



COMMON FIXED POINT THEOREMS UNDER RATIONAL CONTRACTIONS IN COMPLEX VALUED EXTENDED b -METRIC SPACES

V. Vairaperumal¹, J. Carmel Pushpa Raj², J. Maria Joseph³
and M. Marudai⁴

¹PG and Research Department of Mathematics, St. Josephs College(Autonomous)
Tiruchirappalli-620 002, Tamilnadu, India
e-mail: vairamv3@gmail.com

²PG and Research Department of Mathematics, St. Josephs College(Autonomous)
Tiruchirappalli-620 002, Tamilnadu, India
e-mail: carmelsjc@gmail.com

³PG and Research Department of Mathematics, St. Josephs College(Autonomous)
Tiruchirappalli-620 002, Tamilnadu, India
e-mail: joseph80john@gmail.com

⁴Department of Mathematics, Bharathidasan University
Tiruchirappalli-620 024, Tamilnadu, India
e-mail: mmarudai@yahoo.co.in

Abstract. In this paper, we discuss the existence and uniqueness of fixed point and common fixed point theorems in complex valued extended b -metric spaces for a pair of mappings satisfying some rational contraction conditions which generalized and unify some well-known results in the literature.

1. INTRODUCTION

The fixed point theory is one of the important tools in nonlinear analysis, science and engineering. In 1992, Banach [3] introduced the famous Banach

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⁰Corresponding author: V. Vairaperumal(vairamv3@gmail.com).

contraction theorem. After this remarkable contribution, many researchers proved generalization of Banach contraction theorem in metric spaces and generalized metric spaces [8, 12]. Bakhtin [2] and Czerwik [5] extended fixed point theorems in b -metric spaces which were generalizations of the Banach contraction principle. Subsequently, Azam et al. [1] introduced the concept of complex valued metric spaces. Then many authors [4, 6, 7, 9, 10, 11, 13, 14] presented fixed point theorems on single-valued and multi-valued mapping in b -metric spaces and complex valued metric spaces. In the article, we prove fixed point theorems in complex valued extended b -metric space using rational contractions.

2. PRELIMINARIES

We give some definitions and their properties for our main results.

Definition 2.1. ([1]) Let \mathbb{C} be the set of complex numbers and $z_1, z_2 \in \mathbb{C}$. Define a partial order \preceq on \mathbb{C} as follows:

$z_1 \preceq z_2$ if and only if $Re(z_1) \leq Re(z_2)$ and $Im(z_1) \leq Im(z_2)$. Thus $z_1 \preceq z_2$ if one of the followings holds:

- (1) $Re(z_1) = Re(z_2)$ and $Im(z_1) = Im(z_2)$;
- (2) $Re(z_1) < Re(z_2)$ and $Im(z_1) = Im(z_2)$;
- (3) $Re(z_1) = Re(z_2)$ and $Im(z_1) < Im(z_2)$;
- (4) $Re(z_1) < Re(z_2)$ and $Im(z_1) < Im(z_2)$.

We will write $z_1 \succ z_2$ if $z_1 \neq z_2$ and one of (2), (3) and (4) is satisfied; also we will write $z_1 \prec z_2$ if only (4) is satisfied.

It follows that

- (i) $0 \preceq z_1 \succ z_2$ implies $|z_1| < |z_2|$;
- (ii) $z_1 \preceq z_2$ and $z_2 \prec z_3$ imply $z_1 \prec z_3$;
- (iii) $0 \preceq z_1 \preceq z_2$ implies $|z_1| \leq |z_2|$;
- (iv) if $a, b \in \mathbb{R}$, $0 \leq a \leq b$ and $z_1 \preceq z_2$, then $az_1 \preceq bz_2$ for all $z_1, z_2 \in \mathbb{C}$.

Definition 2.2. ([1]) Let W be a nonempty set. A function $d_{cv} : W \times W \rightarrow \mathbb{C}$ is called a complex valued metric on W , if for all $l, m, n \in W$, the following conditions are satisfied:

- (i) $0 \preceq d_{cv}(l, m)$ and $d_{cv}(l, m) = 0$ if and only if $l = m$;
- (ii) $d_{cv}(l, m) = d_{cv}(m, l)$;
- (iii) $d_{cv}(l, m) \preceq d_{cv}(l, n) + d_{cv}(n, m)$.

Then, the pair (W, d_{cv}) is called a complex valued metric space.

Example 2.3. ([1]) Let $W = [0, 1]$ and $l, m \in W$. Define $d_{cv} : W \times W \rightarrow \mathbb{C}$ by

$$d_{cv}(l, m) = \begin{cases} 0 & \text{if } l = m, \\ \frac{i}{2} & \text{if } l \neq m. \end{cases} \tag{2.1}$$

Then d_{cv} is a complex valued metric and hence (W, d_{cv}) is a complex valued metric space.

Definition 2.4. ([1]) Let (W, d_{cv}) be a complex valued metric space.

- (i) We say that a point $l \in W$ is an interior point of a set $M \subseteq W$, whenever there exists $0 \prec r \in \mathbb{C}$ such that

$$B(l, r) = \{m \in W : d_{cv}(m, l) \prec r\},$$

- (ii) We say that a point $l \in W$ is a limit point of a set $M \subseteq W$, whenever for every $0 \prec r \in \mathbb{C}$ such that

$$B(l, r) \cap M - \{l\} \neq \emptyset.$$

Definition 2.5. ([2], [5]) Let W be a nonempty set and $s \geq 1$ be a given real number. A function $d_b : W \times W \rightarrow [0, \infty)$ is called a b -metric on W if for all $l, m, n \in W$, the following conditions are satisfied:

- (b1) $d_b(l, m) = 0$ if and only if $l = m$;
- (b2) $d_b(l, m) = d_b(m, l)$;
- (b3) $d_b(l, m) \leq s[d_b(l, n) + d_b(n, m)]$.

Then, the pair (W, d_b) is called a b -metric space.

Example 2.6. ([4]) Let $W = L_p[0, 1]$ be the space of all real functions $l(t)$, $t \in [0, 1]$ such that $\int_0^1 |l(t)|^p dt < \infty$ with $0 < p < 1$. Define $d_b : W \times W \rightarrow \mathbb{R}^+$ as:

$$d_b(l, m) = \left(\int_0^1 |l(t) - m(t)|^p dt \right)^{\frac{1}{p}}$$

then (W, d_b) is a b -metric space with coefficient $s = 2^{\frac{1}{p}}$.

Definition 2.7. ([13]) Let W be a nonempty set and let $s \geq 1$ be a given real number. A function $d_{cvb} : W \times W \rightarrow \mathbb{C}$ is called a complex valued b -metric on W if for all $l, m, n \in W$, the following conditions are satisfied:

- (i) $0 \preceq d_{cvb}(l, m)$ and $d_{cvb}(l, m) = 0$ if and only if $l = m$;
- (ii) $d_{cvb}(l, m) = d_{cvb}(m, l)$;

(iii) $d_{cvb}(l, m) \preceq s[d_{cvb}(l, n) + d_{cvb}(n, m)]$.

Then, the pair (W, d_{cvb}) is called a complex valued b -metric space.

Example 2.8. ([13]) If $W = [0, 1]$, define the mapping $d_{cvb} : W \times W \rightarrow \mathbb{C}$ by

$$d_{cvb}(l, m) = |l - m|^2 + i|l - m|^2$$

for all $l, m \in W$. Then (W, d_{cvb}) is a complex valued b -metric space with $s = 2$.

Definition 2.9. ([10]) Let W be a non-empty set and $\lambda : W \times W \rightarrow [1, \infty)$ be a function. Then $d_\lambda : W \times W \rightarrow [0, \infty)$ is called an extended b -metric if for all $l, m, n \in W$ it satisfies:

- (i) $d_\lambda(l, m) = 0$ if and only if $l = m$;
- (ii) $d_\lambda(l, m) = d_\lambda(m, l)$;
- (iii) $d_\lambda(l, n) \leq \lambda(l, n)[d_\lambda(l, m) + d_\lambda(m, n)]$.

Then, the pair (W, d_λ) is called an extended b -metric space.

Example 2.10. Let $W = \{1, 2, 3\}$. Define $\lambda : W \times W \rightarrow [1, \infty)$ and $d_\lambda : W \times W \rightarrow \mathbb{R}^+$ as:

$$\begin{aligned} \lambda(l, m) &= 1 + l + m, \\ d_\lambda(1, 1) &= d_\lambda(2, 2) = d_\lambda(3, 3) = 0, \\ d_\lambda(1, 2) &= d_\lambda(2, 1) = 80, \quad d_\lambda(1, 3) = d_\lambda(3, 1) = 1000, \\ d_\lambda(2, 3) &= d_\lambda(3, 2) = 600. \end{aligned}$$

Then (W, d_λ) is an extended b -metric space.

Definition 2.11. ([10]) Let (W, d_λ) be an extended b -metric space.

- (i) A sequence $\{l_n\}$ in W is said to converge to $l \in W$, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\lambda(l_n, l) < \epsilon$, for all $n \geq N$. In this case, we write $\lim_{n \rightarrow \infty} l_n = l$.
- (ii) A sequence $\{l_n\}$ in W is said to be Cauchy, if for every $\epsilon > 0$ there exists $N = N(\epsilon) \in \mathbb{N}$ such that $d_\lambda(l_m, l_n) < \epsilon$, for all $m, n \geq N$.
- (iii) If every Cauchy sequence in W is convergent, then (W, d_λ) is said to be a complete extended b -metric space.

Lemma 2.12. ([10]) *Let (W, d_λ) be an extended b -metric space. If d_λ is continuous, then every convergent sequence has a unique limit.*

Definition 2.13. ([14]) Let W be a non-empty set and $\theta : W \times W \rightarrow [1, \infty)$ be a function. Then $d_\theta : W \times W \rightarrow \mathbb{C}$ is known as a complex valued extended b -metric space if the following conditions are satisfied for all $l, m, n \in W$:

- (i) $0 \preceq d_\theta(l, m)$ and $d_\theta(l, m) = 0$ if and only if $l = m$;
- (ii) $d_\theta(l, m) = d_\theta(m, l)$;
- (iii) $d_\theta(l, n) \preceq \theta(l, n)[d_\theta(l, m) + d_\theta(m, n)]$.

Then the pair (W, d_θ) is called a complex valued extended b -metric space.

Example 2.14. If W be a nonempty set and $\theta : W \times W \rightarrow [1, \infty]$ be defined as:

$$\theta(l, m) = \frac{1 + l + m}{l + m},$$

further, let

- (i) $d_\theta(l, m) = \frac{i}{lm}$ for all $l, m \in (0, 1]$;
- (ii) $d_\theta(l, m) = 0 \iff l = m$ for all $l, m \in [0, 1]$;
- (iii) $d_\theta(l, 0) = d_\theta(0, l) = \frac{i}{l}$ for all $l \in (0, 1]$.

Then the pair (W, d_θ) is known as a complex valued extended b -metric space.

Example 2.15. Let $W = [0, \infty)$. $\theta : W \times W \rightarrow [1, \infty)$ be a function defined by $\theta(l, m) = 1 + l + m$ and $d_\theta : W \times W \rightarrow \mathbb{C}$ be given as

$$d_\theta(l, m) = \begin{cases} 0 & \text{if } l = m \\ i & \text{if } l \neq m \end{cases}.$$

Then (W, d_θ) is a complex valued extended b - metric space.

3. MAIN RESULTS

Now, we can state our main results.

Theorem 3.1. *Let (W, d_θ) be a complete complex valued extended b -metric space, let $\theta : W \times W \rightarrow [1, \infty)$ and let U, V be self-mappings from W into itself satisfy the following inequality:*

$$d_\theta(Ul, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta(l, Ul)d_\theta(m, Vm)}{d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m)} + \mu_3 \left[\frac{d_\theta(l, Vl)d_\theta(m, Vm)[1 + d_\theta(m, Ul)d_\theta(l, m)]}{d_\theta(l, Vm) + d_\theta(l, m)} \right] \tag{3.1}$$

for all $l, m \in W$ such that $l \neq m, d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m) \neq 0, d_\theta(l, Vm) + d_\theta(l, m) \neq 0$ where μ_1, μ_2 and μ_3 are non negative reals with $\mu_1 + \theta(l_1, l_2)(\mu_2 + \mu_3) < 1, \zeta = \mu_1 + (\mu_2 + \mu_3)\theta(l_1, l_2)$ where $\zeta \in [0, \infty), \lim_{n, m \rightarrow \infty} \theta(l_n, l_m) < \frac{1}{\zeta}$ or $d_\theta(Ul, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m) = 0, d_\theta(l, Vm) + d_\theta(l, m) = 0$, then U and V have a unique common fixed point in W .

Proof. For any arbitrary point $l_0 \in W$, define a sequence $\{l_n\}$ in W such that

$$l_{2n+1} = Ul_{2n} \text{ and } l_{2n+2} = Vl_{2n+1}, \quad \forall n \geq 0. \tag{3.2}$$

Now we prove that $\{l_n\}$ is a Cauchy sequence.

Let $l = l_0, m = l_1$ in (3.1). Then

$$\begin{aligned} d_\theta(l_1, l_2) &= d_\theta(Ul_0, Vl_1) \\ &\preceq \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, Ul_0)d_\theta(l_1, Vl_1)}{d_\theta(l_0, Vl_1) + d_\theta(l_1, Ul_0) + d_\theta(l_0, l_1)} \\ &\quad + \mu_3 \left[\frac{d_\theta(l_0, Vl_0)d_\theta(l_1, Vl_1)[1 + d_\theta(l_1, Ul_0)d_\theta(l_0, l_1)]}{d_\theta(l_0, Vl_1) + d_\theta(l_0, l_1)} \right] \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, l_1)d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_1, l_1) + d_\theta(l_0, l_1)} \\ &\quad + \mu_3 \left[\frac{d_\theta(l_0, l_1)d_\theta(l_1, l_2)[1 + d_\theta(l_1, l_1)d_\theta(l_0, l_1)]}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} \right] \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta(l_0, l_1)d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} + \mu_3 \left[\frac{d_\theta(l_0, l_1)d_\theta(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} \right]. \end{aligned}$$

Hence, we have

$$\begin{aligned} |d_\theta(l_1, l_2)| &= \mu_1 |d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|} \\ &\quad + \mu_3 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|}. \end{aligned}$$

Using triangle inequality

$$d_\theta(l_1, l_2) \leq \theta(l_1, l_2)[d_\theta(l_1, l_0) + d_\theta(l_0, l_2)].$$

Thus we have

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \mu_1 |d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_1, l_2)|} |\theta(l_1, l_2)| \\ &\quad + \mu_3 \frac{|d_\theta(l_0, l_1)||d_\theta(l_1, l_2)|}{|d_\theta(l_1, l_2)|} |\theta(l_1, l_2)| \\ &= [\mu_1 + (\mu_2 + \mu_3)\theta(l_1, l_2)] |d_\theta(l_0, l_1)|, \end{aligned}$$

that is,

$$|d_\theta(l_1, l_2)| \leq [\mu_1 + (\mu_2 + \mu_3)\theta(l_1, l_2)] |d_\theta(l_0, l_1)|.$$

Since $|d_\theta(l_1, l_2)| < 1 + |d_\theta(l_1, l_2)|$, we have

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \zeta |d_\theta(l_0, l_1)|, \\ |d_\theta(l_2, l_3)| &\leq \zeta^2 |d_\theta(l_0, l_1)|, \\ |d_\theta(l_3, l_4)| &\leq \zeta^3 |d_\theta(l_0, l_1)|, \\ &\vdots \\ |d_\theta(l_n, l_{n+1})| &\leq \zeta^n |d_\theta(l_0, l_1)|. \end{aligned}$$

Now, by triangle inequality, for any $m > n$, $m, n \in \mathbb{N}$, we have

$$\begin{aligned} d_\theta(l_n, l_m) &\preceq \theta(l_n, l_m) \zeta^n d_\theta(l_0, l_1) + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \zeta^{n+1} d_\theta(l_0, l_1) \dots \\ &\quad + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \dots \theta(l_{m-2}, l_m) \theta(l_{m-1}, l_m) \zeta^{m-1} d_\theta(l_0, l_1). \end{aligned}$$

Then

$$\begin{aligned} d_\theta(l_n, l_m) &\preceq d_\theta(l_0, l_1) [\theta(l_n, l_m) \zeta^n + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \zeta^{n+1} \dots \\ &\quad + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \dots \theta(l_{m-2}, l_m) \theta(l_{m-1}, l_m) \zeta^{m-1}]. \end{aligned}$$

Since $\lim_{n, m \rightarrow \infty} \theta(l_n, l_m) \zeta < 1$, so the series $\sum_{n=1}^\infty \zeta^n \prod_{i=1}^n \theta(l_i, l_m)$ converges by ratio test for each $m \in \mathbb{N}$. Let

$$S = \sum_{n=1}^\infty \zeta^n \prod_{i=1}^n \theta(l_i, l_m), \quad S_n = \sum_{j=1}^n \zeta^j \prod_{i=1}^j \theta(l_i, l_m).$$

Thus, for $m > n$, the above can be written as

$$d_\theta(l_n, l_m) \preceq d_\theta(l_0, l_1) [S_{m-1} - S_n]$$

and

$$|d_\theta(l_n, l_m)| \leq |d_\theta(l_0, l_1)| [S_{m-1} - S_n].$$

Letting $m \rightarrow \infty$, we obtain

$$|d_\theta(l_n, l_m)| \rightarrow 0.$$

Thus, $\{l_n\}$ is a Cauchy sequence in W . Since W is complete, there exists some $t \in W$ such that $l_n \rightarrow t$ as $n \rightarrow \infty$.

If not, then there exists $z \in W$ such that

$$|d_\theta(t, Ut)| = |z| > 0. \tag{3.3}$$

So, using the triangular inequality and (3.1), we have

$$\begin{aligned}
 z &= d_\theta(t, Ut) \\
 &\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(l_{2n+2}, Ut) \\
 &= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(Vl_{2n+1}, Ut) \\
 &\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t) \left[\mu_1 d_\theta(t, l_{2n+1}) \right. \\
 &\quad \left. + \mu_2 \frac{d_\theta(t, Ut)d_\theta(l_{2n+1}, Vl_{2n+1})}{d_\theta(t, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ut) + d_\theta(t, l_{2n+1})} \right. \\
 &\quad \left. + \mu_3 \left[\frac{d_\theta(l_{2n+1}, Ul_{2n+1})d_\theta(t, Ut) [1 + d_\theta(t, Vl_{2n+1})d_\theta(l_{2n+1}, t)]}{d_\theta(l_{2n+1}, Ut) + d_\theta(l_{2n+1}, t)} \right] \right] \\
 &= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \mu_1 \theta(t, U_t)d_\theta(t, l_{2n+1}) \\
 &\quad + \mu_2 \theta(t, U_t) \frac{d_\theta(t, Ut)d_\theta(l_{2n+1}, l_{2n+2})}{d_\theta(t, l_{2n+2}) + d_\theta(l_{2n+1}, Ut) + d_\theta(t, l_{2n+1})} \\
 &\quad + \mu_3 \theta(t, U_t) \left[\frac{d_\theta(l_{2n+1}, l_{2n+2})d_\theta(t, Ut) [1 + d_\theta(t, l_{2n+2})d_\theta(l_{2n+1}, t)]}{d_\theta(l_{2n+1}, Ut) + d_\theta(l_{2n+1}, t)} \right].
 \end{aligned}$$

And so,

$$\begin{aligned}
 |z| &= |d_\theta(t, Ut)| \\
 &\leq \theta(t, U_t)|d_\theta(t, l_{2n+2})| + \mu_1 \theta(t, U_t)|d_\theta(t, l_{2n+1})| \\
 &\quad + \mu_2 \theta(t, U_t) \frac{|d_\theta(t, Ut)||d_\theta(l_{2n+1}, l_{2n+2})|}{|d_\theta(t, l_{2n+2})| + |d_\theta(l_{2n+1}, Ut)| + |d_\theta(t, l_{2n+1})|} \\
 &\quad + \mu_3 \theta(t, U_t) \left[\frac{|d_\theta(l_{2n+1}, l_{2n+2})||d_\theta(t, Ut)| [1 + |d_\theta(t, l_{2n+2})||d_\theta(l_{2n+1}, t)]|}{|d_\theta(l_{2n+1}, Ut)| + |d_\theta(l_{2n+1}, t)|} \right].
 \end{aligned}$$

As $n \rightarrow \infty$, we obtain that $|z| = |d_\theta(t, Ut)| \leq 0$, a contradiction. Thus, $|z| = 0$. Hence, $Ut = t$. Similarly, we obtain $Vt = t$.

Now, we show that U and V have a unique common fixed point. To prove this, assume that $t' \neq t$ is another common fixed point of U and V . Then

$$\begin{aligned}
 d_\theta(t, t') &= d_\theta(Ut, Vt') \\
 &\preceq \mu_1 d_\theta(t, t') + \mu_2 \frac{d_\theta(t, Ut)d_\theta(t', Vt')}{d_\theta(t, Vt') + d_\theta(t', Ut) + d_\theta(t, t')} \\
 &\quad + \mu_3 \frac{d_\theta(t, Vt)d_\theta(t', Vt') [1 + d_\theta(t', Ut)d_\theta(t, t')]}{d_\theta(t, Vt') + d_\theta(t, t')}.
 \end{aligned} \tag{3.4}$$

And so, we have

$$|d_\theta(t, t')| \leq \mu_1 |d_\theta(t, t')| + \mu_2 \frac{|d_\theta(t, Ut)||d_\theta(t', Vt')|}{|d_\theta(t, Vt')| + |d_\theta(t', Ut)| + |d_\theta(t, t')|} + \mu_3 \frac{|d_\theta(t, Vt)||d_\theta(t', Vt')|[1 + |d_\theta(t', Ut)||d_\theta(t, t')]|}{|d_\theta(t, Vt')| + |d_\theta(t, t')|},$$

this implies that

$$|d_\theta(t, t')| \leq \mu_1 |d_\theta(t, t')|,$$

which is a contradiction. Hence, $t = t'$ which shows the uniqueness of common fixed point in W .

Now, we consider the second case:

Step 1: $d_\theta(l, Vm) + d_\theta(m, Ul) + d_\theta(l, m) = 0$, $l = l_{2n}$ and $m = l_{2n+1}$. $d_\theta(l_{2n}, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ul_{2n}) + d_\theta(l_{2n}, l_{2n+1}) = 0$, $d_\theta(Ul_{2n}, Vl_{2n+1}) = 0$. So that $l_{2n} = Ul_{2n} = l_{2n+1} = Vl_{2n+1} = l_{2n+2}$. Thus, we have $l_{2n+1} = Ul_{2n} = l_{2n}$, so there exist E_1 and f_1 such that

$$E_1 = Uf_1 = f_1,$$

where $E_1 = l_{2n+1}$ and $f_1 = l_{2n}$.

Using foregoing arguments, we show that there exist E_2 and f_2 such that

$$E_2 = Vf_2 = f_2,$$

where $E_2 = l_{2n+2}$ and $f_2 = l_{2n+1}$.

As, $d_\theta(f_1, Vf_2) + d_\theta(f_2, Uf_1) + d_\theta(f_1, f_2) = 0$ which implies $d_\theta(Uf_1, Vf_2) = 0$. Since $E_1 = Uf_1 = Vf_2 = E_2$, we obtain $E_1 = Uf_1 = UE_1$. Similarly, we have $E_2 = VE_2$.

As $E_1 = E_2$, then $UE_1 = VE_1 = E_1$. Hence $E_1 = E_2$ is common fixed point of U and V .

For uniqueness of common fixed point, assume that E'_1 in W is another common fixed point of U and V . Then we have $UE'_1 = VE'_1 = E'_1$. As $d_\theta(E_1, VE'_1) + d_\theta(E'_1, UE_1) + d_\theta(E_1, E'_1) = 0$, therefore, we have

$$d_\theta(E_1, E'_1) = d_\theta(UE_1, VE'_1) = 0.$$

This implies that $E_1 = E'_1$.

Step 2: $d_\theta(l, Vm) + d_\theta(l, m) = 0$. $l = l_{2n}$ and $m = l_{2n+1}$ in this expression, we get, $d_\theta(l_{2n}, Vl_{2n+1}) + d_\theta(l_{2n}, l_{2n+1}) = 0$, it implies that $d_\theta(Ul_{2n}, Vl_{2n+1}) = 0$. So that $l_{2n} = Ul_{2n} = l_{2n+1} = Vl_{2n+1} = l_{2n+2}$. Thus, we have $l_{2n+1} = Ul_{2n} = l_{2n}$, so there exist E_3 and f_3 such that

$$E_3 = Uf_3 = f_3,$$

where $E_3 = l_{2n+1}$ and $f_3 = l_{2n}$.

Using foregoing arguments, we show that there exist E_4 and f_4 such that

$$E_4 = Vf_4 = f_4,$$

where $E_4 = l_{2n+2}$ and $f_4 = l_{2n+1}$. As, $d_\theta(f_3, Vf_4) + d_\theta(f_3, f_4) = 0$ which implies $d_\theta(Uf_3, Vf_4) = 0$. Hence $E_3 = Uf_3 = Vf_4 = E_4$. Thus we obtain $E_3 = Uf_3 = UE_3$. Similarly, we have $E_4 = VE_4$.

As $E_3 = E_4$, implies $UE_3 = VE_3 = E_3$. Hence $E_3 = E_4$ is common fixed point of U and V .

For uniqueness of common fixed point, assume that E'_3 in W is another common fixed point of U and V . Then we have $UE'_3 = VE'_3 = E'_3$. As $d_\theta(E_3, VE'_3) + d_\theta(E_3, E'_3) = 0$, therefore, we have

$$d_\theta(E_3, E'_3) = d_\theta(UE_3, VE'_3) = 0.$$

This implies that $E_3 = E'_3$. This completes the proof of the theorem. □

Corollary 3.2. *Let (W, d_θ) be a complete complex valued extended b-metric space, let $\theta : W \times W \rightarrow [1, \infty)$ and let V be a self-mapping from W into itself satisfy the following inequality:*

$$d_\theta(Vl, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta(l, Vl)d_\theta(m, Vm)}{d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m)} + \mu_3 \left[\frac{d_\theta(l, Vl)d_\theta(m, Vm)[1 + d_\theta(m, Vl)d_\theta(l, m)]}{d_\theta(l, Vm) + d_\theta(l, m)} \right] \tag{3.5}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m) \neq 0$, $d_\theta(l, Vm) + d_\theta(l, m) \neq 0$ where μ_1, μ_2 and μ_3 are nonnegative reals with $\mu_1 + \theta(l_1, l_2)(\mu_2 + \mu_3) < 1$, $\zeta = \mu_1 + (\mu_2 + \mu_3)\theta(l_1, l_2)$ where $\zeta \in [0, \infty)$, $\lim_{n, m \rightarrow \infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Vl, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Vl) + d_\theta(l, m) = 0$, $d_\theta(l, Vm) + d_\theta(l, m) = 0$, then V has a unique common fixed point in W .

Proof. We can prove this result by applying Theorem 3.1 with the condition $U = V$. □

Theorem 3.3. *Let (W, d_θ) be a complete complex valued extended b-metric space, let $\theta : W \times W \rightarrow [1, \infty)$ and let U, V be self-mappings from W into itself satisfy the following inequality,*

$$d_\theta(Ul, Vm) \preceq \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta^2(l, Vm) + d_\theta^2(m, Ul)}{d_\theta(l, Vm) + d_\theta(m, Ul)} + \mu_3 [d_\theta(l, Ul) + d_\theta(m, Vm)] + \mu_4 [d_\theta(l, m) + d_\theta(m, Ul)] + \mu_5 \frac{d_\theta^2(m, Vm)}{d_\theta(l, Vm) + d_\theta(l, m)} \tag{3.6}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Ul) \neq 0$, $d_\theta(l, Vm) + d_\theta(l, m) \neq 0$ where $\mu_1, \mu_2, \mu_3, \mu_4$ and μ_5 are nonnegative reals with $\mu_1 + 2\theta(l_0, l_2)\mu_2 + 2\mu_3 + \mu_4 + \theta(l_0, l_2)\mu_5 < 1$, $\zeta(1 - (\mu_2 + \mu_5)\theta(l_0, l_2) - \mu_3) = (\mu_1 + \mu_2\theta(l_0, l_2) + \mu_3 + \mu_4)$ where $\zeta \in [0, \infty)$, $\lim_{n, m \rightarrow \infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Ul, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Ul) = 0$, $d_\theta(l, Vm) + d_\theta(l, m) = 0$, then U and V have a unique common fixed point in W .

Proof. For any arbitrary point $l_0 \in W$, define a sequence $\{l_n\}$ in W such that

$$l_{2n+1} = Ul_{2n} \quad \text{and} \quad l_{2n+2} = Vl_{2n}, \quad \forall n \geq 0. \quad (3.7)$$

Now we prove that $\{l_n\}$ is a Cauchy sequence. Let $l = l_0, m = l_1$ in (3.6). Then, we have

$$\begin{aligned} d_\theta(l_1, l_2) &= d_\theta(Ul_0, Vl_1) \\ &\leq \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{[d_\theta^2(l_0, Vl_1) + d_\theta^2(l_1, Ul_0)]}{d_\theta(l_0, Vl_1) + d_\theta(l_1, Ul_0)} \\ &\quad + \mu_3 [d_\theta(l_0, Ul_0) + d_\theta(l_1, Vl_1)] \\ &\quad + \mu_4 [d_\theta(l_0, l_1) + d_\theta(l_1, Ul_0)] + \mu_5 \frac{d_\theta^2(l_1, Vl_1)}{d_\theta(l_0, Vl_1) + d_\theta(l_0, l_1)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{[d_\theta^2(l_0, l_2) + d_\theta^2(l_1, l_1)]}{d_\theta(l_0, l_2) + d_\theta(l_1, l_1)} + \mu_3 [d_\theta(l_0, l_1) + d_\theta(l_1, l_2)] \\ &\quad + \mu_4 [d_\theta(l_0, l_1) + d_\theta(l_1, l_1)] + \mu_5 \frac{d_\theta^2(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)} \\ &= \mu_1 d_\theta(l_0, l_1) + \mu_2 \frac{d_\theta^2(l_0, l_2)}{d_\theta(l_0, l_2)} + \mu_3 [d_\theta(l_0, l_1) + d_\theta(l_1, l_2)] \\ &\quad + \mu_4 d_\theta(l_0, l_1) + \mu_5 \frac{d_\theta^2(l_1, l_2)}{d_\theta(l_0, l_2) + d_\theta(l_0, l_1)}. \end{aligned}$$

And so,

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \mu_1 |d_\theta(l_0, l_1)| + \mu_2 \frac{|d_\theta^2(l_0, l_2)|}{|d_\theta(l_0, l_2)|} + \mu_3 [|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] \\ &\quad + \mu_4 |d_\theta(l_0, l_1)| + \mu_5 \frac{|d_\theta^2(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|}, \end{aligned}$$

that is,

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \mu_1 |d_\theta(l_0, l_1)| + \mu_2 |d_\theta(l_0, l_2)| + \mu_3 [|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] \\ &\quad + \mu_4 |d_\theta(l_0, l_1)| + \mu_5 \frac{|d_\theta^2(l_1, l_2)|}{|d_\theta(l_0, l_2)| + |d_\theta(l_0, l_1)|}. \end{aligned}$$

Using the triangle inequality, we have

$$d_\theta(l_0, l_2) \leq \theta(l_0, l_2)[d_\theta(l_0, l_1) + d_\theta(l_1, l_2)].$$

Thus, we have

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \mu_1 |d_\theta(l_0, l_1)| + \mu_2 \theta(l_0, l_2)[|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] \\ &\quad + \mu_3[|d_\theta(l_0, l_1)| + |d_\theta(l_1, l_2)|] \\ &\quad + \mu_4 |d_\theta(l_0, l_1)| + \mu_5 \theta(l_0, l_2) |d_\theta(l_1, l_2)| \\ &= (\mu_1 + \mu_2 \theta(l_0, l_2) + \mu_3 + \mu_4) |d_\theta(l_0, l_1)| \\ &\quad + ((\mu_2 + \mu_5) \theta(l_0, l_2) + \mu_3) |d_\theta(l_1, l_2)|, \end{aligned}$$

it implies that

$$|d_\theta(l_1, l_2)| \leq \frac{(\mu_1 + \mu_2 \theta(l_0, l_2) + \mu_3 + \mu_4)}{(1 - (\mu_2 + \mu_5) \theta(l_0, l_2) - \mu_3)} |d_\theta(l_0, l_1)|.$$

Then, we obtain

$$|d_\theta(l_1, l_2)| \leq \zeta |d_\theta(l_0, l_1)|.$$

Similarly, we have

$$\begin{aligned} |d_\theta(l_1, l_2)| &\leq \zeta |d_\theta(l_0, l_1)|, \\ |d_\theta(l_2, l_3)| &\leq \zeta^2 |d_\theta(l_0, l_1)|, \\ |d_\theta(l_3, l_4)| &\leq \zeta^3 |d_\theta(l_0, l_1)|, \\ &\vdots \\ |d_\theta(l_n, l_{n+1})| &\leq \zeta^n |d_\theta(l_0, l_1)|. \end{aligned}$$

Now, by triangle inequality, for any $m > n$, $m, n \in \mathbb{N}$ we have

$$\begin{aligned} d_\theta(l_n, l_m) &\preceq \theta(l_n, l_m) \zeta^n d_\theta(l_0, l_1) + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \zeta^{n+1} d_\theta(l_0, l_1) \dots \\ &\quad + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \dots \theta(l_{m-2}, l_m) \theta(l_{m-1}, l_m) \zeta^{m-1} d_\theta(l_0, l_1). \end{aligned}$$

Then

$$\begin{aligned} d_\theta(l_n, l_m) &\preceq d_\theta(l_0, l_1) [\theta(l_n, l_m) \zeta^n + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \zeta^{n+1} \dots \\ &\quad + \theta(l_n, l_m) \theta(l_{n+1}, l_m) \dots \theta(l_{m-2}, l_m) \theta(l_{m-1}, l_m) \zeta^{m-1}]. \end{aligned}$$

Since $\lim_{n,m \rightarrow \infty} \theta(l_n, l_m) \zeta < 1$, series $\sum_{n=1}^{\infty} \zeta^n \prod_{i=1}^n \theta(l_i, l_m)$ converges by ratio test for each $m \in \mathbb{N}$.

Let

$$S = \sum_{n=1}^{\infty} \zeta^n \prod_{i=1}^n \theta(l_i, l_m), \quad S_n = \sum_{j=1}^n \zeta^j \prod_{i=1}^j \theta(l_i, l_m).$$

Then, for $m > n$, the above expression can be written as

$$d_\theta(l_n, l_m) \preceq d_\theta(l_0, l_1)[S_{m-1} - S_n]$$

and

$$|d_\theta(l_n, l_m)| \leq |d_\theta(l_0, l_1)|[S_{m-1} - S_n].$$

Letting $n \rightarrow \infty$, we get

$$|d_\theta(l_n, l_m)| \rightarrow 0.$$

Thus, $\{l_n\}$ is a Cauchy sequence in W . Since W is complete, there exists some $t \in W$ such that $l_n \rightarrow t$ as $n \rightarrow \infty$. If not, then there exists $z \in W$ such that

$$|d_\theta(t, Ut)| = |z| > 0. \quad (3.8)$$

Using the triangle inequality, we have

$$\begin{aligned} z &= d_\theta(t, Ut) \\ &\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(l_{2n+2}, Ut) \\ &= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)d_\theta(Vl_{2n+1}, Ut) \\ &\preceq \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)\mu_1d_\theta(t, l_{2n+1}) \\ &\quad + \theta(t, U_t)\mu_2 \frac{[d_\theta^2(t, Vl_{2n+1}) + d_\theta^2(l_{2n+1}, Ut)]}{d_\theta(t, Vl_{2n+1}) + d_\theta(l_{2n+1}, Ut)} \\ &\quad + \theta(t, U_t)\mu_3[d_\theta(t, Ut) + d_\theta(l_{2n+1}, Vl_{2n+1})] \\ &\quad + \theta(t, U_t)\mu_4[d_\theta(t, l_{2n+1}) + d_\theta(l_{2n+1}, Ut)] \\ &\quad + \theta(t, U_t)\mu_5 \frac{[d_\theta^2(l_{2n+1}, Vl_{2n+1})]}{d_\theta(t, Vl_{2n+1}) + d_\theta(t, l_{2n+1})} \\ &= \theta(t, U_t)d_\theta(t, l_{2n+2}) + \theta(t, U_t)\mu_1d_\theta(t, l_{2n+1}) \\ &\quad + \theta(t, U_t)\mu_2 \frac{[d_\theta^2(t, l_{2n+2}) + d_\theta^2(l_{2n+1}, Ut)]}{d_\theta(t, l_{2n+2}) + d_\theta(l_{2n+1}, Ut)} \\ &\quad + \theta(t, U_t)\mu_3[d_\theta(t, Ut) + d_\theta(l_{2n+1}, l_{2n+2})] \\ &\quad + \theta(t, U_t)\mu_4[d_\theta(t, l_{2n+1}) + d_\theta(l_{2n+1}, Ut)] \\ &\quad + \theta(t, U_t)\mu_5 \frac{[d_\theta^2(l_{2n+1}, l_{2n+2})]}{d_\theta(t, l_{2n+2}) + d_\theta(t, l_{2n+1})}. \end{aligned}$$

Thus, we have

$$\begin{aligned}
 |z| = |d_\theta(t, Ut)| \leq & |\theta(t, Ut)| \left(|d_\theta(t, l_{2n+2})| + \mu_1 |d_\theta(t, l_{2n+1})| \right. \\
 & + \mu_2 \frac{[|d_\theta^2(t, l_{2n+2})| + |d_\theta^2(l_{2n+1}, Ut)]}{|d_\theta(t, l_{2n+2})| + |d_\theta(l_{2n+1}, Ut)|} \\
 & + \mu_3 [|d_\theta(t, Ut)| + |d_\theta(l_{2n+1}, l_{2n+2})|] \\
 & + \mu_4 [|d_\theta(t, l_{2n+1})| + |d_\theta(l_{2n+1}, Ut)|] \\
 & \left. + \mu_5 \frac{|d_\theta^2(l_{2n+1}, l_{2n+2})|}{|d_\theta(t, l_{2n+2})| + |d_\theta(t, l_{2n+1})|} \right).
 \end{aligned}$$

As $n \rightarrow \infty$, we obtain that $|z| = |d_\theta(t, Ut)| \leq 0$, which is a contradiction. Thus, $|z| = 0$. Hence, $Ut = t$. Similarly, we obtain $Vt = t$.

Now, we show that U and V have a unique common fixed point. To prove this, assume that $t' \neq t$ is another common fixed point of U and V . Then

$$\begin{aligned}
 d_\theta(t, t') &= d_\theta(Ut, Vt') \\
 &\leq \mu_1 d_\theta(t, t') + \mu_2 \frac{[d_\theta^2(t, Vt') + d_\theta^2(t', Ut)]}{d_\theta(t, Vt') + d_\theta(t', Ut)} + \mu_3 [d_\theta(t, Ut) + d_\theta(t', Vt')] \\
 &\quad + \mu_4 [d_\theta(t, t') + d_\theta(t', Ut)] + \mu_5 \frac{d_\theta^2(t', Vt')}{d_\theta(t, Vt') + d_\theta(t, t')}.
 \end{aligned}$$

And also, we have

$$\begin{aligned}
 |d_\theta(t, t')| \leq & \mu_1 |d_\theta(t, t')| + \mu_2 \frac{[|d_\theta^2(t, Vt')| + |d_\theta^2(t', Ut)|]}{|d_\theta(t, Vt')| + |d_\theta(t', Ut)|} \\
 & + \mu_3 [|d_\theta(t, Ut)| + |d_\theta(t', Vt')|] \\
 & + \mu_4 [|d_\theta(t, t')| + |d_\theta(t', Ut)|] + \mu_5 \frac{|d_\theta^2(t', Vt')|}{|d_\theta(t, Vt')| + |d_\theta(t, t')|},
 \end{aligned}$$

that is,

$$|d_\theta(t, t')| \leq (\mu_1 + \mu_2 + 2\mu_4) |d_\theta(t, t')|,$$

which is a contradiction. Hence $t = t'$ which shows the uniqueness of common fixed point in W .

For the second case, $d_\theta(Ul, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Ul) = 0$, the proof of uniqueness of common fixed point can be completed in the line of Theorem 3.1. This completes the proof of the theorem. □

Corollary 3.4. *Let (W, d_θ) be a complete complex valued extended b -metric space, let $\theta : W \times W \rightarrow [1, \infty)$ and let V be a self-mapping from W into itself satisfy the following inequality:*

$$\begin{aligned} d_\theta(Vl, Vm) \leq & \mu_1 d_\theta(l, m) + \mu_2 \frac{d_\theta^2(l, Vm) + d_\theta^2(m, Vl)}{d_\theta(l, Vm) + d_\theta(m, Vl)} \\ & + \mu_3 [d_\theta(l, Vl) + d_\theta(m, Vm)] + \mu_4 [d_\theta(l, m) + d_\theta(m, Vl)] \quad (3.9) \\ & + \mu_5 \frac{d_\theta^2(m, Vm)}{d_\theta(l, Vm) + d_\theta(l, m)} \end{aligned}$$

for all $l, m \in W$, such that $l \neq m$, $d_\theta(l, Vm) + d_\theta(m, Vl) \neq 0$, $d_\theta(l, Vm) + d_\theta(l, m) \neq 0$ where $\mu_1, \mu_2, \mu_3, \mu_4$ and μ_5 are nonnegative reals with $\mu_1 + 2\theta(l_0, l_2)\mu_2 + 2\mu_3 + \mu_4 + \theta(l_0, l_2)\mu_5 < 1$, $\zeta(1 - (\mu_2 + \mu_5)\theta(l_0, l_2) - \mu_3) = (\mu_1 + \mu_2\theta(l_0, l_2) + \mu_3 + \mu_4)$ where $\zeta \in [0, \infty)$, $\lim_{n, m \rightarrow \infty} \theta(l_n, l_m) < \frac{1}{\zeta}$. or $d_\theta(Vl, Vm) = 0$ if $d_\theta(l, Vm) + d_\theta(m, Vl) = 0$, $d_\theta(l, Vm) + d_\theta(l, m) = 0$, then V has a unique common fixed point in W .

Proof. By using Theorem 3.3 with $U = V$, we can prove this result. \square

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