < \gamma, \forall \varepsilon > 0$. Then, according to Corollary 2.1, the theoretical convergence rate of the difference formula for this function should be $r^* = 2 + \gamma_- - \beta$ if $t^* = 1/2$ is a mesh point. Table 5.2 gives the numerical results for different parameters β and γ. We see that the numerical convergence rate $r^c \approx r^*$.

Example 3. In this example, we test the convergence rate given in Theorem 4.1 for the finite difference/finite volume method (3.6).

Consider fractional diffusion wave equation (3.1) on domain $\Omega = [0,1]^2$ with the exact solution $u(t,x) = (t^3+1)\sin(\pi x_1)\sin(\pi x_2)$ and data $a(x) = 1$,

$$f(t,x) = \left(\frac{6t^{3-\beta}}{\Gamma(4-\beta)} + 2\pi^2(t^3+1)\right)\sin(\pi x_1)\sin(\pi x_2).$$

We mainly examine the L_2-error of the finite volume method on spatial domain. Therefore, we take the time step $\triangle t$ small enough so that the dominant numerical error comes from the finite volume method. Let $e_h = \|u(T) - u_h(T)\|$ be the error on mesh T_h at terminal time T. The numerical convergence rate r with respect to mesh size h is computed by formula $r = \ln(e_h/e_{\frac{h}{2}})/\ln 2$. Numerical results are given in Table 5.3 for parameter $\beta = 1.3$ and 1.5. We

TABLE 5.2. Error and convergence rate for piecewise smooth $u_\gamma(t), N = 128$.

β	γ	error	Numer. rate	$2+\gamma_- - \beta$
$\beta = 1.1$	0.3	0.0036	1.200	1.2_-
	0.5	0.0015	1.400	1.4_-
	0.9	0.0003	1.800	1.8_-
$\beta = 1.3$	0.3	0.0105	1.000	1.0_-
	0.5	0.0045	1.200	1.2_-
	0.9	0.0008	1.600	1.6_-
$\beta = 1.6$	0.3	0.0495	0.700	0.7_-
	0.5	0.0219	0.900	0.9_-
	0.9	0.0041	1.300	1.3_-
$\beta = 1.9$	0.3	0.2171	0.400	0.4_-
	0.5	0.1012	0.600	0.6_-
	0.9	0.0207	1.000	1.0_-

TABLE 5.3. Error and convergence rate at $T = 0.5$, $\triangle t = 1/2000$.

β	h	$\|u(T) - u_h(T)\|$	rate
$\beta = 1.5$	1/4	3.2305e-01	–
	1/8	0.8354e-01	1.9512
	1/16	2.1071e-02	1.9872
	1/32	5.2921e-03	1.9932
	1/64	1.3267e-03	1.9959
$\beta = 1.3$	1/4	6.4532e-01	–
	1/8	1.7276e-01	1.9012
	1/16	4.4492e-02	1.9572
	1/32	1.1183e-02	1.9922
	1/64	2.8096e-03	1.9929

observe that the convergence rate is of $O(h^2)$-order which is consistent with the theoretical prediction.

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