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ON THE SOLVABILITY OF A NONLINEAR LANGEVIN EQUATION INVOLVING TWO FRACTIONAL ORDERS IN DIFFERENT INTERVALS

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Abstract. This paper deals with a nonlinear Langevin equation involving two fractional orders with three-point boundary conditions. Our aim is to find the existence of solutions for the proposed Langevin equation by using the Banach contraction mapping principle and the Krasnoselskii's fixed point theorem. Three examples are also given to show the significance of our results.

1. Introduction

Fractional differential equations occur in various contexts, such as biology, physics, biophysics, geophysics, fluid dynamics, control theory, etc. For additional subtleties and applications, we allude the peruser to the books [12, 13, 14, 16].

The Langevin equation has been broadly used to depict the evolution of physical processes in fluctuating conditions (see more details in [1, 2, 4]). For example, if white noise is believed to be the random fluctuating force, then the Langevin equation describes the Brownian motion. Nonetheless,

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the ordinary Langevin equations do not grant an accurate description of the elements for frameworks in complicated networks. Therefore, it is easier to substitute the ordinary derivative with a fractional derivative and analyze the fractional Langevin equation. The reader can see published research articles concerning the fractional Langevin equation in [2, 3, 5, 6, 10, 11, 15, 17] and references therein.

Here, we consider the following boundary value problem of the Langevin equation with two different fractional orders

$$\begin{cases} {}^{C}\mathfrak{D}^{\omega}\left({}^{C}\mathfrak{D}^{\vartheta}+\kappa\right)x(t)=\phi(t,x(t)), \quad 0<t<1, \\ x(0)=r_1,\ x(\eta)=r_2,\ x(1)=r_3, \end{cases} \tag{1.1}$$

where $x:[0,1]\to\mathbb{R}$ is an unknown function, ${}^{C}\mathfrak{D}$ is the Caputo fractional derivative, $\phi:[0,1]\times\mathbb{R}\to\mathbb{R}$ is a given continuous function, $0<\vartheta\leq 1$, $1<\omega\leq 2$, $0<\eta<1$ and $\kappa, r_1, r_2, r_3\in\mathbb{R}$.

Our aim is to find the existence of solutions for the Langevin equation (1.1) by using the Banach contraction mapping principle and the Krasnoselskii's fixed point theorem. For this, we first prove a new result for a linear Langevin equation involving two fractional orders in different intervals with three-point boundary conditions. After that, we investigate the existence results for the three-point boundary value problem of nonlinear Langevin equation involving two fractional orders. The first conclusion is based on the idea of the Banach contraction mapping principle, whereas the second finding has relied on the fixed point theorem of the Krasnoselskii. At the end, three examples are also presented which show the significance of our results in this area.

2. Preliminaries

Let $\mathcal{J}=[0,1]$ and $\mathbb{F}=C(\mathcal{J},\mathbb{R})$ denotes the Banach space of all continuous functions from $\mathcal{J}$ into $\mathbb{R}$ endowed with a norm $\|\cdot\|$ defined by $\|x\|=\sup_{t\in\mathcal{J}}|x(t)|$ for all $x\in\mathcal{J}$.

Following definitions and known results will be needed in the sequel.

Definition 2.1. ([7]) The Riemann-Lioville fractional integral of order $\vartheta>0$ for a continuous function $\phi:[0,\infty)\to\mathbb{R}$, denoted by $I^{\vartheta}\phi$, is defined by

$$I^{\vartheta}\phi(\tau)=\frac{1}{\Gamma(\vartheta)}\int_0^{\tau}(\tau-\varsigma)^{\vartheta-1}\phi(\varsigma)d\varsigma$$

equipped with that the integral in the right-hand-side exists, where $\Gamma(\vartheta)$ is defined by

$$\Gamma(\vartheta):=\int_0^{\infty}\tau^{\vartheta-1}e^{-\tau}d\tau.$$

Definition 2.2. ([13]) The Caputo fractional derivative of order $\vartheta > 0$ for a function $\phi : [0,\infty) \to \mathbb{R}$, denoted by ${}^{C}\mathfrak{D}^{\vartheta}\phi$, is defined by

$$ {}^{C}\mathfrak{D}^{\vartheta}\phi(\tau) = \frac{1}{\Gamma(n-\vartheta)} \int_0^{\tau} (\tau-\varsigma)^{n-\vartheta-1}\phi^{(n)}(\varsigma)d\varsigma, $$

where $n \in \mathbb{N}$ with $n-1 < \vartheta \leq n$.

Lemma 2.3. ([7, 13]) *Let $\phi \in L_1(0,1)$ and $\vartheta, \omega > 0$.*

(i) *If $\vartheta \in \mathbb{N}$, then $I^{\vartheta}\phi(\tau) = \frac{1}{(\vartheta-1)!}\int_0^{\tau}(\tau-\varsigma)^{\vartheta-1}\phi(\varsigma)d\varsigma$.*

(ii) *If $\vartheta = n \in \mathbb{N}$, then ${}^{C}\mathfrak{D}^{\vartheta}\phi(\tau) = \phi^{(\vartheta)}(\tau)$.*

(iii) ${}^{C}\mathfrak{D}^{\vartheta}I^{\vartheta}\phi(\tau) = \phi(\tau)$.

(iv) $I^{\vartheta}I^{\omega}\phi(\tau) = I^{\vartheta+\omega}\phi(\tau)$.

Lemma 2.4. ([7]) *For each $\vartheta > 0$, the general solution of the fractional differential equation ${}^{C}\mathfrak{D}^{\vartheta}u(\tau) = 0$ is given by*

$$ u(\tau) = c_0 + c_1\tau + c_2\tau^2 + ... + c_{n-1}\tau^{n-1}, $$

where $c_i \in \mathbb{R}$ for all $i = 0, 1, 2, \ldots, n-1$ and $n = [\vartheta] + 1$.

Lemma 2.5. ([7]) *For each $\vartheta > 0$, we have*

$$ I^{\vartheta}\, {}^{C}\mathfrak{D}^{\vartheta}u(\tau) = u(\tau) + c_0 + c_1\tau + c_2\tau^2 + ... + c_{n-1}\tau^{n-1}, $$

where $c_i \in \mathbb{R}$ for all $i = 0, 1, 2, \ldots, n-1$ and $n = [\vartheta] + 1$.

For solving (1.1), we now discuss the following linear problem for the Langevin equation involving two fractional orders in different intervals:

$$ \begin{cases} {}^{C}\mathfrak{D}^{\omega}\left({}^{C}\mathfrak{D}^{\vartheta} + \kappa\right)x(\tau) = \psi(\tau), \quad 0 < \tau < 1, \\ x(0) = r_1, \; x(\eta) = r_2, \; x(1) = r_3, \end{cases} \tag{2.1} $$

where $x : [0,1] \to \mathbb{R}$ is an unknown function, $\psi \in \mathbb{F}$, $0 < \vartheta \leq 1$, $1 < \omega \leq 2$ and $0 < \eta < 1$.

Lemma 2.6. *The boundary value problem* (2.1) *is equivalent to the integral equation given by*

$$ \begin{aligned} x(\tau) \;=\;& \int_0^{\tau} \frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\xi} \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma - \kappa x(\xi) \right) d\xi \\ & - \frac{\tau^{\vartheta}(1-\tau)}{\eta^{\vartheta}(1-\eta)} \left[\int_0^{\eta} \frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\xi} \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma - \kappa x(\xi) \right) d\xi \right] \\ & + \frac{\tau^{\vartheta}(\eta-\tau)}{(1-\eta)} \left[\int_0^{1} \frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\xi} \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma - \kappa x(\xi) \right) d\xi \right] \\ & + \frac{\tau^{\vartheta}}{(1-\eta)} \left[(\eta-\tau)(r_1-r_3) - \frac{1}{\eta^{\vartheta}}(1-\tau)(r_1-r_2) + \frac{(1-\eta)}{\tau^{\vartheta}} r_1 \right]. \end{aligned} \tag{2.2} $$

Proof. As laid out in [9], the following problem

$$^{C}\mathfrak{D}^{\omega}\left(^{C}\mathfrak{D}^{\vartheta}+\kappa\right)x(\tau)=\psi(\tau)$$

is equivalent to

$$\begin{aligned} x(\tau) &= \int_0^{\tau}\frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma-\kappa x(\xi)\right)d\xi \\ &\quad -c_1\frac{\tau^{\vartheta+1}}{\Gamma(\vartheta+2)}-c_2\frac{\tau^{\vartheta}}{\Gamma(\vartheta+1)}-c_3. \end{aligned} \tag{2.3}$$

Using the boundary conditions for (2.1), we find that

$$\begin{aligned} c_3 &= -r_1, \\ c_2 &= \frac{\vartheta\eta}{(1-\eta)}\left[\frac{1}{\eta^{\vartheta+1}}\int_0^{\eta}(\eta-\xi)^{\vartheta-1}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma-\kappa x(\xi)\right)d\xi\right. \\ &\quad \left.-\int_0^{1}(1-\xi)^{\vartheta-1}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma-\kappa x(\xi)\right)d\xi\right] \\ &\quad +\frac{\Gamma(\vartheta+1)\eta}{(1-\eta)}\left[\frac{1}{\eta^{\vartheta+1}}(r_1-r_2)-(r_1-r_3)\right], \\ c_1 &= \frac{\vartheta(\vartheta+1)}{(1-\eta)}\left[\int_0^{1}(1-\xi)^{\vartheta-1}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma-\kappa x(\xi)\right)d\xi\right. \\ &\quad \left.-\frac{1}{\eta^{\vartheta}}\int_0^{\eta}(\eta-\xi)^{\vartheta-1}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\psi(\varsigma)d\varsigma-\kappa x(\xi)\right)d\xi\right] \\ &\quad +\frac{\Gamma(\vartheta+2)}{(1-\eta)}\left[(r_1-r_3)-\frac{1}{\eta^{\vartheta}}(r_1-r_2)\right]. \end{aligned}$$

Substituting the values of c_1, c_2, c_3 in (2.3), we obtain the result in this theorem. □

Keeping Lemma 2.6 in mind, we mutate Problem (1.1) as the following fixed point problem

$$x=\mathcal{Z}x, \tag{2.4}$$

where $\mathcal{Z}:\mathbb{F}\to\mathbb{F}$ is defined for each $x\in\mathbb{F}$ by

$$
\begin{aligned}
&(\mathcal{Z}x)(\tau)\\
&= \int_0^\tau \frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \\
&\quad - \frac{\tau^\vartheta (1-\tau)}{\eta^\vartheta (1-\eta)} \left[\int_0^\eta \frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \right] \\
&\quad + \frac{\tau^\vartheta (\eta-\tau)}{(1-\eta)} \left[\int_0^1 \frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \right] \\
&\quad + \frac{\tau^\vartheta}{(1-\eta)} \left[(\eta-\tau)(r_1-r_3) - \frac{1}{\eta^\vartheta}(1-\tau)(r_1-r_2) + \frac{(1-\eta)}{\tau^\vartheta} r_1 \right].
\end{aligned}
$$

Observe that Problem (1.1) has solutions if the operator $\mathcal{Z}$ has fixed points.

We now give two famous fixed point results which are main tools for solving the existence of at least one solution of our proposed model (1.1).

Theorem 2.7. (Banach contraction mapping principle) *Let $(\mathcal{X}, \mathcal{D})$ be a complete metric space. Suppose that $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ is a Banach contraction mapping, that is,*

$$
\mathcal{D}(\mathcal{T}x, \mathcal{T}y) \le k\mathcal{D}(x, y) \tag{2.5}
$$

for all $x, y \in \mathcal{X}$, where $k \in [0, 1)$. Then $\mathcal{T}$ has a unique fixed point $a \in \mathcal{X}$, that is, $\mathcal{T}a = a$.

Theorem 2.8. (Krasnoselskii fixed point theorem, [8]) *Let $\mathcal{O}$ be a nonempty, closed and convex subset of a Banach space $\mathcal{B}$. Suppose that $\mathcal{Z}_1, \mathcal{Z}_2 : \mathcal{O} \to \mathcal{B}$ are two operators satisfying the following conditions:*

(1) *$\mathcal{Z}_1 a + \mathcal{Z}_2 b \in \mathcal{O}$ for all $a, b \in \mathcal{O}$;*
(2) *$\mathcal{Z}_1$ is compact and continuous on $\mathcal{O}$;*
(3) *$\mathcal{Z}_2$ is a Banach contraction mapping on $\mathcal{O}$.*

Then $\mathcal{Z}_1 + \mathcal{Z}_2$ has a fixed point, that is, there exists $a \in \mathcal{O}$ such that $\mathcal{Z}_1 a + \mathcal{Z}_2 a = a$.

3. Main results

In this section, we investigate the existence of solutions for the Langevin equation (1.1). For the ease of computing, we set

$$
\mathcal{A} \ : \ = \frac{(\vartheta)^\vartheta}{(1-\eta)(1+\vartheta)^{1+\vartheta}}, \tag{3.1}
$$

$$
\mathcal{F} \ : \ = \frac{\mathcal{A}}{\eta^\vartheta} \left[\eta^{2\vartheta+1} |r_1 - r_3| + |r_1 - r_2| + \frac{\eta^\vartheta}{\mathcal{A}} r_1 \right], \tag{3.2}
$$

$$
\Delta \ : \ = L\Delta_1 + \Delta_2, \tag{3.3}
$$

where

$$\Delta_1 \quad : \quad = \frac{1 + \mathcal{A}(\eta^{\vartheta+1} + \eta^{\omega})}{\Gamma(\vartheta+\omega+1)}, \tag{3.4}$$

$$\Delta_2 \quad : \quad = \frac{|\kappa| \left[1 + \mathcal{A}(1 + \eta^{\vartheta+1})\right]}{\Gamma(\vartheta+1)}. \tag{3.5}$$

Theorem 3.1. *Consider the boundary value problem of the Langevin equation* (1.1)*. Suppose that*

$$|\phi(\tau, s) - \phi(\tau, t)| \leq L\,|s - t| \tag{3.6}$$

for all $\tau \in \mathcal{J}$ and $s, t \in \mathbb{R}$, where $L > 0$ is a constant. Then the Langevin equation (1.1) *has a unique solution provided that $\Delta < 1$, where Δ is defined in* (3.3)*.*

Proof. We first define $\mathcal{Z} : \mathbb{F} \to \mathbb{F}$ for each $x \in \mathbb{F}$ by a function $\mathcal{Z}x$ which is given for each $\tau \in \mathcal{J}$ by

$$\begin{aligned}
&(\mathcal{Z}x)(\tau) \\
&\quad = \int_0^{\tau} \frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\xi} \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \\
&\qquad - \frac{\tau^{\vartheta}(1-\tau)}{\eta^{\vartheta}(1-\eta)} \left[\int_0^{\eta} \frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\xi} \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \right] \\
&\qquad + \frac{\tau^{\vartheta}(\eta-\tau)}{(1-\eta)} \left[\int_0^{1} \frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)} \left(\int_0^{\xi} \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \right] \\
&\qquad + \frac{\tau^{\vartheta}}{(1-\eta)} \left[(\eta-\tau)(r_1 - r_3) - \frac{1}{\eta^{\vartheta}}(1-\tau)(r_1 - r_2) + \frac{(1-\eta)}{\tau^{\vartheta}} r_1 \right].
\end{aligned}$$

Let us set $M := \sup_{\tau \in \mathcal{J}} |\phi(\tau, 0)|$ and choose

$$r \geq \frac{M\Delta_1 + \mathcal{F}}{1 - \Xi}, \tag{3.7}$$

where $\Delta \leq \Xi < 1$.

Now, we will show that $\mathcal{Z}B_r \subset B_r$, where $B_r = \{x \in \mathbb{F} : \|x\| \leq r\}$. For each $x \in B_r$, we have

$$
\begin{aligned}
& \|\mathcal{Z}x\| \\
= & \sup_{\tau\in\mathcal{J}}\left|\int_0^\tau \frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\phi(\varsigma,x(\varsigma))d\varsigma-\kappa x(\xi)\right)d\xi\right. \\
& -\frac{\tau^\vartheta(1-\tau)}{\eta^\vartheta(1-\eta)}\left[\int_0^\eta \frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\phi(\varsigma,x(\varsigma))d\varsigma-\kappa x(\xi)\right)d\xi\right] \\
& +\frac{\tau^\vartheta(\eta-\tau)}{(1-\eta)}\left[\int_0^1 \frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\phi(\varsigma,x(\varsigma))d\varsigma-\kappa x(\xi)\right)d\xi\right] \\
& \left.+\frac{\tau^\vartheta}{(1-\eta)}\left[(\eta-\tau)(r_1-r_3)-\frac{1}{\eta^\vartheta}(1-\tau)(r_1-r_2)+\frac{(1-\eta)}{\tau^\vartheta}r_1\right]\right| \\
\leq & \sup_{\tau\in\mathcal{J}}\left\{\int_0^\tau \frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right. \\
& \times\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\left(|\phi(\varsigma,x(\varsigma)-\phi(\varsigma,0)|+|\phi(\varsigma,0)|\right)d\varsigma+|\kappa x(\xi)|\right)d\xi \\
& +\left|\frac{\tau^\vartheta(\eta-\tau)}{(1-\eta)}\right|\left[\int_0^1 \frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right. \\
& \left.\times\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\left(|\phi(\varsigma,x(\varsigma)-\phi(\varsigma,0)|+|\phi(\varsigma,0)|\right)d\varsigma+|\kappa x(\xi)|\right)d\xi\right] \\
& +\left|\frac{\tau^\vartheta(1-\tau)}{\eta^\vartheta(1-\eta)}\right|\left[\int_0^\eta \frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right. \\
& \times\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\left(|\phi(\varsigma,x(\varsigma)-\phi(\varsigma,0)|+|\phi(\varsigma,0)|\right)d\varsigma+|\kappa x(\xi)|\right)d\xi \\
& \left.+\left|\frac{\tau^\vartheta}{1-\eta}\right|\left(|(\eta-\tau)|\,|r_1-r_3|+\left|\frac{1-\tau}{\eta^\vartheta}\right||r_1-r_2|+\left|\frac{1-\eta}{\tau^\vartheta}\right|r_1\right)\right\} \\
\leq & \sup_{\tau\in\mathcal{J}}\left\{\int_0^\tau \frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right. \\
& \times\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\left(L\,|x(\varsigma)|+|\phi(\varsigma,0)|\right)d\varsigma+|\kappa x(\xi)|\right)d\xi \\
& +\left|\frac{\tau^\vartheta(\eta-\tau)}{(1-\eta)}\right|\left[\int_0^1 \frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right. \\
& \left.\times\left(\int_0^\xi \frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\left(L\,|x(\varsigma)|+|\phi(\varsigma,0)|\right)d\varsigma+|\kappa x(\xi)|\right)d\xi\right]
\end{aligned}
$$

$$
\begin{aligned}
& +\left|\frac{\tau^{\vartheta}(1-\tau)}{\eta^{\vartheta}(1-\eta)}\right|\left[\int_0^{\eta}\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right.\\
& \left.\times\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}\left(L\left|x(\varsigma)\right|+\left|\phi(\varsigma,0)\right|\right)d\varsigma+\left|\kappa x(\xi)\right|\right)d\xi\right]\\
& \left.+\left(\frac{\tau^{\vartheta}}{1-\eta}\right)\left(\left|(\eta-\tau)\right|\left|r_1-r_3\right|+\left(\frac{1-\tau}{\eta^{\vartheta}}\right)\left|r_1-r_2\right|+\left(\frac{1-\eta}{\tau^{\vartheta}}\right)r_1\right)\right\}\\
\leq\ & \sup_{\tau\in\mathcal{J}}\left\{\int_0^{\tau}\frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}d\varsigma\right)d\xi\right\}\left(L\left\|x\right\|+\left|M\right|\right)\\
& +\left|\kappa\right|\left\|x\right\|\left(\sup_{\tau\in\mathcal{J}}\int_0^{\tau}\frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}d\xi\right)\\
& +\sup_{\tau\in\mathcal{J}}\left|\frac{\tau^{\vartheta}(\eta-\tau)}{1-\eta}\right|\left\{\int_0^{1}\frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}d\varsigma\right)d\xi\left(L\left\|x\right\|+\left|M\right|\right)\right.\\
& \left.+\left|\kappa\right|\left\|x\right\|\left(\int_0^{1}\frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}d\xi\right)\right\}\\
& +\sup_{\tau\in\mathcal{J}}\left|\frac{\tau^{\vartheta}(1-\tau)}{\eta^{\vartheta}(1-\eta)}\right|\left\{\int_0^{\eta}\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}d\varsigma\right)d\xi\left(L\left\|x\right\|+\left|M\right|\right)\right.\\
& \left.+\left|\kappa\right|\left\|x\right\|\left(\int_0^{\eta}\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}d\xi\right)\right\}\\
& +\sup_{\tau\in\mathcal{J}}\left|\frac{\tau^{\vartheta}(\eta-\tau)}{1-\eta}\right|\left|r_1-r_3\right|+\sup_{\tau\in\mathcal{J}}\left(\frac{\tau^{\vartheta}}{1-\eta}\right)\left(\frac{1-\tau}{\eta^{\vartheta}}\right)\left|r_1-r_2\right|\\
& \left.+\sup_{\tau\in\mathcal{J}}\left(\frac{\tau^{\vartheta}}{1-\eta}\right)\left(\frac{1-\eta}{\tau^{\vartheta}}\right)r_1\right\}\\
\leq\ & (Lr+M)\left(1+\frac{\eta(\vartheta\eta)^{\vartheta}}{(1-\eta)(1+\vartheta)^{1+\vartheta}}\right)\int_0^{1}\frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}d\varsigma d\xi\\
& +\left|\kappa\right|r\left(1+\frac{\eta(\vartheta\eta)^{\vartheta}}{(1-\eta)(1+\vartheta)^{1+\vartheta}}\right)\int_0^{1}\frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}d\xi\\
& +\left(\frac{(Lr+M)(\vartheta)^{\vartheta}}{(1-\eta)\eta^{\vartheta}(1+\vartheta)^{1+\vartheta}}\right)\int_0^{\eta}\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\int_0^{\xi}\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}d\varsigma d\xi\\
& +\left(\frac{\left|\kappa\right|r(\vartheta)^{\vartheta}}{(1-\eta)\eta^{\vartheta}(1+\vartheta)^{1+\vartheta}}\right)\int_0^{\eta}\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}d\xi\\
& +\frac{\eta(\vartheta\eta)^{\vartheta}}{(1-\eta)(1+\vartheta)^{1+\vartheta}}\left|r_1-r_3\right|+\frac{(\vartheta)^{\vartheta}}{(1-\eta)\eta^{\vartheta}(1+\vartheta)^{1+\vartheta}}\left|r_1-r_2\right|+r_1.
\end{aligned}
$$

Using (3.1)-(3.5), (3.7) and the following relations:

$$B(\omega+1,\vartheta)=\int_0^1(1-\xi)^{\vartheta-1}\xi^{\omega}d\xi=\frac{\Gamma(\vartheta)\Gamma(\omega+1)}{\Gamma(\vartheta+\omega+1)}$$

and

$$\int_0^\eta(\eta-\xi)^{\vartheta-1}\xi^{\omega}d\xi=\frac{\Gamma(\vartheta)\Gamma(\omega+1)}{\Gamma(\vartheta+\omega+1)}\eta^{\vartheta+\omega},$$

where $B(\cdot,\cdot)$ is a Beta function, then we obtain that

$$\begin{aligned}\|\mathcal{Z}x\| &\le (\Delta+1-\Xi)r\\ &\le r.\end{aligned}$$

Now, for each $x,y\in\mathbb{F}$, we obtain

$$\begin{aligned}
&\|\mathcal{Z}x-\mathcal{Z}y\|\\
=\;& \sup_{\tau\in\mathcal{J}}|(\mathcal{Z}x)(\tau)-(\mathcal{Z}y)(\tau)|\\
\le\;& \sup_{\tau\in\mathcal{J}}\left\{\int_0^\tau\frac{(\tau-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right.\\
&\times\left(\int_0^\xi\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}|\phi(\varsigma,x(\varsigma))-\phi(\varsigma,y(\varsigma))|\,d\varsigma+|\kappa|\,|x(\xi)-y(\xi)|\right)d\xi\\
&+\left|\frac{\tau^\vartheta(1-\tau)}{\eta^\vartheta(1-\eta)}\right|\left[\int_0^\eta\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right.\\
&\times\left.\left(\int_0^\xi\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}|\phi(\varsigma,x(\varsigma))-\phi(\varsigma,y(\varsigma))|\,d\varsigma+|\kappa|\,|x(\xi)-y(\xi)|\right)d\xi\right]\\
&+\left|\frac{\tau^\vartheta(\eta-\tau)}{(1-\eta)}\right|\left[\int_0^1\frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\right.\\
&\times\left.\left.\left(\int_0^\xi\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}|\phi(\varsigma,x(\varsigma))-\phi(\varsigma,y(\varsigma))|\,d\varsigma+|\kappa|\,|x(\xi)-y(\xi)|\right)d\xi\right]\right\}\\
\le\;& L\left(1+\frac{\eta(\vartheta\eta)^\vartheta}{(1-\eta)(1+\vartheta)^{1+\vartheta}}\right)\int_0^1\frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\int_0^\xi\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}d\varsigma d\xi\,\|x-y\|\\
&+|\kappa|\left(1+\frac{\eta(\vartheta\eta)^\vartheta}{(1-\eta)(1+\vartheta)^{1+\vartheta}}\right)\int_0^1\frac{(1-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}d\xi\,\|x-y\|\\
&+\left(\frac{(\vartheta)^\vartheta L}{(1-\eta)\eta^\vartheta(1+\vartheta)^{1+\vartheta}}\right)\int_0^\eta\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}\int_0^\xi\frac{(\xi-\varsigma)^{\omega-1}}{\Gamma(\omega)}d\varsigma d\xi\,\|x-y\|\\
&+\left(\frac{|\kappa|\,(\vartheta)^\vartheta}{(1-\eta)\eta^\vartheta(1+\vartheta)^{1+\vartheta}}\right)\int_0^\eta\frac{(\eta-\xi)^{\vartheta-1}}{\Gamma(\vartheta)}d\xi\,\|x-y\|\\
=\;& \Delta\,\|x-y\|,
\end{aligned}$$

where Δ is given by (3.3) and relies only on the parameters involved in the problem. As $\Delta < 1$, therefore $\mathcal{Z}$ is a Banach contraction mapping. Hence, the theorem's inference is accompanied by the idea of the Banach contraction mapping principle. □

Theorem 3.2. *Consider the boundary value problem of the Langevin equation* (1.1)*. Suppose that ϕ maps a bounded subset of $\mathcal{J} \times \mathbb{R}$ into relatively compact subset of $\mathbb{R}$. Furthermore, assume that the following conditions hold:*

(H_1) *there exists $L > 0$ such that*

$$|\phi(\tau, s) - \phi(\tau, t)| \leq L\,|s - t| \tag{3.8}$$

for all $\tau \in \mathcal{J}$ and $s, t \in \mathbb{R}$;

(H_2) *there exists $\sigma \in \mathbb{F}$ such that*

$$|\phi(\tau, s)| \leq \sigma(\tau) \tag{3.9}$$

for all $\tau \in \mathcal{J}$ and $s \in \mathbb{R}$.

If

$$\Theta := \frac{L\mathcal{A}}{\Gamma(\vartheta + \omega + 1)}(\eta^{\vartheta+1} + \eta^{\omega}) + \frac{|\kappa|\,\mathcal{A}}{\Gamma(\vartheta + 1)}(1 + \eta^{\vartheta+1}) < 1, \tag{3.10}$$

where $\mathcal{A}$ is given in (3.1)*, then the Langevin equation* (1.1) *has at least one solution.*

Proof. In view of (H_2), let us fix

$$r \geq [\|\sigma\|\,\Delta_1 + \mathcal{F}]\left(\left|1 - \frac{|\kappa|\left[1 + \mathcal{A}(1 + \eta^{\vartheta+1})\right]}{\Gamma(\vartheta + 1)}\right|\right)^{-1}. \tag{3.11}$$

Let $B_r := \{x \in \mathbb{F} : \|x\| \leq r\}$. We define operators $\mathcal{Z}_1, \mathcal{Z}_2 : B_r \to \mathbb{F}$ for each $x \in B_r$ by

$$(\mathcal{Z}_1 x)(\tau) = \int_0^\tau \frac{(\tau - \xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\xi \frac{(\xi - \varsigma)^{\omega-1}}{\Gamma(\omega)}\phi(\varsigma, x(\varsigma))d\varsigma - \kappa x(\xi)\right)d\xi,$$

$$\begin{aligned}
&(\mathcal{Z}_2 x)(\tau)\\
&= \frac{\tau^{\vartheta}(\eta - \tau)}{(1 - \eta)}\left[\int_0^1 \frac{(1 - \xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\xi \frac{(\xi - \varsigma)^{\omega-1}}{\Gamma(\omega)}\phi(\varsigma, x(\varsigma))d\varsigma - \kappa x(\xi)\right)d\xi\right]\\
&\quad - \frac{\tau^{\vartheta}(1 - \tau)}{\eta^{\vartheta}(1 - \eta)}\left[\int_0^\eta \frac{(\eta - \xi)^{\vartheta-1}}{\Gamma(\vartheta)}\left(\int_0^\xi \frac{(\xi - \varsigma)^{\omega-1}}{\Gamma(\omega)}\phi(\varsigma, x(\varsigma))d\varsigma - \kappa x(\xi)\right)d\xi\right]\\
&\quad + \frac{\tau^{\vartheta}}{(1 - \eta)}\left[(\eta - \tau)(r_1 - r_3) - \frac{1}{\eta^{\vartheta}}(1 - \tau)(r_1 - r_2) + \frac{(1 - \eta)}{\tau^{\vartheta}}r_1\right].
\end{aligned}$$

Then, for each $x, y \in B_r$, it follows from (3.11) that

$$\|\mathcal{Z}_1 x + \mathcal{Z}_2 y\| \leq \left(\|\sigma\| \, \Delta_1 + \frac{|\kappa| \, r \left[1 + \mathcal{A}(1 + \eta^{\vartheta+1})\right]}{\Gamma(\vartheta+1)} + \mathcal{F} \right) \leq r.$$

Thus $\mathcal{Z}_1 x + \mathcal{Z}_2 y \in B_r$. In view of condition (3.10), it can be shown that $\mathcal{Z}_2$ is a Banach contraction mapping. The continuity of ϕ implies that the operator $\mathcal{Z}_1$ is continuous. Also, $\mathcal{Z}_1$ is uniformly bounded on B_r as

$$\|\mathcal{Z}_1 x\| \leq \frac{\|\sigma\|}{\Gamma(\vartheta+\omega+1)} + \frac{|\kappa| \, r}{\Gamma(\vartheta+1)}.$$

Now, we will prove the compactness of the operator $\mathcal{Z}_1$. Setting $\Omega := \mathcal{J} \times B_r$, we define $\bar{\phi} := \sup\limits_{(\tau,x)\in\Omega} |\phi(\tau, x(\tau))|$. Consequently, we have

$$\begin{aligned}
|(\mathcal{Z}_1 x)(\tau_1) - (\mathcal{Z}_1 x)(\tau_2)| = & \left| \int_0^{\tau_1} \frac{(\tau_1 - \xi)^{\vartheta-1}}{\Gamma(\vartheta)} \right. \\
& \times \left(\int_0^{\xi} \frac{(\xi - \varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \\
& - \int_0^{\tau_2} \frac{(\tau_2 - \xi)^{\vartheta-1}}{\Gamma(\vartheta)} \\
& \left. \times \left(\int_0^{\xi} \frac{(\xi - \varsigma)^{\omega-1}}{\Gamma(\omega)} \phi(\varsigma, x(\varsigma)) d\varsigma - \kappa x(\xi) \right) d\xi \right| \\
\leq & \frac{\bar{\phi}}{\Gamma(\vartheta+\omega+1)} \left| \tau_1^{\vartheta+\omega} - \tau_2^{\vartheta+\omega} \right| + \frac{|\kappa| \, r}{\Gamma(\vartheta+1)} \left| \tau_1^{\vartheta} - \tau_2^{\vartheta} \right|,
\end{aligned}$$

which is independent of x and tends to zero as $\tau_2 \to \tau_1$. Thus, $\mathcal{Z}_1$ is relatively compact on B_r. Hence, by the Arzelá-Ascoli theorem, $\mathcal{Z}_1$ is compact on B_r. Thus all assumptions of Theorem 2.8 are satisfied and the conclusion of Theorem 2.8 implies that the boundary value problem of the Langevin equation (1.1) has at least one solution. □

Example 3.3. Consider the following boundary value problem:

$$\begin{cases} {}^{C}\mathfrak{D}^{\frac{5}{2}} \left({}^{C}\mathfrak{D}^{\frac{1}{3}} + \frac{1}{6} \right) x(\tau) = \frac{1}{(\tau+2)^2} \frac{[x(\tau)]^2}{1+[x(\tau)]^2}, \quad 0 < \tau < 1, \\ x(0) = r_1, \quad x\left(\frac{1}{8}\right) = r_2, \quad x(1) = r_3. \end{cases} \tag{3.12}$$

Now, we set a function $\phi : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ by

$$\phi(\tau, s) = \frac{1}{(\tau+2)^2} \frac{s^2}{1+s^2}$$

for all $\tau \in \mathcal{J}$ and $s \in \mathbb{R}$. By setting $\vartheta = \frac{1}{3}$, $\omega = \frac{5}{2}$, $\eta = \frac{1}{8}$ and $\kappa = \frac{1}{6}$, the boundary value problem (3.12) has the form of (1.1). Clearly,

$$|\phi(\tau, s) - \phi(\tau, t)| \leq L\,|s - t|$$

for all $\tau \in \mathcal{J}$ and $s, t \in \mathbb{R}$, where $L = \frac{1}{4}$. Further, we have

$$\Delta = L\Delta_1 + \Delta_2 \approx 0.34 < 1.$$

Thus, by Theorem 3.1, the boundary value problem (3.12) has a unique solution.

Example 3.4. Consider the following boundary value problem:

$$\begin{cases} {}^{C}\mathfrak{D}^{\frac{3}{2}}\left({}^{C}\mathfrak{D}^{\frac{1}{2}} + \frac{1}{4}\right)x(\tau) = \frac{1}{15}\left(x(\tau)\cos\tau\right) - 1, & 0 < \tau < 1, \\ x(0) = 0, \quad x\left(\frac{1}{5}\right) = 0, \quad x(1) = 0. \end{cases} \tag{3.13}$$

Here, we set a function $\phi : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ by

$$\phi(\tau, s) = \frac{1}{15}\left(s\cos\tau\right) - 1$$

for all $\tau \in \mathcal{J}$ and $s \in \mathbb{R}$. By setting $\vartheta = \frac{1}{2}$, $\omega = \frac{3}{2}$, $\eta = \frac{1}{5}$ and $\kappa = \frac{1}{2}$, the boundary value problem (3.13) has the form of (1.1). Clearly,

$$|\phi(\tau, x) - \phi(\tau, y)| \leq L\,|x - y|$$

for all $\tau \in \mathcal{J}$ and $s, t \in \mathbb{R}$, where $L = \frac{1}{15}$. Further, we have

$$\Delta = L\Delta_1 + \Delta_2 \approx 0.90 < 1$$

Thus, by Theorem 3.1, the boundary value problem (3.13) has a unique solution.

Example 3.5. Consider the following boundary value problem:

$$\begin{cases} {}^{C}\mathfrak{D}^{\frac{6}{5}}\left({}^{C}\mathfrak{D}^{\frac{4}{5}} + \frac{1}{4}\right)x(\tau) = \frac{1}{(\tau+3)^2}\frac{[x(\tau)]^2}{1+[x(\tau)]^2}, & 0 < \tau < 1, \\ x(0) = \frac{3}{10}, \quad x\left(\frac{3}{10}\right) = \frac{9}{10}, \quad x(1) = \frac{23}{10}. \end{cases} \tag{3.14}$$

Here, we set a function $\phi : \mathcal{J} \times \mathbb{R} \to \mathbb{R}$ by

$$\phi(\tau, s) = \frac{1}{(\tau + 3)^2}\frac{s^2}{1 + s^2}$$

for all $\tau \in \mathcal{J}$ and $s \in \mathbb{R}$. By setting $\vartheta = \frac{4}{5}$, $\omega = \frac{6}{5}$, $\eta = \frac{3}{10}$ and $\kappa = \frac{1}{4}$, the boundary value problem (3.14) has the form of (1.1). Clearly

$$|\phi(\tau, s) - \phi(\tau, t)| \leq L\,|s - t|$$

for all $\tau \in \mathcal{J}$ and $s, t \in \mathbb{R}$, where $L = \frac{1}{9}$. Further, we have $\mathcal{A} \approx 0.415$ and

$$\begin{aligned} \Theta &= \frac{L\mathcal{A}}{\Gamma(\vartheta+\omega+1)}(\eta^{\vartheta+1}+\eta^{\omega}) + \frac{|\kappa|\,\mathcal{A}}{\Gamma(\vartheta+1)}(1+\eta^{\vartheta+1}) \\ &\approx 0.4955 \\ &< 1. \end{aligned}$$

Thus, by Theorem 3.2, the boundary value problem (3.14) has at least one solution.

4. Conclusion

In this paper, we have discussed the existence and uniqueness of solutions of nonlinear Langevin equation for a three-point boundary value problem involving two fractional orders in different intervals. We used the Banach and Krasnoselskii fixed point theorems to find out the desire results. Indeed, our method is straightforward and can be quickly extended to several real-world situations. Three examples illustrating our approach are also discussed.

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