

## BLOW-UP OF SOLUTIONS FOR WAVE EQUATIONS WITH STRONG DAMPING AND VARIABLE-EXPONENT NONLINEARITY

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ABSTRACT. In this paper we consider the following strongly damped wave equation with variable-exponent nonlinearity

$$u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) = |u(x, t)|^{p(x)-2}u(x, t),$$

where the exponent  $p(\cdot)$  of nonlinearity is a given measurable function. We establish finite time blow-up results for the solutions with non-positive initial energy and for certain solutions with positive initial energy. We extend the previous results for strongly damped wave equations with constant exponent nonlinearity to the equations with variable-exponent nonlinearity.

### 1. Introduction

In this paper, we are concerned with the following wave equation with strong damping and variable-exponent nonlinearity

- (1)  $u_{tt}(x, t) - \Delta u(x, t) - \Delta u_t(x, t) = |u(x, t)|^{p(x)-2}u(x, t)$  in  $\Omega \times (0, T)$ ,
- (2)  $u(x, t) = 0$  on  $\partial\Omega \times (0, T)$ ,
- (3)  $u(x, 0) = u_0(x)$ ,  $u_t(x, 0) = u_1(x)$  on  $\Omega$ ,

where  $\Omega \subset \mathbb{R}^n$ ,  $n \geq 1$ , is a bounded domain with smooth boundary  $\partial\Omega$ , and the exponent  $p(\cdot)$  is a given measurable function.

During the past decades, the following wave equations

$$u_{tt} - \Delta u - \omega \Delta u_t + \mu u_t = |u|^{p-2}u$$

have been studied extensively on existence, nonexistence, stability, and blow up of solutions [5, 10, 12, 15, 17, 26–28]. When  $\omega = \mu = 0$ , Sattinger [27] discussed the existence of local as well as global solutions by introducing the concepts

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of stable and unstable sets. Ball [5] established a finite time blow-up result of solutions with negative initial energy. When  $\omega = 0$  and  $\mu > 0$ , Ikehata [15] gave a characterization of the existence of blow-up solutions for sufficiently small  $\mu > 0$ . Later, Esquivel-Avila [10] extended the result of [15] to the case of any  $\mu > 0$ . Gazzola and Squassina [12] proved the global existence and finite time blow up of solutions under the conditions  $\omega \geq 0$  and  $\mu > -\omega\lambda_1$ , where  $\lambda_1$  is the first eigenvalue of the operator  $-\Delta$  with homogeneous Dirichlet boundary conditions.

Recently, researchers have much interested in nonlinear models of hyperbolic, parabolic, and elliptic equations with variable-exponent nonlinearities [2, 22–25]. This kind of systems appears in electro-rheological fluids or fluids with temperature-dependent viscosity, nonlinear viscoelasticity, filtration processes through a porous media and image processing. We refer [1, 3, 18] for more details on these problems. Antontsev [2] studied a quasilinear equation of the form

$$u_{tt} - \operatorname{div}(a(x, t)|\nabla u|^{r(x, t)-2}\nabla u) - \alpha\Delta u_t = b(x, t)|u|^{p(x, t)-2}u + f(x, t).$$

He proved the existence and blow up of weak solutions with negative initial energy under some conditions on  $a, b, f, r$ , and  $p$ . Messaoudi et al. [25] considered the following wave equations of variable-exponent nonlinearities

$$u_{tt}(x, t) - \Delta u(x, t) + a|u_t(x, t)|^{m(x)-2}u_t = b|u(x, t)|^{p(x)-2}u(x, t),$$

where  $a$  and  $b$  are positive constants, and  $m(\cdot)$  and  $p(\cdot)$  are given measurable functions. They proved the existence of a unique weak solution under suitable assumptions on the variable exponents  $m(\cdot)$  and  $p(\cdot)$  by using the Faedo-Galerkin method. Then, they established the finite time blow-up of solutions when the initial energy is negative. Regarding nonlinear wave equations with constant-exponent nonlinearities, we also refer [6, 13, 14, 20, 21] and references therein. Messaoudi and Talahmeh [23] discussed a quasilinear wave equation of the form

$$\begin{aligned} & u_{tt}(x, t) - \operatorname{div}(|\nabla u(x, t)|^{r(x)-2}\nabla u(x, t)) + a|u_t(x, t)|^{m(x)-2}u_t \\ & = b|u(x, t)|^{p(x)-2}u(x, t). \end{aligned}$$

They showed finite time blow-up results for the solution with negative initial energy and for certain solutions with positive initial energy. Motivated these results, we investigate finite time blow-up results of the solutions with positive initial energy as well as non-positive initial energy for problem (1)-(3). As far as we know, there are few works on wave equations with strong damping and variable-exponent nonlinearity. Moreover, this work extends the previous results for strongly damped wave equations with constant-exponent nonlinearity to the equations with variable-exponent nonlinearity.

The outline of this paper is as follows. In Section 2, we give materials needed for our work. In Section 3, we prove finite time blow-up results.

### 2. Preliminaries

In this section, we present some material needed for the statement and proof of our results. First, we give preliminary facts about Lebesgue and Sobolev spaces with variable exponents (see [8, 9, 11]). Let  $p : \Omega \rightarrow [1, \infty]$  be a measurable function. The Lebesgue space  $L^{p(\cdot)}(\Omega)$  with variable exponent  $p(\cdot)$  is defined by

$$L^{p(\cdot)}(\Omega) = \{u : \Omega \rightarrow \mathbb{R} \mid u \text{ is measurable in } \Omega, \int_{\Omega} |\lambda u(x)|^{p(x)} dx < \infty \text{ for some } \lambda > 0\}.$$

This space is a Banach space with the Luxembourg-type norm

$$\|u\|_{p(\cdot)} = \inf\{\lambda > 0 \mid \int_{\Omega} \left| \frac{u(x)}{\lambda} \right|^{p(x)} dx \leq 1\}.$$

In general, variable-exponent Lebesgue spaces are similar to classical Lebesgue spaces in many aspects (see [16]). As usual,  $(\cdot, \cdot)$  and  $\langle \cdot, \cdot \rangle$  denote the inner product in the space  $L^2(\Omega)$  and the duality pairing between  $H_0^1(\Omega)$  and  $H^{-1}(\Omega)$ , respectively.  $\|\cdot\|_q$  denotes the norm of the space  $L^q(\Omega)$ . For brevity, we denote  $\|\cdot\|_2$  by  $\|\cdot\|$ .

Let us list some properties of the space  $L^{p(\cdot)}(\Omega)$  which will be used in this work.

**Lemma 2.1** ([7]). *If  $p : \Omega \rightarrow [1, \infty)$  is a measurable function satisfying*

$$2 \leq p(x) < \infty \text{ if } n = 1, 2;$$

$$2 \leq \text{ess inf}_{x \in \Omega} p(x) \leq p(x) \leq \text{ess sup}_{x \in \Omega} p(x) < \frac{2n}{n-2} \text{ if } n \geq 3,$$

*then the embedding  $H_0^1(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$  is continuous and compact.*

**Lemma 2.2** ([4]). *If*

$$1 < p_1 := \text{ess inf}_{x \in \Omega} p(x) \leq p(x) \leq p_2 := \text{ess sup}_{x \in \Omega} p(x) < \infty,$$

*then*

$$\min\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\} \leq \int_{\Omega} |u(x)|^{p(x)} dx \leq \max\{\|u\|_{p(\cdot)}^{p_1}, \|u\|_{p(\cdot)}^{p_2}\}$$

*for any  $u \in L^{p(\cdot)}(\Omega)$ .*

**Definition.** Let  $T > 0$ . We say that a function  $u$  is a solution of problem (1)-(3) on  $\Omega \times (0, T)$  if

$$u \in C([0, T]; H_0^1(\Omega)) \cap C^1([0, T]; L^2(\Omega)) \cap C^2([0, T]; H^{-1}(\Omega))$$

with  $u_t \in L^2(0, T; H_0^1(\Omega))$  and satisfies

$$(4) \quad \langle u_{tt}(t), w \rangle + (\nabla u(t), \nabla w) + (\nabla u_t(t), \nabla w) = (|u(t)|^{p(\cdot)-2}u(t), w)$$

for any  $w \in H_0^1(\Omega)$ , and

$$u(0) = u_0 \text{ in } H_0^1(\Omega), \quad u_t(0) = u_1 \text{ in } L^2(\Omega).$$

We state the well-posedness which can be established by the arguments of [19, 25].

**Theorem 2.3.** *Assume that  $p(\cdot)$  satisfies*

$$2 < p_1 \leq p(x) \leq p_2 < \infty \quad \text{if } n = 1, 2;$$

$$2 < p_1 \leq p(x) \leq p_2 < \frac{2(n-1)}{n-2} \quad \text{if } n \geq 3,$$

where

$$p_1 := \text{ess inf}_{x \in \Omega} p(x), \quad p_2 := \text{ess sup}_{x \in \Omega} p(x),$$

and the log-Hölder continuity condition:

$$(5) \quad |p(x) - p(y)| \leq -\frac{A}{\log|x-y|} \quad \text{for a.e. } x, y \in \Omega, \quad \text{with } |x-y| < \delta,$$

here  $A > 0$  and  $0 < \delta < 1$ . Then, for every  $u_0 \in H_0^1(\Omega)$  and  $u_1 \in L^2(\Omega)$ , problem (1)-(3) has a unique local solution.

### 3. Finite time blow-up of solutions

In this section we prove that the solution to problem (1)-(3) blows up in finite time when the initial energy is positive as well as non-positive. In order to state and prove our results, we need the following lemma:

**Lemma 3.1** ([17]). *Let  $G(t)$  be a positive, twice differentiable function satisfying the inequality*

$$G(t)G''(t) - (1 + \delta)(G'(t))^2 \geq 0 \quad \text{for } t > 0,$$

where  $\delta$  is a positive constant. If  $G(0) > 0$  and  $G'(0) > 0$ , then there exists a time  $T_* \leq \frac{G(0)}{\delta G'(0)}$  such that

$$\lim_{t \rightarrow T_*^-} G(t) = +\infty.$$

We start by defining the energy of the solution to problem (1)-(3) as

$$(6) \quad E(t) = \frac{1}{2} \|u_t(t)\|^2 + \frac{1}{2} \|\nabla u(t)\|^2 - \int_{\Omega} \frac{|u(x,t)|^{p(x)}}{p(x)} dx.$$

Then, multiplying (1) by  $u_t$  and integrating it over  $\Omega$ , we find

$$(7) \quad E'(t) = -\|\nabla u_t(t)\|^2 \leq 0 \quad \text{for } 0 \leq t < T_{\max},$$

where  $T_{\max}$  is the maximal existence time of the solution  $u$  of problem (1)-(3). Moreover, we also get

$$(8) \quad E(t) + \int_0^t \|\nabla u_t(s)\|^2 ds = E(0) \quad \text{for } 0 \leq t < T_{\max}.$$

### 3.1. Blow up for non-positive initial energy

In this subsection, we show that the solution with non-positive initial energy blows up in finite time.

**Theorem 3.2.** *Let the conditions of Theorem 2.1 hold and  $E(0) \leq 0$ . Moreover, assume that  $(u_0, u_1) > 0$  when  $E(0) = 0$ . Then the solution  $u$  of problem (1)-(3) blows up in finite time .*

*Proof.* By contradiction, suppose that the solution  $u$  is global. For any  $T > 0$ , we consider  $G : [0, T] \rightarrow \mathbb{R}^+$  defined by

$$(9) \quad G(t) = \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds + (T - t)\|\nabla u_0\|^2 + b(t + T_0)^2,$$

where  $T_0 > 0$  and  $b \geq 0$  to be specified later. Then

$$(10) \quad G(t) > 0 \quad \text{for } t \in [0, T]$$

and

$$(11) \quad \begin{aligned} G'(t) &= 2(u(t), u_t(t)) + \|\nabla u(t)\|^2 - \|\nabla u_0\|^2 + 2b(t + T_0) \\ &= 2(u(t), u_t(t)) + 2 \int_0^t (\nabla u(s), \nabla u_t(s)) ds + 2b(t + T_0). \end{aligned}$$

From (1), we obtain

$$(12) \quad \begin{aligned} G''(t) &= 2\|u_t(t)\|^2 + 2\langle u_{tt}(t), u(t) \rangle + 2(\nabla u(t), \nabla u_t(t)) + 2b \\ &= 2\|u_t(t)\|^2 - 2\|\nabla u(t)\|^2 + 2 \int_{\Omega} |u(x, t)|^{p(x)} dx + 2b. \end{aligned}$$

By Cauchy-Schwartz inequality and (9), we get

$$(13) \quad \begin{aligned} \frac{(G'(t))^2}{4} &= \left( (u(t), u_t(t)) + \int_0^t (\nabla u(s), \nabla u_t(s)) ds + b(t + T_0) \right)^2 \\ &\leq \left( \|u(t)\|^2 + \int_0^t \|\nabla u(s)\|^2 ds + b(t + T_0)^2 \right) \\ &\quad \times \left( \|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + b \right) \\ &= \left( G(t) - (T - t)\|\nabla u_0\|^2 \right) \left( \|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + b \right) \\ &\leq G(t) \left( \|u_t(t)\|^2 + \int_0^t \|\nabla u_t(s)\|^2 ds + b \right). \end{aligned}$$

From (12) and (13), we have the following differential inequality

$$(14) \quad G(t)G''(t) - \frac{p_1 + 2}{4}(G'(t))^2 \geq G(t)F(t),$$

where

$$(15) \quad \begin{aligned} F(t) = & -p_1 \|u_t(t)\|^2 - 2 \|\nabla u(t)\|^2 + 2 \int_{\Omega} |u(x, t)|^{p(x)} dx \\ & - (p_1 + 2) \int_0^t \|\nabla u_t(s)\|^2 ds - p_1 b. \end{aligned}$$

Using (6) and (8), we find

$$(16) \quad \begin{aligned} F(t) = & -2p_1 E(t) + (p_1 - 2) \|\nabla u(t)\|^2 - 2p_1 \int_{\Omega} \frac{|u(x, t)|^{p(x)}}{p(x)} dx \\ & + 2 \int_{\Omega} |u(x, t)|^{p(x)} dx - (p_1 + 2) \int_0^t \|\nabla u_t(s)\|^2 ds - p_1 b \\ = & -2p_1 E(0) + (p_1 - 2) \|\nabla u(t)\|^2 - 2p_1 \int_{\Omega} \frac{|u(x, t)|^{p(x)}}{p(x)} dx \\ & + 2 \int_{\Omega} |u(x, t)|^{p(x)} dx + (p_1 - 2) \int_0^t \|\nabla u_t(s)\|^2 ds - p_1 b \\ \geq & -2p_1 E(0) + (p_1 - 2) \|\nabla u(t)\|^2 + (p_1 - 2) \int_0^t \|\nabla u_t(s)\|^2 ds - p_1 b. \end{aligned}$$

We now consider two cases  $E(0) < 0$  and  $E(0) = 0$ .

*Case 1:*  $E(0) < 0$ .

Taking  $0 < b \leq -2E(0)$ , we have from (16) that

$$(17) \quad F(t) \geq 0.$$

*Case 2:*  $E(0) = 0$ .

Choosing  $b = 0$ , we get from (16) that

$$(18) \quad F(t) \geq 0.$$

Adapting (17) and (18) to (14), we infer

$$(19) \quad G(t)G''(t) - \frac{p_1 + 2}{4}(G'(t))^2 \geq 0.$$

Now, it remains to show  $G'(0) > 0$ . When  $E(0) < 0$ , we take  $T_0$  large enough such that

$$G'(0) = 2(u_0, u_1) + 2bT_0 > 0.$$

If  $E(0) = 0$ , the condition  $(u_0, u_1) > 0$  gives

$$G'(0) = 2(u_0, u_1) > 0.$$

Thus, we conclude from Lemma 3.1 that

$$(20) \quad \lim_{t \rightarrow T_*^-} G(t) = +\infty$$

for

$$(21) \quad T_* \leq \frac{4G(0)}{(p_1 - 2)G'(0)} = \frac{2\|u_0\|^2 + 2T\|\nabla u_0\|^2 + 2bT_0^2}{(p_1 - 2)((u_0, u_1) + bT_0)}.$$

Thus, we deduce that

$$(22) \quad T_* \leq \frac{2\|u_0\|^2 + 2bT_0^2}{(p_1 - 2)(u_0, u_1) + (p_1 - 2)bT_0 - 2\|\nabla u_0\|^2}.$$

From (9), (20) and (22), we have

$$(23) \quad \lim_{t \rightarrow T_*^-} \left( \|u(t)\| + \int_0^t \|\nabla u(s)\|^2 ds \right) = +\infty.$$

This contradicts our assumption that the solution is global. Therefore, we conclude that the solution  $u$  to problem (1)-(3) blows up in finite time.  $\square$

### 3.2. Blow up for positive initial energy

In this subsection, we establish a finite time blow-up result for certain solutions with positive energy. For this, we set

$$(24) \quad \tilde{B} = \max\{1, B\}, \quad \xi_1 = \left(\frac{1}{\tilde{B}}\right)^{\frac{p_1}{p_1-2}}, \quad d = \frac{(p_1 - 2)\xi_1^2}{2p_1},$$

where  $B$  is the optimal constant of the embedding inequality

$$(25) \quad \|v\|_{p(\cdot)} \leq B\|\nabla v\| \quad \text{for } v \in H_0^1(\Omega).$$

Next, we define a functional  $h$  by

$$(26) \quad h(\xi) = \frac{1}{2}\xi^2 - \frac{\tilde{B}^{p_1}}{p_1}\xi^{p_1}.$$

We can easily check that  $h$  is continuous,  $h(0) = 0$ , and  $\lim_{\xi \rightarrow +\infty} h(\xi) = -\infty$ . Moreover, we see that  $h$  is increasing on  $(0, \xi_1)$  and decreasing on  $(\xi_1, \infty)$ . So,  $h$  has the maximum value  $h(\xi_1) = d$ .

**Lemma 3.3.** *Let  $u$  be the solution of problem (1)-(3). Assume that*

$$(27) \quad E(0) < d \quad \text{and} \quad \xi_1 < \|\nabla u_0\| \leq \frac{1}{\tilde{B}}.$$

*Then there exists a constant  $\xi_* > \xi_1$  such that*

$$(28) \quad \|\nabla u(t)\|^2 \geq \xi_*^2 \quad \text{for } 0 \leq t < T_{\max}.$$

*Proof.* From (6), Lemma 2.2, (25) and (24), we have

$$\begin{aligned} E(t) &\geq \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{p_1} \int_{\Omega} |u(x, t)|^{p(x)} dx \\ &\geq \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{p_1} \max\{\|u(t)\|_{p(\cdot)}^{p_1}, \|u(t)\|_{p(\cdot)}^{p_2}\} \\ &\geq \frac{1}{2}\|\nabla u(t)\|^2 - \frac{1}{p_1} \max\{\tilde{B}^{p_1}\|\nabla u(t)\|^{p_1}, \tilde{B}^{p_2}\|\nabla u(t)\|^{p_2}\} \\ (29) \quad &= g(\|\nabla u(t)\|), \end{aligned}$$

where

$$g(\xi) = \frac{1}{2}\xi^2 - \frac{1}{p_1} \max\{\tilde{B}^{p_1}\xi^{p_1}, \tilde{B}^{p_2}\xi^{p_2}\}.$$

It is noted that

$$h(\xi) = g(\xi) \quad \text{for } 0 \leq \xi \leq \frac{1}{B},$$

where  $h$  is the function given in (26). Owing to  $E(0) < d$ , there exists  $\xi_* > \xi_1$  such that

$$(30) \quad E(0) = h(\xi_*).$$

Considering  $t = 0$  in the inequality (29), we have from (27) and (30) that

$$(31) \quad h(\xi_*) = E(0) \geq g(\|\nabla u_0\|) = h(\|\nabla u_0\|).$$

Since  $h$  is decreasing on  $(\xi_1, \infty)$ , we see

$$(32) \quad \xi_* \leq \|\nabla u_0\|.$$

From (27), we also know

$$(33) \quad \xi_* \leq \frac{1}{B}.$$

Now, we want to show that

$$(34) \quad \|\nabla u(t)\| \geq \xi_* \quad \text{for all } t \in [0, T_{\max}).$$

By contradiction, suppose that there exists  $t_0 \in [0, T_{\max})$  such that

$$(35) \quad \|\nabla u(t_0)\| < \xi_*.$$

Because the solution  $u$  is continuous in  $t$ , there exists  $t_1 > 0$  such that

$$(36) \quad \xi_1 < \|\nabla u(t_1)\| < \xi_*.$$

Noting that  $h$  is decreasing on  $(\xi_1, \infty)$ , we have from (33), (36) and (29) that

$$(37) \quad E(0) = h(\xi_*) < h(\|\nabla u(t_1)\|) = g(\|\nabla u(t_1)\|) \leq E(t_1) \leq E(0),$$

we used the fact  $E$  is nonincreasing in the last inequality. But, this is contradiction. Thus we complete the proof.  $\square$

**Theorem 3.4.** *Under the conditions of Lemma 3.2, the solution  $u$  of problem (1)-(3) blows up in finite time.*

*Proof.* Suppose that the solution  $u$  is global. For any  $T > 0$ , we consider the function  $G$  defined in (9). Then, (10), (14), (15), and (16) hold. First, we show that the function  $F(t)$  given in (15) is non-negative. Indeed, the fact that the solution  $u$  is continuous on  $[0, T]$  and Lemma 3.2 ensure the existence of  $\epsilon > 0$  satisfying

$$(38) \quad \xi_1^2 + \epsilon < \xi_*^2 \leq \|\nabla u(t)\|^2 \quad \text{for all } t \in [0, T].$$

From (16), (38), and (24), we observe

$$(39) \quad F(t) > -2p_1d + (p_1 - 2)(\xi_1^2 + \epsilon) - p_1b = (p_1 - 2)\epsilon - p_1b.$$

Choosing  $b > 0$  sufficiently small such that  $(p_1 - 2)\epsilon - p_1b \geq 0$ , we obtain

$$(40) \quad F(t) > 0.$$

From (40) and (14), we have

$$(41) \quad G(t)G''(t) - \frac{p_1 + 2}{4}(G'(t))^2 > 0.$$

Choosing  $T_0$  large enough such that

$$G'(0) = 2(u_0, u_1) + 2bT_0 > 0,$$

we can get

$$G'(0) > 0.$$

The remainder of the proof can be established by repeating the steps (20) to (23) of the proof of Theorem 3.1.  $\square$

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