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EXISTENCE AND UNIQUENESS RESULTS FOR SYSTEM OF FRACTIONAL DIFFERENTIAL EQUATIONS WITH INITIAL TIME DIFFERENCE

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Abstract. Existence and uniqueness results for solutions of system of Riemann-Liouville (R-L) fractional differential equations with initial time difference are obtained. Monotone technique is developed to obtain existence and uniqueness of solutions of system of R-L fractional differential equations with initial time difference.

1. Introduction

Theory of fractional differential equations [7, 9, 17] parallel to the well-known theory of ordinary differential equations [5, 6] has been attracted researchers. Due to wide range of applications of fractional calculus in sciences, engineering, nature and social sciences numerous methods of solving fractional differential equations are developed [11, 12]. Lakshmikantham et al. [8] studied

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local and global existence results for solutions of Riemann-Liouville fractional differential equations. Monotone iterative method for Riemann-Liouville fractional differential equations with initial conditions is studied by McRae [10]. Devi obtained [1] the general monotone method for periodic boundary value problem of Caputo fractional differential equations. The Caputo fractional differential equation with periodic boundary conditions have been studied in [2, 3] and developed monotone method for the problem. Existence and uniqueness of solution of Riemann-Liouville fractional differential equation with integral boundary conditions is proved in [14, 15].

Recently, initial value problems involving Riemann-Liouville fractional derivative was studied by authors [4, 16]. Yaker et al. studied existence and uniqueness of solutions of fractional differential equations with initial time difference for locally Holder continuous functions [18]. Authors have generalized these results for the class of continuous functions [13].

Monotone iterative technique is a powerful technique to study qualitative properties of solutions such as existence and uniqueness of solutions of fractional differential equations. As population models, pharmacodynamic models and economic models etc.are governed by system of fractional differential equations many researchers attracted towards such models and studied existence and uniqueness of solutions of system of fractional differential equations. This motivates us to study system of nonlinear fractional differential equations with initial time difference.

In this paper, we consider the system of Riemann-Liouville fractional differential equations with initial time difference when the function on the right hand side is quasi-monotone non-decreasing and construct two monotone convergent sequences to obtain existence and uniqueness of solution for the nonlinear system.

The paper is organized as follows: In section 2, basic definitions and results are given. Section 3 is devoted to develop monotone technique to study existence and uniqueness results for the considered system. An example is given to validate the obtained results.

2. Preliminaries

Basic definitions and results required to develop monotone technique for the system are given in this section.

Definition 2.1. ([17]) The Riemann-Liouville fractional derivative of order q $(0 < q < 1)$ is defined as

$$D_a^q u(t) = \frac{1}{\Gamma(n-q)} \left(\frac{d}{dt}\right)^n \int_a^t (t-\tau)^{n-q-1} u(\tau) d\tau, \quad \text{for } a \leq t \leq b.$$

Lemma 2.2. ([1]) *Let* $m \in C_p(J, \mathbb{R})$ *and for any* $t_1 \in (t_0, T]$ *we have* $m(t_1) = 0$ *and* $m(t) < 0$ *for* $t_0 \le t \le t_1$. *Then* $D^q m(t_1) \ge 0$.

Theorem 2.3. ([14]) *Let* $v, w \in C_p([t_0, T], \mathbb{R}), f \in C([t_0, T] \times \mathbb{R}, \mathbb{R})$ *and*

$$D^q v(t) \le f(t, v(t)), \quad D^q w(t) \ge f(t, w(t)), \quad t_0 < t \le T.$$

Assume $f(t, u)$ *satisfy one sided Lipschitz condition*

$$f(t, u) - f(t, v) \le L(u - v), \quad u \ge v, \ L > 0.$$

Then $v^0 < w^0$, *where* $v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}$ *and* $w^0 = w(t)(t-t_0)^{1-q}|_{t=t_0}$, *implies* $v(t) \le w(t), t \in [t_0, T]$.

Corollary 2.4. ([14]) *The function* $f(t, u) = \sigma(t)u$, *where* $\sigma(t) \le L$, *is admissible in Theorem* 2.3 *to yield* $u(t) \le 0$ *on* $t_0 \le t \le T$.

The results proved by Yakar et al. for the following problem:

$$D^q u(t) = f(t, u), \ u(t)(t-t_0)^{1-q}|_{t=t_0} = u^0, \tag{2.1}$$

where $0 < q < 1$, $f \in C[\mathbb{R}^+ \times \mathbb{R}, \mathbb{R}]$, are generalized by authors [13] for the class of continuous functions $u(t)$. These results will be stated in Theorem 2.5 and Theorem 2.6.

The corresponding Volterra fractional integral equation is given by

$$u(t) = u^0(t) + \frac{1}{\Gamma(q)} \int_{t_0}^{t} (t-s)^{q-1} f(s, u(s)) ds, \tag{2.2}$$

where

$$u^0(t) = \frac{u(t)(t-t_0)^{1-q}}{\Gamma(q)}$$

and that every solution of (2.2) is a solution of (2.1).

Theorem 2.5. ([13]) *Assume that*

(i) $v \in C_p[J, \mathbb{R}], t_0, T > 0, w \in C_p^*[J^*, \mathbb{R}]$ *is continuous and* $p = 1 - q$ *where*

$$C_p(J, \mathbb{R}) = \{u(t) \in C(J, \mathbb{R}) \ \textit{and} \ u(t)(t-t_0)^p \in C(J, \mathbb{R})\},$$

$$C_p^*(J^*, \mathbb{R}) = \{u(t) \in C(J^*, \mathbb{R}) \ \textit{and} \ u(t)(t-\tau_0)^p \in C(J^*, \mathbb{R})\},$$

$$f \in C[[t_0, \tau_0 + T] \times \mathbb{R}, \mathbb{R}], J = [t_0, t_0 + T], J^* = [\tau_0, \tau_0 + T] \ \textit{and}$$

$$D^q v(t) \le f(t, v(t)), \quad t_0 \le t \le t_0 + T,$$

$$D^q w(t) \ge f(t, w(t)), \quad \tau_0 \le t \le \tau_0 + T,$$

$$v^0 \le u^0 \le w^0,$$

$$\textit{where } v^0 = v(t)(t-t_0)^{1-q}|_{t=t_0}, w^0 = w(t)(t-\tau_0)^{1-q}|_{t=\tau_0},$$

(ii) $f(t,u)$ *satisfies Lipschitz condition:*

$$f(t,u) - f(t,v) \leq L[u-v], \textit{for } u \geq v, \textit{ and } L \geq 0,$$

(iii) $\tau_0 > t_0$ *and* $f(t,u)$ *is nondecreasing in* t *for each* u.

Then we have

(a) $v(t) \leq w(t+\eta), t_0 \leq t \leq t_0 + T,$

(b) $v(t-\eta) \leq w(t), \tau_0 \leq t \leq \tau_0 + T,$ *where* $\eta = \tau_0 - t_0$.

Theorem 2.6. ([13]) *Assume that*

(i) *Assumption* (i) *of Theorem* 2.5 *holds.*

(ii) $f(t,u)$ *is nondecreasing in* t *for each* u *and* $v(t) \leq w(t+\eta)$, $t_0 \leq t \leq t_0 + T$, *where* $\eta = \tau_0 - t_0$.

Then there exists a solution $u(t)$ *of* (2.1) *with* $u^0 = u(t)(t-t_0)^{1-q}|_{t=t_0}$ *satisfying* $v(t) \leq u(t) \leq w(t+\eta)$ *on* $[t_0, t_0+T]$.

In this paper, we develop monotone technique coupled with lower and upper solutions for the class of continuous functions for the following system of Riemann-Liouville fractional differential equations with initial time difference and obtain existence and uniqueness of solution for the system using monotone technique.

$$\begin{aligned} D^q u_1(t) &= f_1(t, u_1(t), u_2(t)), \quad u_1(t)(t-t_0)^{1-q}|_{t=t_0} = {u_1}^0, \\ D^q u_2(t) &= f_2(t, u_1(t), u_2(t)), \quad u_2(t)(t-\tau_0)^{1-q}|_{t=t_0} = {u_2}^0, \end{aligned} \tag{2.3}$$

where $t \in J = [t_0, t_0+T]$ f_1, f_2 in $C(J \times \mathbb{R}^2, \mathbb{R}), 0 < q < 1$.

Definition 2.7. A pair of functions $v = (v_1, v_2)$ and $w = (w_1, w_2)$ in $C_p(J, \mathbb{R}^2)$, $p = 1-q$ are said to be ordered lower and upper solutions $(v_1, v_2) \leq (w_1, w_2)$ of the problem (2.3) if

$$D^q v_i(t) \leq f_i(t, v_1(t), v_2(t)), \qquad v_i(t)(t-t_0)^{1-q}|_{t=t_0} = {v_i}^0$$

and

$$D^q w_i(t) \geq f_i(t, w_1(t), w_2(t)), \quad w_i(t)(t-\tau_0)^{1-q}|_{t=\tau_0} = {w_i}^0.$$

Definition 2.8. A function $f_i = f_i(t, u_1, u_2)$ in $C(J \times \mathbb{R}^2, \mathbb{R})$ is said to be quasi-monotone non-decreasing if

$$f_i(t, u_1(t), u_2(t)) \leq f_i(t, v_1(t), v_2(t)) \quad \text{if} \quad u_i = v_i \quad \text{and} \quad u_i \leq v_j, \\ i \neq j,\ i = j = 1, 2.$$

Definition 2.9. A function $f_i = f_i(t, u_1, u_2)$ in $C(J \times \mathbb{R}^2, \mathbb{R})$ is said to be quasi-monotone non-increasing if

$$f_i(t, u_1(t), u_2(t)) \geq f_i(t, v_1(t), v_2(t)) \quad \text{if} \quad u_i = v_i \quad \text{and} \quad u_i \leq v_j, \\ i \neq j,\ i = j = 1, 2.$$

3. Existence and uniqueness results

This section is devoted to develop monotone technique for system of Riemann-Liouville fractional differential equations with initial time difference and obtain existence and uniqueness of solution of the problem (2.3).

Theorem 3.1. *Assume that*

(E_1) $v = (v_1, v_2) \in C_p[J, \mathbb{R}], t_0, T > 0$ *and* $w = (w_1, w_2) \in C_p^*[J^*, \mathbb{R}]$ *are continuous functions and* $p = 1 - q$, *where*

$$C_p(J, \mathbb{R}^2) = \{u(t) \in C(J, \mathbb{R}^2) \ \text{and} \ u(t)(t - t_0)^p \in C(J, \mathbb{R}^2)\},$$
$$C_p^*(J^*, \mathbb{R}^2) = \{u(t) \in C(J^*, \mathbb{R}^2) \ \text{and} \ u(t)(t - \tau_0)^p \in C(J^*, \mathbb{R}^2)\},$$
$$f_i \in C[[t_0, t_0 + T] \times \mathbb{R}^2, \mathbb{R}], \quad J = [t_0, t_0 + T], J^* = [\tau_0, \tau_0 + T] \ \text{and}$$
$$D^q v(t) \leq f_i(t, v_1(t), v_2(t)), \quad t_0 \leq t \leq t_0 + T,$$
$$D^q w(t) \geq f_i(t, w_1(t), w_2(t)), \quad \tau_0 \leq t \leq \tau_0 + T,$$
$$v^0 \leq u^0 \leq w^0,$$
$$\text{for } v^0 = v(t)(t - t_0)^{1-q}|_{t=t_0} \ \text{and} \ w^0 = w(t)(t - \tau_0)^{1-q}|_{t=\tau_0},$$

(E_2) $f_i(t, u_1, u_2)$ *is quasi-monotone nondecreasing in* t *for each* u_i *and* $v(t) \leq w(t + \eta), t_0 \leq t \leq t_0 + T$, *where* $\eta = \tau_0 - t_0$,

(E_3) f_i *satisfies one-sided Lipschitz condition,*

$$f_i(t, u_1, u_2) - f_i(t, \overline{u}_1, \overline{u}_2) \geq -M_i[u_i - \overline{u}_i], \quad \text{for } \overline{u}_i \leq u_i, M_i \geq 0.$$

Then there exist monotone sequences $\{v^n(t)\}$ *and* $\{w^n(t)\}$ *such that*

$$v^n(t) \to v(t) = (v_1, v_2) \quad \text{and} \quad w^n(t) \to w(t) = (w_1, w_2) \ \text{as } n \to \infty,$$

where $v(t)$ *and* $w(t)$ *are minimal and maximal solutions of the problem* (2.3), *respectively.*

Proof. Let $w_{i0}(t) = w_i(t + \eta)$ and $v_{i0}(t) = v_i(t)$ $i = 1, 2$ for $t_0 \leq t \leq t_0 + T$, where $\eta = \tau_0 - t_0$. Since $f_i(t, u_1, u_2)$ is nondecreasing in t for each u_i we have

$$D^q w_{i0}(t) = D^q w_i(t + \eta) \geq f_i(t + \eta, w_1(t + \eta), w_2(t + \eta)) \geq f_i(t, w_1(t), w_2(t))$$

and

$$w_{i0}^0 = w_{i0}(t)(t - \tau_0)^{1-q}|_{t=\tau_0} = w_i(t + \eta)(t - \tau_0)^{1-q}|_{t=\tau_0} = w_i(t)(t - \tau_0)^{1-q}|_{t=\tau_0} = w^0.$$

Also,

$$D^q v_{i0}(t) = D^q v_i(t) \leq f_i(t, v_{10}(t), v_{20}(t))$$

and

$$v_{i0}^0 = v_{i0}(t)(t-t_0)^{1-q}|_{t=t_0} = v_i(t)(t-t_0)^{1-q}|_{t=t_0} = v^0, v^0 \leq u^0 \leq w^0,$$

which proves that v_0 and w_0 are lower and upper solutions of IVP (2.3) respectively.

For any $\theta(t) = (\theta_1, \theta_2)$ in $C_p(J, \mathbb{R}^2)$ such that for $\alpha_{10} \leq \theta_1 \leq \beta_{10}, \alpha_{20} \leq \theta_2 \leq \beta_{20}$ on J, consider the following linear system of fractional differential equations:

$$\begin{aligned} D^q u_i(t) &= f_i(t, \theta_1(t), \theta_2(t)) - M_i[u_i(t) - \theta_i(t)], \\ u_i^0 &= u_i(t)(t-t_0)^{1-q}|_{t=t_0}. \end{aligned} \tag{3.1}$$

Since the right hand side of IVP (3.1) satisfies Lipschitz condition, unique solution of IVP (3.1) exists on J.

For each $\eta(t)$ and $\mu(t)$ in $C_p(J, \mathbb{R}^2)$ such that $v_i^0(0) \leq \eta_i(t), w_i^0(0) \leq \mu_i(t)$, define a mapping A by $A[\eta, \mu] = u(t)$ where $u(t)$ is the unique solution of the problem (3.1).

Firstly, we prove that

(A_1) $v^0 \leq A[v^0, w^0], \quad w^0 \geq A[w^0, v^0]$,

(A_2) A possesses the monotone property on the segment

$$[v^0, w^0] = \left\{ (t, u) \in C(J, \mathbb{R}^2) : v_1^0 \leq u_1 \leq w_1^0, v_2^0 \leq u_2 \leq w_2^0 \right\}.$$

Set $A[v^0, w^0] = v_i^1$, where $v_i^1 = (v_1^1, v_2^1)$ is the unique solution of system (3.1) with $\eta_i = v_i^0(t)$. Setting $p_i(t) = v_i^0(t) - v_i^1(t)$, then we see that

$$\begin{aligned} D^q p_i(t) &= D^q v_i^0(t) - D^q v_i^1(t) \\ &= f_i(t, v_1^0(t), v_2^0(t)) - f_i(t, \theta_1^1(t), \theta_2^1(t)) + M_i(v_i^1(t) - \theta_i(t)) \\ &\leq -M_i(v_i^0(t) - v_i^1(t)) + M_i(v_i^1(t) - \theta_i(t)) \\ &\leq -M_i[v_i^0(t) - v_i^1(t)] \\ &\leq -M_i p_i(t). \end{aligned}$$

By Corollary 2.4, we get $p_i(t) \leq 0 \quad \text{on} \quad 0 \leq t \leq T$ and hence $v_i^0(t) - v_i^1(t) \leq 0$ which implies $v_i^0 \leq A[v^0, w^0]$. Set $A[v^0, w^0] = w_i^1$, where $w_i^1 = (w_1^1, w_2^1)$ is the unique solution of the problem (3.1) with $\mu_i = w_i^0(t)$.

Similarly, by Corollary 2.4, setting $p_i(t) = w_i^0(t) - w_i^1(t)$, we have $w_i^0 \geq w_i^1$. Hence $w^0 \geq A[w^0, v^0]$. This proves (A_1). Let $\eta, \beta, \mu \in [v^0, w^0]$ with $\eta \leq \beta$. Suppose that $A[\eta, \mu] = u(t), A[\beta, \mu] = v(t)$. Then setting $p_i(t) = u_i(t) - v_i, (t)$

we find that $p_i(t) \le 0$ and

$$\begin{aligned} D^q p_i(t) &= D^q u_i(t) - D^q v_i(t) \\ &= f_i(t, \eta_1, \eta_2) - M_i[u_i(t) - \eta_i(t)] - f_i(t, \beta_1, \beta_2) \\ &\quad + M_i[v_i(t) - \beta_i(t)] \\ &\le -M_i p_i(t). \end{aligned}$$

As before in (A_1), we have $A[\eta, \mu] \le A[\beta, \mu]$.

Similarly, if $\eta(t), \gamma(t), \mu(t) \in [v^0, w^0]$ satisfying $\gamma(t) \le \mu(t)$ and $A[\eta, \gamma] = u(t), A[\eta, \mu] = v(t)$ we can prove that $A[\eta, \gamma] \ge A[\eta, \mu]$. Thus the mapping A possesses monotone property on $[v^0, w^0]$. Define the sequences

$$v_i^n(t) = A[v_i^{n-1}, w_i^{n-1}], \quad w_i^n(t) = A[w_i^{n-1}, v_i^{n-1}],$$

on the segment $[v^0, w^0]$ by

$$\begin{aligned} D^q v_i^n(t) &= f_i(t, v_1^{n-1}, v_2^{n-1}) - M_i[v_i^n - v_i^{n-1}], \quad v_i^n(t)(t - t_0)^{1-q}|_{t=t_0} = v_i^{n0}, \\ D^q w_i^n(t) &= f_i(t, w_1^{n-1}, w_2^{n-1}) - M_i[w_i^n - w_i^{n-1}], \quad w_i^n(t)(t - \tau_0)^{1-q}|_{t=\tau_0} = w_i^{n0}. \end{aligned}$$

From (A_1), we have $v_i^0 \le v_i^1,\ w_i^0 \ge w_i^1$. To prove $v_i^k \le v_i^{k+1},\ w_i^k \ge w_i^{k+1}$ and $v_i^k \ge w_i^k$, define $p_i(t) = v_i^k(t) - v_i^{k+1}(t)$ and assume $v_i^{k-1} \le v_i^k,\ w_i^{k-1} \ge w_i^k$. Thus

$$\begin{aligned} D^q p_i(t) &= f_i(t, v_1^{k-1}, v_2^{k-1}) - M_i[v_i^k - v_i^{k-1}] \\ &\quad - \{f_i(t, v_1^k(t), v_2^k(t)) - M_i[v_i^{k+1}(t) - v_i^k(t)]\} \\ &\le -M_i[v_i^{k-1} - v_i^k] - M_i[v_i^k - v_i^{k-1}] + M_i[v_i^{k+1}(t) - v_i^k(t)] \\ &\le -M_i[v_i^k(t) - v_i^{k+1}(t)] \\ &\le -M_i p_i(t). \end{aligned}$$

It follows from Corollary 2.4 that $p_i(t) \le 0$, which gives $v_i^k(t) \le v_i^{k+1}(t)$.

Similarly we can prove $w_i^k(t) \ge w_i^{k+1}(t)$ and $v_i^k(t) \ge w_i^k(t)$. By induction, it follows that

$$v_i^0(t) \le v_i^1(t) \le v_i^2(t) \le ... \le v_i^n(t) \le w_i^n(t) \le w_i^{n-1}(t) \le ... \le w_i^1(t) \le w_i^0(t).$$

Thus the sequences $\{v^n(t)\}$ and $\{w^n(t)\}$ are bounded from below and bounded from above respectively and monotonically nondecreasing and monotonically nonincreasing on J. Hence point-wise limit exist and are given by

$$\lim_{n\to\infty} v_i^n(t) = v_i(t), \quad \lim_{n\to\infty} w_i^n(t) = w_i(t) \text{ on } J.$$

Using corresponding Volterra fractional integral equations

$$v_i^n(t) = v_i^0 + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \Big\{ f_i(s, v_1^n(s), v_2^n(s)) - M_i[v_i^n - v_i^{n-1}] \Big\} ds,$$
$$w_i^n(t) = w_i^0 + \frac{1}{\Gamma(q)} \int_0^T (t-s)^{q-1} \Big\{ f_i(s, w_1^n(s), w_2^n(s)) - M_i[w_1^n - w_1^{n-1}] \Big\} ds,$$

as $n \to \infty$, we get

$$v_i(t) = \frac{v_i^0 (t-t_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{t_0}^T (t-s)^{q-1} f_i(s, v_1^n(s), v_2^n(s)) ds,$$
$$w_i(t) = \frac{w_i^0 (t-\tau_0)^{q-1}}{\Gamma(q)} + \frac{1}{\Gamma(q)} \int_{\tau_0}^T (t-s)^{q-1} f_i(s, w_1^n(s), w_2^n(s)) ds,$$

where $v_i^0 = v_i(t)(t-t_0)^{1-q}|_{t=t_0}$, $\quad w_i^0 = w_i(t)(t-\tau_0)^{1-q}|_{t=\tau_0}$. It follows that $v(t)$ and $w(t)$ are solutions of system (2.3).

Lastly, we prove $v(t)$ and $w(t)$ are the minimal and maximal solutions of the problem (2.3). Let $u(t) = (u_1, u_2)$ be any solution of (2.3) other than $v(t)$ and $w(t)$, so that there exists k such that $v_i^k(t) \leq u_i(t) \leq w_i^k(t)$ on J and setting $p_i(t) = v_i^{k+1}(t) - u_i(t)$, then we have $p_i(t) \leq 0$ and

$$\begin{aligned} D^q p_i(t) &= f_i(t, v_1^k, v_2^k) - M_i[v_i^{k+1} - v_i^k] - f_i(t, u_1, u_2) \\ &\leq -M_i p_i(t). \end{aligned}$$

Thus $v_i^{k+1}(t) \leq u_i(t)$ on J. Since $v_i^0(t) \leq u_i(t)$ on J, by induction it follows that $v_i^k(t) \leq u_i(t)$ for all k. Similarly, we can prove $u_i \leq w_i^k$ for all k on J. Hence $v_i^k(t) \leq u_i(t) \leq w_i^k(t)$ on J. Taking limit as $n \to \infty$, it follows that $v_i(t) \leq u_i(t) \leq w_i(t)$ on J. □

Now, we obtain the uniqueness of solution of the problem (2.3) in the following:

Theorem 3.2. *Assume that*

(U_1) *Assumptions* (E_1) *and* (E_3) *of Theorem* 3.1 *hold.*
(U_2) $f_i = f_i(t, u_1, u_2)$ *satisfies Lipschitz condition (two-sided),*

$$|f_i(t, u_1, u_2) - f_i(t, \overline{u}_1, \overline{u}_2)| \geq -M_i |u_i - \overline{u}_i|, M_i \geq 0.$$

Then the solution of the problem (2.3) *is unique.*

Proof. It is sufficient to prove $v(t) \geq w(t)$. If $p_i(t) = w_i(t) - v_i(t)$, then $p_i(t) = 0$ and

$$\begin{aligned} D^q p_i(t) &= D^q w_i(t) - D^q v_i(t) \\ &= f_i(t, w_1(t), w_2(t)) - f_i(t, v_1(t), v_2(t)) \\ &\leq -M_i(w_i(t) - v_i(t)) \\ &\leq -M_i p_i(t). \end{aligned}$$

Thus, by Corollary 2.4, we get $p_i(t) \leq 0$ implies $w_i(t) \leq v_i(t)$. Hence $v(t) = u(t) = w(t)$ is the unique solution of (2.3) on $[t_0, t_0 + T]$. □

Example 3.3. We validate obtained results for the following system of R-L fractional differential equations with initial time difference:

$$\begin{aligned} D^q u_1(t) &= 2t^q(1-t)^{\frac{1}{2}} - \frac{1}{4}u_1^3 + u_2, \quad u_1(t)(t-t_0)^{1-q}|_{t=t_0} = 0, \\ D^q u_2(t) &= 5t^q(1-t)^{\frac{1}{3}} + u_1 - \frac{1}{2}u_2^2, \quad u_2(t)(t-t_0)^{1-q}|_{t=t_0} = 1, \end{aligned} \tag{3.2}$$

where $t \in J = [t_0, t_0 + T]$. We have

$$\begin{aligned} |f_1(t, u_1, u_2) - f_1(t, \overline{u_1}, \overline{u_2})| &= |-\frac{1}{4}u_1^3 + u_2 - \frac{1}{4}\overline{u_1}^3 - \overline{u_2}| \\ &\leq \frac{1}{4}|u_1 - \overline{u_1}| \end{aligned}$$

and

$$\begin{aligned} |f_2(t, u_1, u_2) - f_2(t, \overline{u_1}, \overline{u_2})| &= |u_1 - \frac{1}{2}u_2^2 - \overline{u_1} + \frac{1}{2}\overline{u_2}^2| \\ &\leq \frac{1}{2}|u_2 - \overline{u_2}|. \end{aligned}$$

Thus, assumptions of the Theorem 3.1 hold with Lipschitz constants $\frac{1}{4}$ and $\frac{1}{2}$. The unique solution $u(t) = (u_1, u_2)$ of the system (3.2) exists satisfying $v(t) \leq u(t) \leq w(t)$ where $v(t) = (0, 0)$ is lower solution and $w(t) = (2t^{q-1}, 5t^{q-1})$ is upper solution of the system (3.2).

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