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KENMOTSU MANIFOLDS SATISFYING THE FISCHER-MARSDEN EQUATION

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Abstract. The present paper deals with the study of Fischer-Marsden conjecture on a Kenmotsu manifold. It is proved that if a Kenmotsu metric satisfies $\mathfrak{L}_g^*(\lambda) = 0$ on a $(2n+1)$-dimensional Kenmotsu manifold M^{2n+1}, then either $\xi\lambda = -\lambda$ or M^{2n+1} is Einstein. If $n = 1$, M^3 is locally isometric to the hyperbolic space $H^3(-1)$.

1. Introduction

The contact geometry plays a major role in science, medical science and technology, although it has broad applications in physics, for example geometric optics, geometric quantization, control theory, thermodynamics, integrable systems and to the classical mechanics. Due to its broad applications in different era, it becomes the center of attraction for the researchers. Boothby and Wang [5], in 1958, considered an odd dimensional differentiable manifold equipped with the contact and almost contact structures and studied their properties from topological approach. In 1960, Sasaki [30] characterized the properties of an odd dimensional differentiable manifold along with the contact structures by using the tensor calculus. They called such manifolds as the contact manifolds. Since then, many researchers characterized the several classes of the contact manifolds and studied their properties. In this series, Kenmotsu [20] considered a class of contact metric manifold satisfying the certain tensorial relations and called it a Kenmotsu manifold. He proved that a semisymmetric Kenmotsu manifold $(R(U,V)\cdot R = 0)$ is a manifold of constant negative curvature -1, where R denotes the Riemannian curvature tensor and $R(U,V)$ acts as a derivation of the tensor algebra at each point of the tangent space of the manifold. Various properties of the Kenmotsu manifolds have been studied by

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many researchers, for instance we refer to [1], [2], [7], [9], [10], [16], [19], [26], [23–25], [28], [31–34] and the references there in.

Let g be a Riemannian metric of an n-dimensional compact orientable manifold (M^n, g). We denote the set of all Riemannian metrics of unit volume on (M^n, g) by $\mathfrak{g}$ and the symmetric bilinear tensor of type $(0, 2)$ on M^n by g^*. The linearization of the scalar curvature $\mathfrak{L}_g g^*$ is given by

$$\mathfrak{L}_g g^* = -\triangle_g(tr_g g^*) + div(div(g^*)) - g(g^*, S_g),$$

where $'div'$ stands for the divergence, $\triangle_g$ denotes the negative Laplacian of the Riemannian metric g and S is the Ricci tensor of M^n. If $\mathfrak{L}_g^*$ denotes the formal L^2-adjoint of the linearized scalar curvature operator $\mathfrak{L}_g$, then it is defined by

$$\mathfrak{L}_g^*(\lambda) = -(\triangle_g \lambda)g - \lambda S_g + Hess_g \lambda. \tag{1}$$

Here $Hess_g$ denotes the Hessian of the smooth function λ and is defined as $Hess_g\lambda(U, V) = g(\nabla_U D\lambda, V)$, $\forall\, U, V \in \chi(M^n)$, where ∇ is the Levi-Civita connection, D, the gradient operator of g and $\chi(M^n)$ represents the set of all smooth vector fields of M^n. We call the equation $\mathfrak{L}_g^*(\lambda) = 0$ as the Fischer-Marsden equation and the pair (g, λ) for which $\mathfrak{L}_g^*(\lambda) = 0$ on M^n is known as the solution of the Fischer-Marsden equation. Bourguignon [6] and Fischer and Marsden [17] proved that if (g, λ) is a non-trivial solution of the equation $\mathfrak{L}_g^*(\lambda) = 0$ on a complete Riemannian manifold (M^n, g), then the scalar curvature of g is constant. Corvino [14] proved that the warped product metric $g^* = g - \lambda^2 dt^2$ on a compact Riemannian manifold (M^n, g) is Einstein if and only if (g, λ) is a non-trivial solution of the equation $\mathfrak{L}_g^*(\lambda) = 0$. We also recall the following Fischer-Marsden conjecture [17] as:

"A compact Riemannian manifold that admits a non-trivial solution of the equation $\mathfrak{L}_g^*(\lambda) = 0$ is necessarily an Einstein manifold."

Kobayashi presented a counterexample of the Fischer-Marsden conjecture in [21], whereas Lafontain [22] studied the same conjecture when g is conformally flat. Cernea and Guan [8] showed that if (g, λ) is a non-trivial solution of the equation $\mathfrak{L}_g^*(\lambda) = 0$ on a closed homogeneous Riemannian manifold (M^n, g), then (M^n, g) assumes the form $S^m \times N$, where S^m, N represent the Euclidean sphere and the Einstein manifold. In 2017, Patra and Ghosh [27] considered the Fischer-Marsden conjecture on K-contact and (κ, μ)-contact metric manifolds and proved that if a complete K-contact manifold satisfies $\mathfrak{L}_g^*(\lambda) = 0$, then it is Einstein and locally isometric to a unit sphere S^{2n+1}. Recently, in [15], the second author with Mandal studied the Fischer-Marsden conjecture within the framework of almost Kenmotsu manifold and proved that if a 3-dimensional non-Kenmotsu $(\kappa, \mu)'$-almost Kenmotsu manifold satisfies the Fischer-Marsden conjecture, then the manifold is locally isometric to the product space $H^2(-4) \times \mathbb{R}$. They also established that if the metric of a complete almost Kenmotsu manifold with conformal Reeb foliation satisfies the Fischer-Marsden conjecture, then the manifold is Einstein. In [29], Prakasha, Veeresha and Venkatesha proved that if the metric of a non-Kenmotsu $(\kappa, \mu)'$-almost

Kenmotsu manifold M^{2n+1} satisfies the equation $\mathfrak{L}_g^*(\lambda) = 0$, then M^{2n+1} is locally isometric to the warped products $\mathbb{H}^{n+1}(\alpha) \times_f \mathbb{R}^n$ or $\mathbb{B}^{n+1}(\alpha') \times_{f'} \mathbb{R}^n$, where $\mathbb{H}^{n+1}$ is the hyperbolic space of constant curvature $\alpha = -1 - \frac{2}{n} - \frac{1}{n^2}$, tangent to the distribution $[\xi] \otimes [\gamma]$, $\mathbb{B}^{n+1}(\alpha')$ is a space of constant curvature $\alpha' = -1 + \frac{2}{n} - \frac{1}{n^2}$, tangent to the distribution $[\xi] \otimes [-\gamma]$, $f = ce^{(1-\frac{1}{n})t}$ and $f' = c'e^{(1+\frac{1}{n})t}$, where c, c' are positive constants.

The above studies motivate us to characterize a Kenmotsu manifold if the Kenmotsu metric is the non-trivial solution of the Fischer-Marsden equation $\mathfrak{L}_g^*(\lambda) = 0$. Throughout the paper, we consider $\mathfrak{L}_g^*(\lambda) = 0$ as a Fischer-Marsden equation on a Kenmotsu manifold. After brief introduction in Section 1, we list the essential basic results of a Kenmotsu manifold to prove our main results in Section 2. Next we characterize Kenmotsu manifold satisfying Fischer-Marsden equation. Precisely, we prove the following Theorems:

Theorem 1.1. *Suppose the Ricci operator Q of a $(2n+1)$-dimensional Kenmotsu manifold M^{2n+1} is a Reeb flow invariant. If (g, λ) is a non-trivial solution of the Fischer-Marsden equation $\mathfrak{L}_g^*(\lambda) = 0$, then either $\xi\lambda = -\lambda$ or M^{2n+1} is an Einstein manifold.*

Theorem 1.2. *Let (g, λ) be a non-trivial solution of the equation $\mathfrak{L}_g^*(\lambda) = 0$ on a 3-dimensional Kenmotsu manifold M^3. Then the manifold M^3 is locally isometric to the hyperbolic space form $H^3(-1)$.*

2. Kenmotsu manifolds

A quadruple (ϕ, ξ, η, g) defined on a $(2n+1)$-dimensional Riemannian manifold M^{2n+1} of class C^∞ is known as an almost contact metric structure if

$$\phi^2(U) = -U + \eta(U)\xi, \quad \eta(\xi) = 1 \tag{2}$$

and

$$g(U, \xi) = \eta(U), \quad g(\phi U, \phi V) + \eta(U)\eta(V) = g(U, V) \tag{3}$$

hold for all $U, V \in \chi(M^{2n+1})$, where ϕ is a $(1,1)$-type vector field, ξ, the structure vector field of type $(1,0)$, η, 1-form and g is the Riemannian metric on M^{2n+1}. The manifold M^{2n+1} equipped with the structure (ϕ, ξ, η, g) is called an almost contact metric manifold of dimension $(2n+1)$ [4]. It is obvious from (2) that

$$\phi\xi = 0, \ \ \eta(\phi U) = 0, \ \ \operatorname{rank}\phi = 2n \tag{4}$$

for all $U \in \chi(M^{2n+1})$. An almost contact metric manifold M^{2n+1} is said to be a contact metric manifold if $d\eta(U, V) = g(U, \phi V)$, where d represents the exterior derivative. An almost contact metric manifold M^{2n+1} together with $[\phi, \phi] = -2d\eta \otimes \eta$, where $[\phi, \phi]$ is the Nijenhuis tensor of ϕ, is said to be a contact normal metric manifold. If M^{2n+1} satisfies the condition

$$(\nabla_U \phi)(V) = g(\phi U, V)\xi - \eta(V)\phi(U) \tag{5}$$

or equivalently

$$\nabla_U \xi = U - \eta(U)\xi \tag{6}$$

for all $U,\ V \in \chi(M^{2n+1})$, then M^{2n+1} is known as a Kenmotsu manifold [20]. In view of the equations (3) and (6), we obtain

$$(\nabla_U \eta)(V) = g(U,V) - \eta(U)\eta(V). \tag{7}$$

It is well known that a Kenmotsu manifold M^{2n+1} satisfies the following:

$$R(U,V)\xi = \eta(U)V - \eta(V)U, \tag{8}$$

$$R(\xi,U)V = \eta(V)U - g(U,V)\xi, \tag{9}$$

$$S(U,\xi) = -2n\,\eta(U), \tag{10}$$

$$S(\phi U,\phi V) = S(U,V) + 2n\eta(U)\eta(V) \tag{11}$$

for all $U,\ V \in \chi(M^{2n+1})$ [19]. De and Pathak [16] proved that a three dimensional Kenmotsu manifold satisfies the following:

$$S(U,V) = \frac{1}{2}\{(r+2)g(U,V) - (r+6)\eta(U)\eta(V)\}, \tag{12}$$

which is equivalent to

$$QU = \frac{1}{2}\{(r+2)U - (r+6)\eta(U)\xi\}. \tag{13}$$

Here r is the scalar curvature and Q denotes the Ricci operator corresponding to the Ricci tensor, that is, $S(U,V) = g(QU,V),\ \forall U,\ V \in \chi(M^{2n+1})$. Also we have

$$Q\xi = -2\xi. \tag{14}$$

A Kenmotsu manifold M^{2n+1} $(n > 2)$ is said to be an Einstein manifold if the non-vanishing Ricci tensor S of M^{2n+1} satisfies the condition $S(U,V) = \alpha\, g(U,V),\ \forall\ U,\ V \in \chi(M^{2n+1})$, where α is a constant.

The Ricci operator Q defined on M^{2n+1} is said to be Reeb flow invariant if it satisfies the condition $\mathfrak{L}_\xi Q = 0$, where $\mathfrak{L}_\xi$ denotes the Lie derivative in the direction of the characteristic vector field ξ. For instance, see ([11–13], [35], [37]).

Example 2.1 ([18], Example of a five dimensional Kenmotsu manifold)**.** Let $M^5 = \{(u,v,w,x,y) \in \mathfrak{R}^5\}$ be a five dimensional differentiable manifold, where (u,v,w,x,y) are the standard coordinates in $\mathfrak{R}^5$, and $\mathfrak{R}^5$ denotes the real space of dimension five. If $e_1 = e^{-y}\frac{\partial}{\partial u}$, $e_2 = e^{-y}\frac{\partial}{\partial v}$, $e_3 = e^{-y}\frac{\partial}{\partial w}$, $e_4 = e^{-y}\frac{\partial}{\partial x}$, $e_5 = e^{-y}\frac{\partial}{\partial y}$ are linearly independent vector fields at each point of M^5, then the non-vanishing components of the Lie bracket are given by

$$[e_1,e_5] = e_1, \quad [e_2,e_5] = e_2, \quad [e_3,e_5] = e_3, \quad [e_4,e_5] = e_4.$$

Let g be the Riemannian metric defined by

$$g(e_i, e_j) = \begin{cases} 0 & \text{for} \quad i \neq j, \\ 1 & \text{for} \quad i = j, \quad \forall \;\; i, j = 1, 2, 3, 4, 5. \end{cases}$$

The 1-form η and the $(1, 1)$-tensor ϕ are defined as:

$$g(U, e_5) = \eta(U) \text{ and } \phi e_1 = e_3, \;\; \phi e_2 = e_4, \;\; \phi e_3 = e_1, \;\; \phi e_4 = -e_2, \;\; \phi e_5 = 0$$

for all U of M^5.

From the linearity of ϕ and g, we can easily observed that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(e_5) = 1 \text{ and } g(\phi U, \phi V) = g(U, V) - \eta(U)\eta(V)$$

for all vector fields U and V on M^5. Thus we can say that M^5 defines an almost contact metric manifold. It is obvious that the 1-form η is closed and $\Omega(\frac{\partial}{\partial u}, \frac{\partial}{\partial v}) = -e^{2y}$, where Ω denotes the 2-form defined by $\Omega(U, V) = g(U, \phi V)$ for all vector fields U and V on M^5. Hence $\Omega = -e^{2y} du \wedge dw \implies d\Omega = -2e^{2y} dy \wedge du \wedge dw = 2\eta \wedge \Omega$. Thus $M^5(\phi, \xi, \eta, g)$ is an almost Kenmotsu manifold. Since $M^5(\phi, \xi, \eta, g)$ is normal therefore it is a Kenmotsu manifold.

3. Proof of the main Theorems

Let us suppose that a Kenmotsu manifold M^{2n+1} of dimension $(2n + 1)$ admits a Reeb flow invariant Ricci operator Q, that is,

$$\begin{aligned} 0 &= (\mathfrak{L}_\xi Q)(V), \quad \forall \;\; V \in \chi(M^{2n+1}) \\ &= [\xi, QV] - Q[\xi, V] \\ &= (\nabla_\xi Q)(V) - \nabla_{QV}\xi + Q\nabla_V \xi \\ &= (\nabla_\xi Q)(V), \end{aligned}$$

which shows that the Ricci operator Q is locally symmetric along the vector field ξ. Taking covariant derivative of $Q\xi = -2n\xi$ along the vector field V and then using the equations (6) and (10), we have

$$(\nabla_V Q)(\xi) = -QV - 2nV.$$

The last two equations reveal that

$$(\nabla_\xi Q)(V) - (\nabla_V Q)(\xi) = QV + 2nV. \tag{15}$$

The contraction of the equation (15) along the vector field V together with the result $(div\, Q)(\xi) = \frac{dr(\xi)}{2}$ give

$$dr(\xi) = -2[r + 2n(2n + 1)]. \tag{16}$$

Thus we can state:

Proposition 3.1. *If a* $(2n+1)$*-dimensional Kenmotsu manifold* M^{2n+1} *admits a Reeb flow invariant Ricci operator* Q*, then the equations* (15) *and* (16) *hold on* M^{2n+1}.

Proof of Theorem 1.1. We suppose that M^{2n+1} satisfies the Fischer-Marsden equation $\mathfrak{L}_g^*(\lambda) = 0$, that is, (g, λ) is a non-trivial solution of the equation $\mathfrak{L}_g^*(\lambda) = 0$. Then the equation (1) becomes

$$(\triangle_g \lambda)g + \lambda S_g - Hess_g \lambda = 0,$$

which gives $\triangle_g \lambda = -\frac{r\lambda}{2n}$. Hence we can write the Fischer-Marsden equation as

$$\nabla_U D\lambda = \lambda QU + fU, \ \ U \in \chi(M^{2n+1}),$$

where $f = -\frac{r\lambda}{2n}$. Differentiating the above equation covariantly along the vector field V, we find that

$$R(U, V)D\lambda = (U\lambda)QV - (V\lambda)QU + \lambda\{(\nabla_U Q)(V) - (\nabla_V Q)(U)\} + (Uf)V - (Vf)U. \tag{17}$$

Replacing U by ξ in (17) and then using the equations (10) and (15), we have

$$R(\xi, V)D\lambda = (\xi\lambda)QV + 2n(V\lambda)\xi + \lambda\{QV + 2nV\} + (\xi f)V - (Vf)\xi,$$

which becomes

$$g(R(\xi, V)D\lambda, U) = (\xi\lambda)S(V, U) + 2n(V\lambda)\eta(U) + \lambda\{2ng(V, U) + S(V, U)\} + (\xi f)g(V, U) - (Vf)\eta(U).$$

Again, the equation (9) together with the equation (3) gives

$$g(R(\xi, V)U, D\lambda) = \eta(U)g(V, D\lambda) - g(U, V)g(\xi, D\lambda).$$

The last two equations infer

$$(\xi\lambda)S(V, U) + 2n(V\lambda)\eta(U) + \lambda[2n\, g(V, U) + S(V, U)] - (Vf)\eta(U) + \eta(U)g(V, D\lambda) - g(U, V)g(\xi, D\lambda) + (\xi f)g(V, U) = 0. \tag{18}$$

Let $\{e_i,\ i = 1, 2, 3, \dots, 2n+1\}$ denote a set of orthonormal vectors at each point of the tangent space of M^{2n+1}, then setting $U = V = e_i$ in the equation (18) and taking sum for i, $1 \le i \le 2n+1$, we get

$$2n(\xi f) = -\lambda[r + 2n(2n+1)] - r(\xi\lambda). \tag{19}$$

Since $f = -\frac{r}{2n}\lambda$ and therefore

$$\xi f = -\frac{1}{2n}\{(\xi r)\lambda + r(\xi\lambda)\}. \tag{20}$$

In consequence of the equations (19) and (20), we conclude that

$$\lambda(\xi r) = \lambda[r + 2n(2n+1)].$$

Since we are interested in the non-trivial solution of the Fischer-Marsden equation, therefore $\lambda \neq 0$. Thus the above equation becomes

$$\xi r = r + 2n(2n+1). \tag{21}$$

The equation (16) along with the equation (21) becomes

$$r = -2n(2n+1) \tag{22}$$

and hence

$$Vf = (2n+1)V\lambda, \quad \xi f = (2n+1)\xi\lambda. \tag{23}$$

In view of the equation (23), the equation (18) takes the form

$$[\xi\lambda + \lambda]\{S(V,U) + 2ng(V,U)\} = 0, \tag{24}$$

which reflects that either

$$\xi\lambda = -\lambda \tag{25}$$

or

$$S(V,U) = -2ng(V,U). \tag{26}$$

This completes the proof. □

Proof of Theorem 1.2. The covariant derivative of the equation (13) with respect to the Levi-Civita connection ∇ along the vector field ξ and then using the equations (2), (3), (6) and (7), we get

$$(\nabla_\xi Q)(V) = \frac{dr(\xi)}{2}(V - \eta(V)\xi). \tag{27}$$

Also, differentiating the equation (14) covariantly along the vector field U, we conclude that

$$(\nabla_V Q)(\xi) = -(Q + 2I)\{V - \eta(V)\xi\}. \tag{28}$$

From the equations (27) and (28), we obtain

$$\begin{aligned} g((\nabla_\xi Q)(V) - (\nabla_V Q)(\xi), U) = {} & \frac{dr(\xi)}{2}\{g(V,U) - \eta(V)\eta(U)\} \\ & + S(V,U) + 2g(V,U). \end{aligned} \tag{29}$$

Replacing U by ξ in the equation (17) and then using the equations (2), (3) and (29), we get

$$\begin{aligned} g(R(\xi,V)D\lambda, U) = {} & \frac{(\xi\lambda)}{2}[(r+2)g(V,U) - (r+6)\eta(V)\eta(U)] \\ & + 2(V\lambda)\eta(U) + \lambda\frac{dr(\xi)}{2}(g(V,U) - \eta(V)\eta(U)) \\ & + (\xi f)g(V,U) - (Vf)\eta(U) + \lambda S(V,U) + 2\lambda g(V,U). \end{aligned} \tag{30}$$

Also the equation (9) gives

$$g(R(\xi,V)U, D\lambda) = (V\lambda)\eta(U) - (\xi\lambda)g(U,V). \tag{31}$$

The equation (30) along with the equation (31) takes the form

$$\begin{aligned} & \frac{(\xi\lambda)}{2}[(r+2)g(V,U) - (r+6)\eta(V)\eta(U)] + 3(V\lambda)\eta(U) \\ & + \lambda\frac{dr(\xi)}{2}\{g(V,U) - \eta(V)\eta(U)\} + (\xi f)g(V,U) \\ & - (Vf)\eta(U) + \lambda S(V,U) + 2\lambda g(V,U) - (\xi\lambda)g(U,V) = 0. \end{aligned} \tag{32}$$

Setting $U = V = e_i$ in (32), where $\{e_i, i = 1, 2, 3\}$ is a set of orthonormal vector fields of the tangent space of M^3, and then summing for i, $1 \leq i \leq 3$, we get

$$r(\xi\lambda) + \lambda dr(\xi) + 2(\xi f) = -\lambda(r + 6). \tag{33}$$

Using the equation (33) in the equation (32), we find

$$\begin{aligned}\lambda S(U, V) = {} & \frac{\lambda}{2}(r + 2)g(U, V) + \frac{1}{2}\{6(\xi\lambda) - (r + 6)\lambda - 2(\xi f)\}\eta(U)\eta(V) \\ & - 3(V\lambda)\eta(U) + (V f)\eta(U).\end{aligned} \tag{34}$$

In the light of the equation (12), the equation (34) becomes

$$\{V f - (\xi f)\eta(V)\}\eta(U) - 3\{V\lambda - (\xi\lambda)\eta(V)\}\eta(U) = 0. \tag{35}$$

Replacing U by ξ in the equation (35) and then using the equation (2), we get

$$\{V f - 3(V\lambda)\} - \{\xi f - 3(\xi\lambda)\}\eta(V) = 0. \tag{36}$$

Contracting the equation (28) along the vector field V and using the relation $(divQ)(\xi) = \frac{1}{2}dr(\xi)$, we obtain

$$dr(\xi) = -2(r + 6). \tag{37}$$

Using the equation (37) in the equation (33), we have

$$r(\xi\lambda) + 2(\xi f) = \lambda(r + 6). \tag{38}$$

Since $f = -\frac{r}{2}\lambda$ and therefore with the help of the equation (37), we conclude that

$$2(\xi f) = 2\lambda(r + 6) - r(\xi\lambda). \tag{39}$$

In view of the equations (38) and (39), we obtain

$$r = -6, \tag{40}$$

provided $\lambda \neq 0$. In [16], De and Pathak proved that a 3-dimensional Kenmotsu manifold is a manifold of constant negative curvature if and only if the scalar curvature $r = -6$ (see, p. 160, Lemma 2.1). It is noticed that in a three dimensional Riemannian manifold, the curvature tensor R can be expressed in the following form:

$$\begin{aligned}R(U, V)Z = {} & g(V, Z)QU - g(U, Z)QV + S(V, Z)U - S(U, Z)V \\ & - \frac{r}{2}\{g(V, Z)U - g(U, Z)V\}\end{aligned} \tag{41}$$

for all U, V, $Z \in \chi(M^{2n+1})$. In consequence of the equations (12), (13), (40) and (41), the curvature tensor takes the form

$$R(U, V)Z = -\{g(V, Z)U - g(U, Z)V\}, \tag{42}$$

which reflects that the manifold M^3 under consideration is a hyperbolic space form $H^3(-1)$. This completes the proof. □

A Kenmotsu manifold is said to be locally ϕ-symmetric if $\phi^2(\nabla_W R)(U,V)Z = 0$ for all vector fields U, V, Z, W orthogonal to ξ.

This notion was introduced by Takahashi [33] for Sasakian manifolds.

In [16], De and Pathak proved that a 3-dimensional Kenmotsu manifold is locally ϕ-symmetric if and only if the scalar curvature r is constant. Hence with the help of Theorem 1.2 we state the following:

Corollary 3.2. *Every three dimensional Kenmotsu manifold satisfying the Fischer-Marsden equation $\mathfrak{L}_g^*(\lambda) = 0$ is locally ϕ-symmetric.*

In [16], De and Pathak showed that a 3-dimensional Kenmotsu manifold M^3 possesses a constant scalar curvature r if and only if M^3 satisfies $R(U,V)\cdot S = 0$ (see, p. 162, [16]). These facts together with Theorem 1.2 state the following:

Corollary 3.3. *Suppose that the pair (g,λ) is a non-trivial solution of the Fischer-Marsden equation $\mathfrak{L}_g^*(\lambda) = 0$ on the 3-dimensional Kenmotsu manifold M^3. Then the manifold M^3 is Ricci semisymmetric.*

Note that the Weyl tensor vanishes on any three dimensional Riemannian manifold. Therefore we may consider another conformal invariant of a three dimensional Riemannian manifold, the $(1,1)$ Cotton tensor $\mathfrak{C}(U,V)$, defined by

$$\mathfrak{C}(U,V) = (\nabla_U Q)(V) - (\nabla_V Q)(U) - \frac{1}{4}\{dr(U)(V) - dr(V)(U)\}$$

for all vector fields U and V on M^3 (see, [3, 36]). A three dimensional Riemannian manifold is said to be conformally flat if the Cotton tensor $\mathfrak{C}(U,V)$ vanishes.

Since the manifold under consideration is of constant curvature, that is, the scalar curvature r is constant, therefore the Cotton tensor vanishes. From the above discussions, we conclude the following:

Corollary 3.4. *If a three dimensional Kenmotsu manifold M^3 satisfies the equation $\mathfrak{L}_g^*(\lambda) = 0$, then the Cotton tensor $\mathfrak{C}$ vanishes on M^3.*

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