



## NEW SUBCLASS OF MEROMORPHIC MULTIVALENT FUNCTIONS ASSOCIATED WITH HYPERGEOMETRIC FUNCTION

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**Abstract.** As hypergeometric meromorphic multivalent functions of the form

$$L_{\varpi, \sigma}^{t, \rho} f(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} a_{\kappa+\rho} \zeta^{\kappa+\rho}$$

contains a new subclass in the punctured unit disk  $\Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$  for  $-1 \leq D < S \leq 1$ , this paper aims to determine sufficient conditions, distortion properties and radii of starlikeness and convexity for functions in the subclass  $L_{\varpi, \sigma}^{t, \rho} f(\zeta)$ .

### 1. INTRODUCTION

Let  $\Sigma_\rho$  denote the class of meromorphic multivalent functions  $f(\zeta)$  normalized by

$$f(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} a_{\kappa+\rho} \zeta^{\kappa+\rho}; \rho \in \mathbb{N} \setminus \{0\} \quad (1.1)$$

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which are analytic in the punctured unit disk

$$\mathbb{U}^* = \{ \zeta : \zeta \in \mathbb{C} \text{ and } 0 < |\zeta| < 1 \} = \mathbb{U} \setminus \{0\},$$

where  $\mathbb{C}$  is the set of complex numbers.

The starlike and convex functions are most important subclass of meromorphic functions, as they have very useful characteristics. See for example (Aouf *et al.* [4], Ghanim and Darus [8], Srivastava [12], Kulkarni *et al.* [13], Morga [17], Owa *et al.* [18], Srivastava and Owa [19], Uralegaddi and Somantha [20], and Yang [21]).

The functions  $f_v(\zeta)$ , ( $v = 1, 2$ ) are defined by:

$$f_v(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} a_{\kappa+\rho,v} \zeta^{\kappa+\rho}, (v = 1, 2).$$

The convolution (or Hadamard product) of  $f_1(\zeta)$  and  $f_2(\zeta)$  are defined by :

$$(f_1 * f_2)(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} a_{\kappa+\rho,1} a_{\kappa+\rho,2} \zeta^{\kappa+\rho} = (f_2 * f_1)(\zeta).$$

Let the function  $\tilde{\rho}(\varpi, \sigma; \zeta)$  be defined by:

$$\tilde{\rho}(\varpi, \sigma; \zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \zeta^{\kappa+\rho}, (\sigma \in \mathbb{C} / (\mathbb{Z}^- \cup \{0\}) ; \varpi \in \mathbb{C}), \tag{1.2}$$

where  $(\tau)_\kappa$  is defined by:

$$(\tau)_\kappa := \frac{\Gamma(\tau + \kappa)}{\Gamma(\tau)} = \begin{cases} \tau(\tau + 1) \dots (\tau + \kappa - 1), & (\kappa = n \in \mathbb{N}; \tau \in \mathbb{C}) \\ 1, & (\kappa = 0; \tau \in \mathbb{C} \setminus \{0\}), \end{cases}$$

$(\tau)_\kappa$  is called the Pochhammer symbol and  $\Gamma$  is Gamma function.

The meromorphic functions with the generalized hypergeometric functions were considered recently by Al-Janaby and Ghanim [2], Al-Janaby *et al.* [3], Cho and Kim [5], Dziok and Srivastava [6], El-Ashwah [7], Ghanim and Darus [9-11], Liu [14], and Liu and Srivastava [15].

Let the function  $f$  which is defined in (1.1) belong to the subclass  $\Sigma_\rho$ , we start by reintroducing  $c^t f(\zeta)$  which was studied by [16]:

$$\begin{aligned} c^t f(\zeta) &= (1 - t)f(\zeta) + \frac{t \zeta(-f(\zeta))'}{\rho} \\ &= \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\rho - (\kappa + 2\rho)t)}{\rho} a_{\kappa+\rho} \zeta^{\kappa+\rho}, (t \geq 0). \end{aligned} \tag{1.3}$$

Using the convolution between (1.2) and (1.3), we will introduce a new function  $L_{\varpi, \sigma}^{t, \rho}$  defined on  $\Sigma_\rho$  by:

$$\begin{aligned}
 L_{\varpi, \sigma}^{t, \rho} f(\zeta) &= \tilde{\rho}(\varpi, \sigma; \zeta) * c^t f(\zeta) \\
 &= \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa+2\rho)t)}{\rho} a_{\kappa+\rho} \zeta^{\kappa+\rho}, (\zeta \in \mathbb{U}^*).
 \end{aligned}
 \tag{1.4}$$

For all  $\zeta \in \mathbb{U}^*$  and  $-1 \leq D < S \leq 1$ , the function  $f \in \Sigma_\rho$  is said to be a member of the subclass  $\Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$  if it satisfies:

$$\left| \frac{\zeta (L_{\varpi, \sigma}^{t, \rho} f(\zeta))' + \rho L_{\varpi, \sigma}^{t, \rho} f(\zeta)}{D\zeta (L_{\varpi, \sigma}^{t, \rho} f(\zeta))' + S\rho (L_{\varpi, \sigma}^{t, \rho} f(\zeta))} \right| < 1.
 \tag{1.5}$$

See for example [1] and [8].

2. COEFFICIENT ESTIMATES AND DISTORTION THEOREMS

In this section, our first result will be concerning the coefficient estimates and distortion theorem for the subclass  $\Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$ .

**Theorem 2.1.** *Let the function  $f$  be defined by (1.4) and satisfies (1.5). Then*

$$\begin{aligned}
 &\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| \left| [(\kappa + 2\rho) - (\kappa + \rho)D + S\rho] |a_{\kappa+\rho}| \right. \\
 &\leq \rho(S - D),
 \end{aligned}
 \tag{2.1}$$

where  $-1 \leq D < S \leq 1$ .

*Proof.* Suppose that (1.5) holds. Let

$$\begin{aligned}
 M &= \zeta \left( \frac{-\rho}{\zeta^{\rho+1}} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} (\kappa + \rho) a_{\kappa+\rho} \zeta^{k+\rho-1} \right), \\
 N &= \frac{\rho}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (k + 2\rho)t)}{\rho} \rho a_{\kappa+\rho} \zeta^{\kappa+\rho},
 \end{aligned}$$

$$P = D\zeta \left( \frac{-\rho}{\zeta^{\rho+1}} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} (\kappa + \rho) a_{\kappa+\rho} \zeta^{\kappa+\rho-1} \right)$$

and

$$Q = S\rho \left( \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} a_{\kappa+\rho} \zeta^{\kappa+\rho} \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right).$$

Then

$$\left| \frac{M + N}{P + Q} \right| \leq 1,$$

it implies that

$$\left| \frac{\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa+2\rho)t)}{\rho} \{ \kappa + 2\rho \} a_{\kappa+\rho} \zeta^{\kappa+\rho}}{\frac{S\rho - \rho D}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa+2\rho)t)}{\rho} \{ (\kappa + \rho)D + S\rho \} a_{\kappa+\rho} \zeta^{\kappa+\rho}} \right| \leq 1.$$

Thus we have

$$\left| \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \{ \kappa + 2\rho \} a_{\kappa+\rho} \zeta^{\kappa+\rho} \right| - \left| \frac{S\rho - \rho D}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \{ (\kappa + \rho)D + S\rho \} a_{\kappa+\rho} \zeta^{\kappa+\rho} \right| \leq 0.$$

Therefore,

$$\left| \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \{ \kappa + 2\rho \} a_{\kappa+\rho} \zeta^{\kappa+2\rho} \right| - \left| S\rho - \rho D + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \frac{(\rho - (k + 2\rho)t)}{\rho} \{ (k + \rho)D + S\rho \} a_{\kappa+\rho} \zeta^{\kappa+2\rho} \right| \leq 0.$$

Hence we have

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{\rho - (\kappa + 2\rho) t}{\rho} \right| (\kappa + 2\rho) |a_{\kappa+\rho}| |\zeta^{\kappa+2\rho}| - S\rho + \rho D - \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{\rho - (\kappa + 2\rho)t}{\rho} \right| ((\kappa + \rho)D + S\rho) |a_{\kappa+\rho}| |\zeta^{\kappa+2\rho}| \leq 0.$$

If  $|\zeta| = r \rightarrow 1$ , then we have

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho) t)}{\rho} \right| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)] |a_{\kappa+\rho}| \leq \rho(S - D).$$

Hence it follows from (1.5) that  $f \in \Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$  and the intended result is achieved.  $\square$

**Corollary 2.2.** *Let the function  $f$  be defined by (1.4). If  $f \in \Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$ , then*

$$|a_{\kappa+\rho}| \leq \frac{\rho^2 (\sigma)_{\kappa+2} (S - D)}{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}, (\kappa \geq 0). \tag{2.2}$$

Our assertion in Theorem 2.1 is sharp for functions of the form:

$$f_{\kappa}(\zeta) = \frac{1}{\zeta^{\rho}} + \frac{\rho^2(\sigma)_{\kappa+2}(S - D)}{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]} \zeta^{\kappa+\rho}. \tag{2.3}$$

**Corollary 2.3.** *Let  $S = 1$  and  $D = -1$  in Theorem 2.1. Then we have*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| (\kappa + \rho) |a_{\kappa+\rho}| \leq \rho$$

and therefore the function  $L_{\varpi,\sigma}^{t,\rho} f(\zeta)$  is starlike in  $\mathbb{U}^*$ .

**Corollary 2.4.** *If  $t = 0, S = 1$  and  $D = -1$  in Theorem 2.1, then  $f \in \Sigma_{\rho}$  satisfying the following condition:*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} (\kappa + \rho) |a_{\kappa+\rho}| \leq \rho$$

and it is starlike in  $\mathbb{U}^*$ .

**Corollary 2.5.** *For  $S = 1$  and  $D = -1$  in Theorem 2.1, we have:*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| (\kappa + \rho)^2 |a_{\kappa+\rho}| \leq \rho$$

and therefore the function  $L_{\varpi,\sigma}^{t,\rho} f(\zeta)$  is convex in  $\mathbb{U}^*$ .

**Corollary 2.6.** *If  $t = 0, S = 1$  and  $D = -1$  in Theorem 2.1, then  $f \in \Sigma_{\rho}$  satisfies the following condition:*

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} (\kappa + \rho)^2 |a_{\kappa+\rho}| \leq \rho$$

and it is convex in  $\mathbb{U}^*$ .

A distortion property for functions in the subclass  $\Sigma_{\varpi,\sigma}^{S,D}(t, \kappa, \rho)$  is given in the following result:

**Theorem 2.7.** *If the function  $f$  is defined by (1.4) in the subclass  $\Sigma_{\varpi,\sigma}^{E,S}(t, \kappa, \rho)$ , then for  $0 < |\zeta| = r < 1$ , we have*

$$\frac{1}{r^{\rho}} - r^{\rho} \frac{(S - D)}{[2 - (S + D)]} \leq |f(\zeta)| \leq \frac{1}{r^{\rho}} + r^{\rho} \frac{(S - D)}{[2 - (S + D)]} \tag{2.4}$$

and

$$\frac{\rho}{r^{\rho+1}} - \frac{\rho(S - D)}{[2 - (S + D)]} r^{\rho-1} \leq |f'(\zeta)| \leq \frac{\rho}{r^{\rho+1}} + \frac{\rho(S - D)}{[2 - (S + D)]} r^{\rho-1} \tag{2.5}$$

with equality for

$$f(\zeta) = \frac{1}{\zeta^\rho} + \frac{(S - D)}{[2 - (D + S)]} \zeta^\rho.$$

*Proof.* Let  $f \in \Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$ . Then

$$\begin{aligned} |f(\zeta)| &\leq \frac{1}{|\zeta^\rho|} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| |\zeta|^{\kappa+\rho} \\ &\leq \frac{1}{r^\rho} + r^\rho \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (k + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}|. \end{aligned}$$

Theorem 2.1 readily yields the inequality:

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| \leq \frac{\rho(S - D)}{[(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]} \tag{2.6}$$

thus, for  $0 < |\zeta| = r < 1$ , we have

$$|f(\zeta)| \leq \frac{1}{r^\rho} + r^\rho \frac{\rho(S - D)}{[(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}, (\kappa \geq 0).$$

Hence, we have

$$|f(\zeta)| \leq \frac{1}{r^\rho} + r^\rho \frac{(S - D)}{[2 - (S + D)]} \tag{2.7}$$

and the other side of the inequality in (2.4) can be obtained using similar procedure. For (2.5),

$$|f'(\zeta)| \leq \frac{\rho}{|\zeta^{\rho+1}|} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| (\kappa + \rho) |a_{\kappa+\rho}| |\zeta|^{\kappa+\rho-1}.$$

It follows from Theorem 2.1 that

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| (\kappa + \rho) |a_{\kappa+\rho}| \leq \frac{(\kappa + \rho)\rho(SD)}{[(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]} \tag{2.8}$$

thus, for  $0 < |\zeta| = r < 1$ , we have

$$|f'(\zeta)| \leq \frac{\rho}{r^{\rho+1}} + \frac{(\kappa + \rho)\rho(S - D)}{[(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]} r^{\kappa+\rho-1}, (\kappa \geq 0).$$

Hence, we have

$$|f'(\zeta)| \leq \frac{\rho}{r^{\rho+1}} + \frac{\rho(S - D)}{[2 - (S + D)]} r^{\rho-1} \tag{2.9}$$

By similarity, the other side of the inequality follows and the proof is complete.  $\square$

3. STARLIKENESS AND RADII OF CONVEXITY

**Theorem 3.1.** *If the function  $f$  is in the subclass  $\Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$ , then  $f$  is meromorphically starlike of order  $\chi$  ( $0 \leq \chi < 1$ ) in  $|\zeta| < r_1$ ,*

$$r_1 = r_1(S, D, \kappa, \rho) = \inf_{\kappa \geq 0} \left\{ \frac{(\rho - \chi)[(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}{(\kappa + 3\rho - \chi)\rho(S - D)} \right\}^{\frac{1}{\kappa + 2\rho}}, \quad (3.1)$$

where the result is sharp for the function  $f_\kappa$  given by (2.3).

*Proof.* It suffices to prove that

$$\left| \frac{\zeta (L_{\varpi, \sigma}^{t, \rho} f(\zeta))'}{L_{\varpi, \sigma}^{t, \rho} f(\zeta)} + \rho \right| \leq \rho - \chi, \quad (3.2)$$

for  $|\zeta| < r_1$ . We note that

$$\begin{aligned} & \left| \frac{\zeta (L_{\varpi, \sigma}^{t, \rho} f(\zeta))'}{L_{\varpi, \sigma}^{t, \rho} f(\zeta)} + \rho \right| \\ &= \left| \frac{\frac{-\rho}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} (\kappa + \rho) a_{\kappa+\rho} \zeta^{\kappa+\rho}}{\frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} a_{\kappa+\rho} \zeta^{\kappa+\rho}} + \rho \right| \\ &= \left| \frac{\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} a_{\kappa+\rho} \zeta^{\kappa+\rho} (\kappa + 2\rho)}{\frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \cdot \frac{(\rho - (\kappa + 2\rho)t)}{\rho} a_{\kappa+\rho} \zeta^{\kappa+\rho}} \right| \\ &\leq \frac{\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| (\kappa + 2\rho) |\zeta|^{\kappa+2\rho}}{\frac{1}{|\zeta|^\rho} - \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| |\zeta|^{\kappa+2\rho}} \\ &\leq \frac{\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| (\kappa + 2\rho) |\zeta|^{\kappa+2\rho}}{1 - \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| |\zeta|^{\kappa+2\rho}}}. \end{aligned} \quad (3.3)$$

Hence, if

$$\begin{aligned} & \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| (\kappa + 2\rho) |\zeta|^{\kappa+2\rho} \\ & \leq (\rho - \chi) \left[ 1 - \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| |\zeta|^{\kappa+2\rho} \right] \end{aligned} \quad (3.4)$$

or

$$\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| |\zeta|^{\kappa+2\rho} \left( \frac{\kappa + 3\rho - \chi}{\rho - \chi} \right) \leq 1,$$

that is,

$$\frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |a_{\kappa+\rho}| |\zeta|^{\kappa+2\rho} \left( \frac{\kappa + 3\rho - \chi}{\rho - \chi} \right) \leq 1, \text{ for } \kappa \geq 0 \quad (3.5)$$

with the aid of (2.2) and (3.5), for all  $\kappa \geq 0$ ,

$$\begin{aligned} & \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left| \frac{(\rho - (\kappa + 2\rho)t)}{\rho} \right| |\zeta|^{\kappa+2\rho} \left( \frac{\kappa + 3\rho - \chi}{\rho - \chi} \right) \\ & \leq \frac{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}{\rho^2 (\sigma)_{\kappa+2} (S - D)}. \end{aligned} \quad (3.6)$$

Solving (3.6) for  $|\zeta|$ , we obtain

$$|\zeta| \leq \left\{ \frac{(\rho - \chi)[(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}{(\kappa + 3\rho - \chi)\rho(S - D)} \right\}^{\frac{1}{\kappa+2\rho}}, \quad \forall \kappa \geq 0. \quad (3.7)$$

□

**Theorem 3.2.** *If the function  $f$  is defined by (1.4) in the  $\Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$ , then  $f$  is meromorphically convex of order  $\chi$  ( $0 \leq \chi < 1$ ) in  $|\zeta| < r_2$ , where*

$$r_2 = r_2(S, D, \kappa, \rho) = \inf_{\kappa \geq 0} \left\{ \frac{(\rho - \chi)[(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}{(\kappa + \rho)(\kappa + 3\rho - \chi)(S - D)} \right\}^{\frac{1}{\kappa+2\rho}}, \quad (3.8)$$

the result is sharp for the function  $f_\kappa$  given by (2.3).

*Proof.* By using the technique employed in the proof of Theorem 3.2, we can show that

$$\left| \frac{\zeta (L_{\varpi, \sigma}^{t, \rho} f(\zeta))''}{(L_{\varpi, \sigma}^{t, \rho} f(\zeta))'} + \rho + 1 \right| \leq (\rho - \chi), \quad (3.9)$$

for  $|\zeta| < r_2$ , with the aid of Theorem 2.1. Thus, we have the assertion of Theorem 3.2. □

#### 4. CONVEX LINEAR COMBINATIONS

Our next results involve a linear combination of several functions of the type (2.3).

**Theorem 4.1.** *Let*

$$f_{\rho-1}(\zeta) = \frac{1}{\zeta^\rho} \quad (4.1)$$

and for  $\kappa \geq 0$ ,

$$f_{\kappa+\rho}(\zeta) = \frac{1}{\zeta^\rho} + \frac{\rho^2 (\sigma)_{\kappa+2} (S - D)}{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]} \zeta^{\kappa+\rho}. \quad (4.2)$$



Then  $f \in \Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$  if and only if it can be expressed in the form

$$f(\zeta) = \sum_{\kappa=0}^{\infty} \eta_{\kappa+\rho-1} f_{\kappa+\rho-1}(\zeta) \tag{4.3}$$

where  $\eta_{\kappa+\rho-1} \geq 0$  and  $\sum_{\kappa=0}^{\infty} \eta_{\kappa+\rho-1} = 1$ .

*Proof.* From (4.1), (4.2) and (4.3), it is easily seen that

$$\begin{aligned} f(\zeta) &= \sum_{\kappa=0}^{\infty} \eta_{\kappa+\rho-1} f_{\kappa+\rho-1}(\zeta) \\ &= \frac{1}{\zeta^{\rho}} + \sum_{\kappa=0}^{\infty} \frac{\rho^2 (\sigma)_{\kappa+2} (S - D) \eta_{\kappa+\rho}}{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]} \zeta^{\kappa+\rho} \\ &=: \frac{1}{\zeta^{\rho}} + \sum_{\kappa=0}^{\infty} d_{\kappa+\rho} \zeta^{\kappa+\rho}. \end{aligned} \tag{4.4}$$

Since

$$\begin{aligned} &\sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}{\varpi \rho^2 (\sigma)_{\kappa+2} (S - D)} \\ &\times \frac{\rho^2 (\sigma)_{\kappa+2} (S - D) \eta_{\kappa+\rho}}{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]} \\ &= \sum_{\kappa=0}^{\infty} \eta_{\kappa+\rho} = 1 - \eta_{\rho-1} \leq 1, \end{aligned}$$

it follows from Theorem 2.1 that the function  $f \in \Sigma_{\varpi, \sigma}^{S, D}(t, k, \rho)$ .

Conversely, suppose that  $f \in \Sigma_{\varpi, \sigma}^{S, D}(t, k, \rho)$ . Du to

$$|a_{\kappa+\rho}| \leq \frac{\rho^2 (\sigma)_{\kappa+2} (S - D)}{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(k + 2\rho) - ((k + \rho)D + S\rho)]}, (\kappa \geq 0).$$

Setting

$$\eta_{\kappa+\rho} = \frac{(\varpi)_{\kappa+2} |(\rho - (\kappa + 2\rho)t)| [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)]}{\rho^2 (\sigma)_{\kappa+2} (S - D)} |a_{\kappa+\rho}|, (\kappa \geq 0)$$

and

$$\eta_{\rho-1} = 1 - \sum_{\kappa=0}^{\infty} \eta_{\kappa+\rho},$$

it follows that  $f(\zeta) = \sum_{\kappa=0}^{\infty} \eta_{\kappa+\rho-1} f_{\kappa+\rho-1}(\zeta)$ . □

**Theorem 4.2.** *The subclass  $\Sigma_{\varpi, \sigma}^{S, D}(t, \kappa, \rho)$  is closed under convex linear combinations.*

*Proof.* Suppose that the functions  $f_1(\zeta)$  and  $f_2(\zeta)$  defined by

$$f_v(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} a_{\kappa+\rho,v} \zeta^{\kappa+\rho}, \quad (v = 1, 2; \zeta \in \mathbb{U}^*) \quad (4.5)$$

are in the subclass  $\Sigma_{\varpi,\sigma}^{S,D}(t, \kappa, \rho)$ . Let

$$f(\zeta) = \mu f_1(\zeta) + (1 - \mu) f_2(\zeta), \quad (0 \leq \mu \leq 1). \quad (4.6)$$

Then, from (4.5), we find that

$$f(\zeta) = \frac{1}{\zeta^\rho} + \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left( \frac{\rho - (\kappa + 2\rho)t}{\rho} \right) \{ \mu a_{\kappa+\rho,1} + (1 - \mu) a_{\kappa+\rho,2} \} \zeta^{\kappa+\rho}, \quad (4.7)$$

$(0 \leq \mu \leq 1; \zeta \in \mathbb{U}^*).$

In view of Theorem 2.1, we have

$$\begin{aligned} & \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left( \frac{\rho - (\kappa + 2\rho)t}{\rho} \right) [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)] \\ & \times |(\mu a_{\kappa+\rho,1} + (1 - \mu) a_{\kappa+\rho,2})| \\ & \leq \mu \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left( \frac{\rho - (\kappa + 2\rho)t}{\rho} \right) [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)] |a_{\kappa+\rho,1}| \\ & \quad + (1 - \mu) \sum_{\kappa=0}^{\infty} \frac{(\varpi)_{\kappa+2}}{(\sigma)_{\kappa+2}} \left( \frac{\rho - (\kappa + 2\rho)t}{\rho} \right) \\ & \quad \times [(\kappa + 2\rho) - ((\kappa + \rho)D + S\rho)] |a_{\kappa+\rho,2}| \\ & \leq \mu \rho (S - D) + (1 - \mu) \rho (S - D) \\ & = \rho (S - D). \end{aligned}$$

This means that  $f \in \Sigma_{\varpi,\sigma}^{S,D}(t, \kappa, \rho)$ . □

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