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REMARKS ON CERTAIN NOTED COINCIDENCE THEOREMS: A UNIFYING AND ENRICHING APPROACH

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Abstract. In this paper, we unify and enrich the well-known classical metrical coincidence theorems on a complete metric space due to Machuca, Goebel and Jungck. We further extend our newly proved results on a subspace Y of metric space X, wherein X need not be complete. Finally, we slightly modify the existing results involving $(E.A)$-property and (CLR_g)-property and apply these results to deduce our coincidence and common fixed point theorems.

1. Introduction and Preliminaries

Throughout the manuscript, the sets $\mathbb{N}$, $\mathbb{N}_0$ and $\mathbb{R}$ stand for the sets of natural numbers, whole numbers and real numbers, respectively. For a self-mapping f on a nonempty set X, "x is fixed point of f" is equivalent to saying that $f(x) = I(x)$ (where I denotes identity mapping on X). This

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fact motivates whether the identity mapping can be replaced by another self-mapping g on X. Henceforth, given two self-mappings f and g on a nonempty set X, consider the problem regarding to find $x, \overline{x} \in X$ such that

$$f(x) = g(x) = \overline{x}. \tag{1.1}$$

Then

- x is called a coincidence point of f and g,
- $\overline{x}$ is called a point of coincidence of f and g,
- x is called a common fixed point of f and g provided $\overline{x} = x$.

Clearly, every common fixed point of f and g is also a coincidence point as well as point of coincidence. It is well known that the coincidence problem [1] is, under appropriate conditions, equivalent to a fixed point problem.

Given two self-mappings f and g on a metric space (X, d), we say that f is g-contraction if there exists $k \in [0, 1)$ such that

$$d(fx, fy) \leq kd(gx, gy), \ \ \forall\, x, y \in X.$$

In 1967, Machuca [16] proved a first metrical coincidence theorem for a pair of mappings $f, g : X \to Y$, where X and Y are complete metric space and T_1-topological space satisfying the first axiom of countability respectively. We particularize Machuca coincidence theorem by taking $Y = X$ besides removing some unnecessary conditions as follows:

Theorem 1.1. *Let (X, d) be a complete metric space and f and g be two self-mappings on X. Suppose that the following conditions hold:*

(i) $f(X) \subseteq g(X)$,
(ii) *f is a g-contraction,*
(iii) *one of $f(X)$ and $g(X)$ is closed.*

Then f and g have a coincidence point.

The condition "$f(X)$ is closed" or "$g(X)$ is closed" was only used to guarantee that (fX, d) or (gX, d) is a complete metric space. The same thesis can be deduced replacing "(X, d) is complete and $g(X)$ is closed" by the weaker condition "$g(X)$ a complete". Using this fact, in 1968, Goebel [6] enrich Theorem 1.1 for a pair of mappings $f, g : X \to Y$, where X and Y are complete metric space and an arbitrary set respectively. We particularize Goebel coincidence theorem by taking $Y = X$ as follows:

Theorem 1.2. *Let (X, d) a metric space and f and g be two self-mappings on X. Suppose that the following conditions hold:*

(i) $f(X) \subseteq g(X)$,
(ii) *f is a g-contraction,*
(iii) *one of $f(X)$ and $g(X)$ is a complete subspace of X.*

Then f and g have a coincidence point.

Recall that two self-mappings f and g on a nonempty set X are said to be commuting if $f(gx) = g(fx)$ for all $x \in X$. Eldon Dyer (1954), Allen Lowell Shields (1955) and Lester Dubins (1956) almost simultaneously posed an interesting problem independently. The problem first appears in the literature in 1957 as part of a more general question raised Isbell [11]. This problem states below:

Problem: Let f and g be two commuting continuous self-mappings on a unit interval. Do they have a common fixed point?

This conjecture was settled in negative by Boyce [4, 5] and Huneke [7, 8] independently and the answer was given by constructing a pair of commuting function with no common fixed point employing a limiting process. The functions were discovered as the result of a computer aided search based in part on necessary conditions derived by Baxter [3]. Thus in order to coin a common fixed point theorem, one is required to impose extra conditions either on the space or on the mappings under consideration which is evident in all existing common fixed point theorems. In 1976, Jungck [12] generalized Banach contraction principle to obtain common fixed point for commuting mappings by using a constructive procedure of sequence of iterations.

Theorem 1.3. ([12]) *Let (X, d) a complete metric space and f and g be two self-mappings on X. Suppose that the following conditions hold:*

(i) $f(X) \subseteq g(X)$,
(ii) *f is a g-contraction,*
(iii) *g is continuous,*
(iv) *f and g are commuting.*

Then f and g have a unique common fixed point.

With a view to improve commutativity conditions in Theorem 1.3, in 1982, Sessa [20] introduced the notion of weakly commuting mappings which runs as follows:

Definition 1.4. ([20]) Let (X, d) be a metric space and f and g be two self mappings on X. We say that f and g are weakly commuting if

$$d(gfx, fgx) \leq d(gx, fx), \;\; \forall\, x \in X.$$

Clearly, commuting mappings are weakly commuting but the converse is not true generally as shown by the following example.

Example 1.5. ([20]) Consider $X = [0, 1]$ with usual metric. Define the functions $f : X \to X$ and $g : X \to X$ by

$$f(x) = \frac{x}{x+2} \text{ and } g(x) = \frac{x}{2}, \ \ \forall\, x \in X.$$

Then f and g are weakly commuting but not commuting mappings.

Soon after this definition, Jungck [13] extended the concept of weak commutativity by defining compatible mappings in the following way:

Definition 1.6. ([13]) Let (X, d) be a metric space and f and g be two self-mappings on X. We say that f and g are compatible if for any sequence $\{x_n\} \subset X$ and $z \in X$, $\lim\limits_{n\to\infty} f(x_n) = \lim\limits_{n\to\infty} g(x_n) = z$, then

$$\lim_{n\to\infty} d(gfx_n, fgx_n) = 0.$$

It is well known that two weakly commuting mappings are compatible, but the converse is not true. Some examples supporting this fact can be found in [13].

Example 1.7. ([13]) Consider $X = \mathbb{R}$ with usual metric. Define the functions $f : X \to X$ and $g : X \to X$ by

$$f(x) = x^3 \text{ and } g(x) = 2x^3, \ \ \forall\, x \in X.$$

Then f and g are compatible but not weakly commuting mappings.

Definition 1.8. ([14]) Let (X, d) be a metric space and f and g be two self-mappings on X. We say that f and g are weakly compatible (or partially commuting or coincidentally commuting) if f and g commute at their coincidence points, that, for any $x \in X$, $f(x) = g(x)$, then

$$f(gx) = g(fx).$$

Clearly two compatible mappings are weakly compatible but converse not true in general as substantiated by the following example:

Example 1.9. Consider $X = [1, 7]$ with usual metric. Define two self-mappings f and g on X by

$$f(x) = \begin{cases} 1 & \text{if } x = 1 \text{ or } x \in (3, 7] \\ 5 & \text{if } x \in (1, 3] \end{cases}$$

and

$$g(x) = \begin{cases} x & \text{if } x \in [1, 3], \\ x - 2 & \text{if } x \in (3, 7]. \end{cases}$$

Then 1 is the only coincidence point of f and g and $f(g1) = g(f1) = 1$. Therefore f and g are weakly compatible. But, consider a sequence $\{x_n\} \subset X$, where $x_n = 3 + \frac{1}{n}$ for all $n \in \mathbb{N}$ then $f(x_n) = 1$ and $g(x_n) = 1 + \frac{1}{n}$. Clearly, $\lim_{n\to\infty} g(x_n) = \lim_{n\to\infty} f(x_n) = 1$. Also, $f(gx_n) = f(1 + \frac{1}{n}) = 5$ and $g(fx_n) = g(1) = 1$, which implies that $\lim_{n\to\infty} d(gfx_n, fgx_n) = 4 \neq 0$. It follows that f and g are not compatible.

In subsequent years, various researchers of the domain studied so many weaker forms of compatibility and utilized the same to develop common fixed point theorems. The comprehensive and lucid collections of such conditions and their interplay can be found in Murthy [17], Kadelburg *et al*. [15] and Agarwal *et al*. [2].

On the other hand, fixed point theory for non-compatible mappings is equally interesting. In fact, Pant [18] has initiated the concept of coincidence and fixed point theorems for non-compatible mappings. One can establish fixed point theorems for such mappings pairs not only under non-expansive conditions but also under Lipschitz type conditions even without using the usual contractive method of proof. The best examples of non-compatible maps are found among pairs of mappings which are discontinuous at their common fixed point.

Definition 1.10. ([18]) Let (X, d) be a metric space and f and g be two self-mappings on X. We say that f and g are non-compatible if there exists a sequence $\{x_n\} \subset X$,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z,$$

for some $z \in X$ but $\lim_{n\to\infty} d(gfx_n, fgx_n)$ is either non-zero or non-existence.

Aamri and El Moutawakil [1] generalized the concept of non-compatible mappings by defining the following notion:

Definition 1.11. ([1]) Let (X, d) be a metric space and f and g be two self-mappings on X. We say that f and g satisfy $(E.A)$-property if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z$$

for some $z \in X$.

It may be noticed that the $(E.A)$-property is equivalent to the previously known notion of 'tangential mappings' introduced by Sastry *et al*. [19]. Clearly, a pair of non-compatible mappings satisfies $(E.A)$-property. The concept of $(E.A)$-property allows to replace the completeness requirement of the space with a more natural condition of closeness of the range. In fact the notion of $(E.A)$-property circumvents the most crucial part of fixed point theorems consisting of constructive procedures yielding a Cauchy sequence. For further details of the concept of $(E.A)$-property, we refer [9, 10].

Sintunavarat and Kumam [21] introduced an interesting property, which completely buys the condition of closedness of the ranges of the involved mappings and has an edge over the $(E.A)$-property.

Definition 1.12. ([21]) Let (X, d) be a metric space and f and g be two self-mappings on X. We say that f and g satisfy (CLR_g)-property (common limit in the range of g property) if there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(x)$$

for some $x \in X$.

Clearly, if f and g satisfy (CLR_g)-property, then they satisfy $(E.A)$-property.

Definition 1.13. ([19]) Let (X, d) be a metric space, f and g be two self-mappings on X and $x \in X$. We say that f is g-continuous at x if for all sequence $\{x_n\} \subset X$,

$$g(x_n) \xrightarrow{d} g(x) \;\Rightarrow\; f(x_n) \xrightarrow{d} f(x).$$

Moreover, f is called g-continuous if it is g-continuous at each point of X.

Notice that with $g = I$ (the identity mapping on X) Definition 1.13 reduces to the definition of continuity.

Theorem 1.14. ([19]) *Let (X, d) be a metric space and f and g be two self-mappings on X. Suppose that the following conditions hold:*

(i) *f and g satisfy $(E.A)$-property,*
(ii) *f is g-continuous,*
(iii) *either $g(X)$ is closed or $\overline{f(X)} \subseteq g(X)$.*

Then f and g have a coincidence point in X.

The main objective of the present article is three-fold:

- To unify the classical theorems of Machuca, Goebel and Jungck.
- To prove enriched and sharpened versions of these results via a new subset Y.
- To generalize results involving $(E.A)$-property and (CLR_g)-property besides filling the gaps in existing results.

2. Auxiliary results

In this section, we present some relevant results related to this section, which are needed in our main results. For a pair of self-mappings f and g on a nonempty set X, we denote the following sets:

$$\mathrm{C}(f,g)=\{x\in X: gx=fx\},$$

that is, the set of all coincidence points of f and g.

$$\overline{\mathrm{C}}(f,g)=\{\overline{x}\in X:\overline{x}=gx=fx,\ x\in X\},$$

that is, the set of all points of coincidence of f and g.

$$\mathrm{F}(f,g)=\{x\in X: x=gx=fx\},$$

that is, the set of all points of common fixed points of f and g.

The following relation is a straightforward fact.

$$\mathrm{F}(f,g)\subseteq \mathrm{C}(f,g)\cap\overline{\mathrm{C}}(f,g).$$

Definition 2.1. Let X be a nonempty set, f and g two self-mappings on X and $\{x_n\}\subset X$ a sequence. We say that $\{x_n\}$ is a sequence of joint iteration of f and g based at a point $x_0\in X$ if

$$g(x_{n+1})=f(x_n),\ \ \forall\ n\in\mathbb{N}_0.$$

Lemma 2.2. *Let f and g be two self-mappings on a nonempty set X such that $f(X)\subseteq g(X)$. Then there exists a sequence of joint iteration of f and g based on each point of X.*

Proof. Choose $x_0\in X$ arbitrarily and then by using assumption $f(X)\subseteq g(X)$, we can construct a sequence inductively $\{x_n\}\subset X$ such that $g(x_{n+1})=f(x_n)$ for all $n\in\mathbb{N}_0$. □

For the sake of completeness, we recall the following two elementary results, which indicates relation between the complete subspace and closed subspace of a metric space.

Lemma 2.3. *A complete subspace of a metric space is closed.*

Lemma 2.4. *A closed subspace of a complete metric space is complete.*

The following result will be utilized to prove our uniqueness results.

Lemma 2.5. *Let f and g be two self-mappings on a nonempty set X such that f and g have a unique point of coincidence. Then*

(i) *the point of coincidence remains a unique common fixed point provided f and g are weakly compatible.*

(ii) *f and g have a unique coincidence point provided one of f and g is one-one.*

Proof. (i) Given that f and g have a unique point of coincidence, say, w. Hence, we have $\overline{\mathrm{C}}(f,g) = \{w\}$. Now, show that w remains a unique common fixed point of f and g. Clearly, for each $x \in \mathrm{C}(f,g)$, we get

$$w = g(x) = f(x).$$

By using weakly compatibility of f and g, we have

$$g(w) = g(fx) = f(gx) = f(w),$$

which implies that $w \in \mathrm{C}(f,g)$ yielding thereby $g(w)(= f(w)) \in \overline{\mathrm{C}}(f,g)$. It follows that

$$w = g(w) = f(w).$$

Hence, w is a common fixed point of f and g, *i.e.*, $w \in \mathrm{F}(f,g)$ so that $\overline{\mathrm{C}}(f,g) \subseteq \mathrm{F}(f,g)$. But, obviously, we have $\mathrm{F}(f,g) \subseteq \overline{\mathrm{C}}(f,g)$, which yields that $\mathrm{F}(f,g) = \overline{\mathrm{C}}(f,g) = \{w\}$ so that w is a unique common fixed point of f and g.

(ii) Take $x, y \in \mathrm{C}(f,g)$, then by using uniqueness of point of coincidence, we have

$$g(x) = f(x) = f(y) = g(y).$$

As f or g is one to one, we have $x = y$, which implies uniqueness of coincidence point. □

Lemma 2.6. *Let (X, d) be a metric space. Also, let f and g be two self-mappings such that f is a g-contraction. Then*

(i) *f is g-continuous.*

(ii) *f is continuous provided g is continuous.*

Proof. The proof of above result is easy and hence we skip it. □

3. On unifying classical coincidence and common fixed point theorems

Now, we are equipped to prove a unified version of Theorems 1.1, 1.2 and 1.3 regarding the existence and uniqueness of point of coincidence as well as coincidence point on a complete metric space.

Theorem 3.1. *Let (X, d) be a complete metric space and f and g be two self-mappings on X. Suppose that the following conditions hold:*

(*a*) $f(X) \subseteq g(X)$,
(*b*) f *is a* g*-contraction,*
(*c*) g *is continuous and* f *and* g *are compatible, or alternately,*
(c') *there exists a closed subspace* Y *of* X *such that* $f(X) \subseteq Y \subseteq g(X)$.

Then f *and* g *have a unique point of coincidence. Moreover, if one of* f *and* g *is one to one, then* f *and* g *have a unique coincidence point.*

Proof. Take arbitrary $x_0 \in X$. Using assumption (a) and Lemma 2.2, we construct a sequence $\{x_n\} \subset X$ such that

$$g(x_{n+1}) = f(x_n), \quad \forall\, n \in \mathbb{N}_0. \tag{3.1}$$

By using Eq. (3.1) and g-contractivity of f, we obtain

$$d(gx_n, gx_{n+1}) = d(fx_{n-1}, fx_n) \leq \alpha d(gx_{n-1}, gx_n), \quad \forall\, n \in \mathbb{N}.$$

By induction, we have, for all $n \in \mathbb{N}$,

$$d(gx_n, gx_{n+1}) \leq \alpha d(gx_{n-1}, gx_n) \leq \alpha^2 d(gx_{n-2}, gx_{n-1}) \leq \cdots \leq \alpha^n d(gx_0, gx_1),$$

so that

$$d(gx_n, gx_{n+1}) \leq \alpha^n d(gx_0, gx_1), \quad \forall\, n \in \mathbb{N}. \tag{3.2}$$

For $n < m$, using Eq. (3.2), we obtain

$$\begin{aligned} d(gx_n, gx_m) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + \cdots + d(gx_{m-1}, gx_m) \\ &\leq (\alpha^n + \alpha^{n+1} + \cdots + \alpha^{m-1}) d(gx_0, gx_1) \\ &= \frac{\alpha^n - \alpha^m}{1-\alpha} d(gx_0, gx_1) \\ &\leq \frac{\alpha^n}{1-\alpha} d(gx_0, gx_1) \\ &\to 0 \text{ as } m, n \to \infty, \end{aligned}$$

which follows that the sequence $\{gx_n\}$ is Cauchy. As X is complete, there exists $z \in X$ such that

$$\lim_{n\to\infty} g(x_n) = z. \tag{3.3}$$

By using (3.1) and (3.3), we obtain

$$\lim_{n\to\infty} f(x_n) = z. \tag{3.4}$$

Now, we use assumptions (c) and (c') to accomplish the proof. Assume that (c) holds. Using (3.3), (3.4) and continuity of g, we obtain

$$\lim_{n\to\infty} g(gx_n) = g(\lim_{n\to\infty} gx_n) = g(z) \tag{3.5}$$

and

$$\lim_{n\to\infty} g(fx_n) = g(\lim_{n\to\infty} fx_n) = g(z). \tag{3.6}$$

Using (3.3), (3.4) and compatibility of f and g, we obtain

$$\lim_{n\to\infty} d(fgx_n, gfx_n) = 0. \tag{3.7}$$

By using assumption (b), we obtain

$$d(fz, fgx_n) \leq \alpha d(gz, ggx_n), \ \ \forall\ k \in \mathbb{N}_0. \tag{3.8}$$

By using triangular inequality, (3.5), (3.6), (3.7) and (3.8), we get

$$\begin{aligned} d(fz, gz) &\leq d(fz, fgx_n) + d(fgx_n, gfx_n) + d(gfx_n, gz) \\ &\leq \alpha d(gz, ggx_n) + d(fgx_n, gfx_n) + d(gfx_n, gz) \\ &\to 0 \ \text{ as } \ n \to \infty \end{aligned}$$

so that

$$f(z) = g(z).$$

Thus, z is a coincidence point of f and g and hence we are through.

Now, assume that (c') holds. As Y is closed and $f(X) \subseteq Y$, using (3.6), we have $z \in Y$. Owing to assumption $Y \subseteq g(X)$, we can find some $u \in X$ such that $z = g(u)$. Hence, (3.3) and (3.4) respectively reduce to

$$\lim_{n\to\infty} g(x_n) = g(u) \tag{3.9}$$

and

$$\lim_{n\to\infty} f(x_n) = g(u). \tag{3.10}$$

By using assumption (b) and (3.9), we obtain

$$d(fx_n, fu) \leq \alpha d(gx_n, gu) \to 0 \ \text{ as } \ n \to \infty$$

so that

$$\lim_{n\to\infty} f(x_n) = f(u). \tag{3.11}$$

By using (3.10), (3.11) and uniqueness of limit, we get

$$f(u) = g(u).$$

Thus, u is a coincidence point of f and g and hence we are done.

To prove uniqueness of point of coincidence, take $\overline{x}, \overline{y} \in \overline{\mathrm{C}}(f, g)$, then there exist $x, y \in \mathrm{C}(f, g)$ such that

$$f(x) = g(x) = \overline{x} \ \text{ and } \ f(y) = g(y) = \overline{y}. \tag{3.12}$$

By using assumption (b) and (3.12), we have

$$d(\overline{x}, \overline{y}) = d(fx, fy) \leq \alpha d(gx, gy) = \alpha d(\overline{x}, \overline{y})$$

so that $\overline{x} = \overline{y}$. It follows that f and g have a unique point of coincidence. Finally, uniqueness of coincidence point is directly followed by using part (ii) of Lemma 2.5. □

Remark 3.2. In view of Theorem 3.1, we can say that the closedness of range subspace ($f(X)$, or $g(X)$) in the hypotheses of Theorem 1.1 is not necessary as it can be alternately replace by the closedness of any arbitrary subspace having the property that $f(X) \subseteq Y \subseteq g(X)$.

Corollary 3.3. *Let (X, d) be a complete metric space and f and g be two self-mappings on X. Suppose that the following conditions hold:*

(i) *either g is onto or $\overline{f(X)} \subseteq g(X)$,*

(ii) *f is a g-contraction.*

Then f and g have a unique point of coincidence. Moreover, if one of f and g is one to one, then f and g have a unique coincidence point.

Proof. Notice that the assumption (ii) of above corollary remains same as assumption (b) of Theorem 3.1. Thus, in order to prove our result, it is enough to prove that rest conditions of the hypotheses of Theorem 3.1 are also satisfied. To prove this, firstly suppose that g is onto, then $g(X) = X$ and assumptions (a) as well as (c') trivially hold (as $f(X) \subseteq Y = g(X) = X$). Secondly, if $\overline{f(X)} \subseteq g(X)$, then again assumptions (a) as well as (c') hold (as $f(X) \subseteq Y = \overline{f(X)} \subseteq g(X)$). Therefore, in both the cases, the conclusion is immediate by using Theorem 3.1. □

In the following lines, we prove a common fixed point theorem corresponding to Theorem 3.1, which is indeed an improved version of classical common fixed point theorem of Jungck (that is, Theorem 1.3).

Theorem 3.4. *Let (X, d) be a complete metric space and f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) *$f(X) \subseteq g(X)$,*

(b) *f is a g-contraction,*

(c) *g is continuous and f and g are compatible,*
or alternately,

(c') *there exists a closed subspace Y of X such that $f(X) \subseteq Y \subseteq g(X)$ and f and g are weakly compatible.*

Then f and g have a unique common fixed point.

Proof. In both the cases (c) and (c'), the mappings f and g are weakly compatible. Hence, using part (i) of Lemma 2.5, our result follows. □

4. Enriched results on coincidence and common fixed points

Now, we prove a sharpened version of foregoing results on a metric space (not necessarily complete) but have a complete subspace.

Theorem 4.1. *Let (X, d) be a metric space and Y a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X) \cap Y$,
(b) f *is a* g*-contraction,*
(c) g *is continuous and* f *and* g *are compatible,*
or alternately,
(c') $Y \subseteq g(X)$.

Then f and g have a unique point of coincidence. Moreover, if one of f and g is one to one, then f and g have a unique coincidence point.

Proof. The proof of above result runs analogously on the lines of the proof of Theorem 3.1. Firstly, we notice that the hypothesis $f(X) \subseteq g(X) \cap Y$ is equivalent to saying that $f(X) \subseteq g(X)$ and $f(X) \subseteq Y$. Following the lines of the proof of Theorem 3.1, we can construct a sequence $\{x_n\} \subset X$ of joint iteration of f and g based at an arbitrary point $x_0 \in X$ (due to availability of $f(X) \subseteq g(X)$) and then, we can show that the sequence $\{gx_n\}$ (and hence $\{fx_n\}$ also) is Cauchy. As $\{gx_n\} \subset f(X) \subseteq Y$, $\{gx_n\}$ is a Cauchy sequence in Y. By completeness of Y, there exists $z \in Y$ such that (3.3) and (3.4) are satisfied.

If assumption (c) holds, then followed by the similar lines of the proof of Theorem 3.1, we can prove that z remains a coincidence point of f and g in Y. On the other hand, if (c') holds, then using assumption $Y \subseteq g(X)$, we can find some $u \in X$ such that $z = g(u)$ such that (3.9) and (3.10) hold. Proceeding on the lines of the proof of Theorem 3.1, one can show that u is a coincidence point of f and g. The proof of uniqueness part is similar to the corresponding part of the proof of Theorem 3.1. $\square$

Remark 4.2. As a consequence, Theorem 3.1 can be deduced from Theorem 4.1. The result corresponding to part (c) follows easily on setting $Y = X$, while the same (result) in the presence of part (c') follows using Lemma 2.4.

Now, we present a common fixed point theorem corresponding to Theorem 4.1 as follows:

Theorem 4.3. *Let (X, d) be a metric space and Y be a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq g(X) \cap Y$,
(b) f *is a* g*-contraction,*
(c) g *is continuous and* f *and* g *are compatible, or alternately,*
(c') $Y \subseteq g(X)$ *and* f *and* g *are weakly compatible.*

Then f and g have a unique common fixed point.

Proof. In both the cases (c) and (c'), the mappings f and g are weakly compatible. Hence, using part (i) of Lemma 2.5, the conclusion is immediate. □

Remark 4.4. Using an argument similar to Remark 4.2, we can deduce Theorem 3.4 from Theorem 4.3.

Combining assumptions (a) and (c'), we can rewrite the form of Theorem 4.1 (also Theorem 4.3) corresponding to assumption (c') as follows.

Corollary 4.5. *Let (X, d) be a metric space and Y be a complete subspace of X. Let f and g be two self-mappings on X. Suppose that the following conditions hold:*

(a) $f(X) \subseteq Y \subseteq g(X)$,
(b) *f is g-contraction.*

Then f and g have a unique point of coincidence. Moreover,

(i) *the point of coincidence remains a unique common fixed point provided f and g are weakly compatible,*
(ii) *f and g have a unique coincidence point provided one of f and g is one to one.*

Remark 4.6. Notice that above corollary improves the result corresponding to the part (c') of Theorem 3.1 (also, Theorem 3.4) on a metric space (not necessarily complete) but have a complete subspace. In view of Corollary 4.5, we can say that the completeness of range subspace ($f(X)$, or $g(X)$) in the hypotheses of Theorem 1.2 is not necessary as it can be alternately replace by the completeness of any arbitrary subspace having the property that $f(X) \subseteq Y \subseteq g(X)$.

Remark 4.7. In view of Lemmas 2.3 and 2.4, if X is complete then the notions of 'closedness' and 'completeness' are equivalent. It concludes that the results corresponding to the part (c') of Theorem 3.1 remains a consequence of Corollary 4.5.

5. Results involving $(E.A)$-property and (CLR_g)-property

The following result establishes the superiority of the ideas of $(E.A)$- property and (CLR_g)-property over another assumptions involved in the hypotheses classical coincidence and common fixed point theorems.

Lemma 5.1. *Let (X, d) be a metric space and f and g be two self-mappings on X such that f is a g-contraction. If Y is a complete subspace of X such that $f(X) \subseteq g(X)$, then f and g satisfy $(E.A)$-property. In addition, if there* *exists a closed subspace Y of X such that $f(X) \subseteq Y \subseteq g(X)$, then the pair f and g satisfy (CLR_g)-property.*

Proof. Using the same argument as in the proof of Theorem 3.1, due to availability of assumption $f(X) \subseteq g(X)$, we define a sequence $\{x_n\} \subset X$ of joint iteration of f and g based at an arbitrary point $x_0 \in X$ and then by using g-contractility of f, it can be shown that $\{fx_n\}$ and $\{gx_n\}$ both are Cauchy sequences in X. By completeness of X, there exists $z \in X$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z. \tag{5.1}$$

If follows that f and g satisfy $(E.A)$- property,

Further, suppose that Y is closed. Then $z \in Y$ as $f(X) \subseteq Y$. Due to the relation $Y \subseteq g(X)$, we have $z = g(u)$, for some $u \in X$. Hence, Eq. (5.1) reduces to

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = f(u). \tag{5.2}$$

□

Now, we present the main result of this section, which sharpens Theorem 1.14 and runs as follows:

Theorem 5.2. *Let (X,d) be a metric space. Let f and g be two self-mappings on X satisfying $(E.A)$-property. Suppose that one of the following conditions hold:*

(i) *f and g are compatible as well as continuous,*
(ii) *either $g(X)$ is closed or $\overline{f(X)} \subseteq g(X)$. Also, f is g-continuous.*

Then f and g have a coincidence point.

Proof. As f and g satisfy $(E.A)$- property, there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = z \tag{5.3}$$

for some $z \in X$.

Firstly, suppose that (i) holds. Using (5.3) and continuity of f, we have

$$\lim_{n\to\infty} f(gx_n) = f(\lim_{n\to\infty} gx_n) = f(z). \tag{5.4}$$

Using (5.3) and continuity of g, we have

$$\lim_{n\to\infty} g(fx_n) = g(\lim_{n\to\infty} fx_n) = g(z). \tag{5.5}$$

By using (5.3), (5.4), (5.5), continuity of d and compatibility of f and g, we obtain

$$\begin{aligned} d(fz, gz) &= d(\lim_{n\to\infty} fgx_n, \lim_{n\to\infty} gfx_n) \\ &= \lim_{n\to\infty} d(fgx_n, gfx_n) \\ &= 0 \end{aligned}$$

so that

$$f(z) = g(z).$$

Thus $z \in X$ is a coincidence point of f and g and hence we are through.

Secondly, assume that (ii) holds. Owing to the closedness of $g(X)$, we have

$$z = \lim_{n\to\infty} g(x_n) \in g(X).$$

Otherwise, we obtain

$$z = \lim_{n\to\infty} f(x_n) \in \overline{f(X)} \subseteq g(X).$$

Henceforth, in both the cases, we conclude that $z \in g(X)$, which ensures the existence of $u \in X$ such that $z = g(u)$. Therefore, (5.3) reduces to

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(u). \tag{5.6}$$

Using (5.6) and g-continuity of f, we get

$$\lim_{n\to\infty} f(x_n) = f(u). \tag{5.7}$$

By using (5.6) and (5.7), we get

$$g(u) = f(u).$$

Hence, $u \in X$ is a coincidence point of f and g. This completes the proof. $(E.A)$-property and (CLR_g)-property □

Using the fact that (CLR_g)-property implies $(E.A)$-property, we can say that the term $(E.A)$-property can be replaced by (CLR_g)-property in Theorem 5.2. But main advantage of (CLR_g)-property is that the assumption: "either $g(X)$ is closed or $\overline{f(X)} \subseteq g(X)$" can be relaxed in condition (ii). For the sake of completeness, we present the results involving (CLR_g)-property corresponding to part (ii) as follows:

Theorem 5.3. *Let (X, d) be a metric space. Let f and g be two self-mappings on X satisfying (CLR_g)-property. If f is g-continuous, then f and g have a coincidence point.*

Proof. As f and g satisfy (CLR_g)-property, there exists a sequence $\{x_n\} \subset X$ such that

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} g(x_n) = g(w) \tag{5.8}$$

for some $w \in X$. Next, following the lines similar to the proof of previous result, we use g-continuity of f to prove that $w \in X$ is a coincidence point of f and g. □

Conclusion: Although, Theorems 5.2 and 5.3 admit unnatural weaker conditions, yet the behaviour of such results are similar as topological coincidence theorems on metrical structure rather than metrical coincidence theorems. Recall that topological fixed point results refers those results in which underlying mapping admits topological properties (such as: continuity) rather than geometric property (such as: contraction). A topological space X is said to have the fixed point property if every continuous mapping on X admits a fixed point. Using Lemmas 2.6 and 5.1, Theorem 5.2 and Theorem 5.3 deduce our results proved in Section 3.

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