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NON-INVARIANT HYPERSURFACES OF A (ϵ, δ)-TRANS SASAKIAN MANIFOLDS

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Abstract. The object of this paper is to study non-invariant hypersurface of a (ϵ, δ)-trans Sasakian manifolds equipped with (f, g, u, v, λ)-structure. Some properties obeyed by this structure are obtained. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurface with (f, g, u, v, λ)-structure of a (ϵ, δ)-trans Sasakian manifolds to be totally geodesic. The second fundamental form of a non-invariant hypersurface of a (ϵ, δ)-trans Sasakian manifolds with (f, g, u, v, λ)-structure has been traced under the condition when f is parallel.

1. Introduction

The study of (ϵ)-Sasakian manifolds have been studies by Bejancu and Duggal [2], and Xufeng and Xiaoli [10] studied that these manifolds are real hypersurface of indefinite Kahlerian manifolds. Tripathi et al. [9] introduced and studied (ϵ)-almost para contact manifolds. De and Sarkar [4] also introduced (ϵ)-Kenmotsu manifolds and studied conformally flat, Weyl semisymmetric, ϕ-recurrent (ϵ)-Kenmotsu manifolds. Nagaraja et al. [7] studied (ϵ, δ)-trans Sasakian structure.

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In 1970, Goldberg et al. [5] introduced the notion of a non-invariant hypersurface of an almost contact manifold in which the transform of a tangent vector of the hypersurface by the (1,1)-structure tensor field f defining the almost contact structure is never tangent to the hypersurface. The notion of (f, g, u, v, λ)-structure was given by Yano and Okumura [11]. It is well known ([12] and [3]) that hypersurface of an almost contact metric manifold always admits a (f, g, u, v, λ)-structure. In [5], author proved that there always exists a (f, g, u, v, λ)-structure on a non-invariant hypersurface of an almost contact metric manifold. They also proved that there does not exist invariant hypersurface of a contact manifold. Prasad [8] studied the non-invariant hypersurface of trans Sasakian manifolds. Khan [6] studied the non-invariant hypersurface of Nearly Kenmotsu manifold. Ahmed et el. [1] studied the non-invariant hypersurface of nearly hyperbolic Sasakian manifold. In the present paper, we study the non-invariant hypersurface of (ϵ, δ)-trans Sasakian manifolds.

This paper is organized as follows. In section 2, we give a brief description of (ϵ, δ)-trans Sasakian manifolds. In section 3, introduce the non-invariant hypersurface and induced (f, g, u, v, λ)-structure on non-invariant hypersurface M getting some equation. Some results of non-invariant hypersurface with (f, g, u, v, λ)-structure of (ϵ, δ)-trans Sasakian manifolds. The necessary and sufficient conditions also have been obtained for totally umbilical non-invariant hypersurface with (f, g, u, v, λ)-structure of (ϵ, δ)-trans Sasakian manifolds to be totally geodesic.

2. Preliminaries

Let $\widetilde{M}$ be a n-dimensional almost contact metric manifold with the almost contact metric structure (ϕ, ξ, η, g) where a tensor ϕ of type (1,1), a vector field ξ, called structure vector field and η, the dual 1-form and a Riemannian metric g satisfying the following,

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.1}$$

$$\eta(\xi) = 1, \ \eta(\phi X) = 0, \ \phi\xi = 0. \tag{2.2}$$

An almost contact metric manifold $\widetilde{M}$ is called an (ϵ)-almost contact metric manifold if

$$\eta(X) = \epsilon g(X, \xi), g(\xi, \xi) = \epsilon, \tag{2.3}$$

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon\eta(X)\eta(Y), \tag{2.4}$$

$$g(\phi X, Y) = -g(X, \phi Y), \tag{2.5}$$

for all $X, Y \in TM$ [10], where $\epsilon = g(\xi, \xi) = \pm 1$.

An (ϵ)-almost contact metric manifold is called an (ϵ,δ)-trans Sasakian manifold [9] if

$$(\widetilde{\nabla}_X\phi)Y = \alpha\{g(X,Y)\xi - \epsilon\eta(Y)X\} + \beta\{g(\phi X,Y)\xi - \delta\eta(Y)\phi X\}, \tag{2.6}$$

$$\widetilde{\nabla}_X\xi = -\epsilon\alpha(\phi X) - \delta\beta\phi^2 X, \tag{2.7}$$

hold for some smooth function α and β on $\widetilde{M}$ and $\epsilon = \pm 1$, $\delta = \pm 1$. For $\beta = 0$, $\alpha = 1$ an (ϵ,δ)-tans Sasakian manifold reduces to an (ϵ)-Sasakian manifold and for $\alpha = 0, \beta = 1$, it is reduced to a (δ)-Kenmotsu manifold.

A hypersurface of an almost contact metric manifold $\widetilde{M}$ is called a non-invariant hypersurface, if the transform of a tangent vector of the hypersurface under the action of $(1,1)$ tensor field ϕ defining the contact structure is never tangent to the hypersurface. Let X be tangent vector on non-invariant hypersurface of an almost contact metric manifold $\widetilde{M}$. Then ϕX is never to tangent of the hypersurface. Let $\widetilde{M}$ be a non-invariant hypersurface of an almost contact metric manifold. Now, we define the following:

$$\phi X = fX + u(X)\widetilde{N}, \tag{2.8}$$

$$\phi\widetilde{N} = -U, \tag{2.9}$$

$$\xi = V + \lambda\widetilde{N}, \lambda = \eta(\widetilde{N}), \tag{2.10}$$

$$\eta(X) = v(X), \tag{2.11}$$

where f is (1,1) tensor field, u and v are 1-form, $\widetilde{N}$ is a unit normal to the hypersurface, $X \in TM$ and $u(X) \neq 0$. Then we get an induced (f,g,u,v,λ)-structure on $\widetilde{M}$ satisfying the conditions

$$\left.\begin{array}{c} f^2 = -I + u\otimes U + v\otimes V, \\ uof = \lambda v, vof = -\lambda u, \\ v(V) = 1-\lambda^2, u(V) = v(U) = 0, u(U) = 1-\lambda^2, \\ fV = \lambda U, fU = \lambda V, \\ u(X) = \epsilon g(X,U), v(X) = \epsilon g(X,V), \\ g(fX,fY) = g(X,Y) - u(X)u(Y) - \epsilon v(X)v(Y), \\ g(fX,Y) = -g(X,fY), \end{array}\right\} \tag{2.12}$$

for all $X, Y \in TM$ and $\lambda = \eta(\widetilde{N})$.

The Gauss and Weingarten formula are given by

$$\widetilde{\nabla}_X Y = \widetilde{\nabla}_X Y + h(X,Y)\widetilde{N}, \tag{2.13}$$

$$\widetilde{\nabla}_X\widetilde{N} = -A_{\widetilde{N}}X, \tag{2.14}$$

for all $X, Y \in TM$, where $\widetilde{\nabla}$ and ∇ are the Riemannian and induced connection on $\widetilde{M}$ and M respectively and $\widetilde{N}$ is the unit normal vector in the normal bundle $T^{\perp}M$. In this formula h is the second fundamental form on M related to $A_{\widetilde{N}}$ by

$$h(X,Y) = g(A_{\widetilde{N}}X, Y), \tag{2.15}$$

for all $X, Y \in TM$.

3. Some properties of non-invariant hypersurfaces

Lemma 3.1. *Let M be a non-invariant hypersurface with (f, g, u, v, λ)-structure of (ϵ, δ)-trans Sasakian manifold $\widetilde{M}$. Then*

$$(\widetilde{\nabla}_X \phi)Y = (\nabla_X f)Y - u(Y)A_{\widetilde{N}}X + h(X,Y)U + ((\nabla_X u)Y + h(X, fY))\widetilde{N}, \tag{3.1}$$

$$(\widetilde{\nabla}_X \eta)Y = (\nabla_X v)Y - \lambda h(X,Y), \tag{3.2}$$

$$\widetilde{\nabla}_X \xi = \nabla_X V - \lambda A_{\widetilde{N}}X + (h(X,V) + X\lambda)\widetilde{N}, \tag{3.3}$$

for all $X, Y \in TM$.

Proof. Consider:

$$(\widetilde{\nabla}_X \phi)Y = \widetilde{\nabla}_X \phi Y - \phi(\widetilde{\nabla}_X Y).$$

Using (2.8) and (2.13), we have

$$\begin{aligned}
(\widetilde{\nabla}_X \phi)Y &= \widetilde{\nabla}_X (fX + u(Y)\widetilde{N}) - \phi(\widetilde{\nabla}_X Y + h(X,Y)\widetilde{N}) \\
(\widetilde{\nabla}_X \phi)Y &= \widetilde{\nabla}_X fX + \widetilde{\nabla}_X (u(Y)\widetilde{N}) - \phi\widetilde{\nabla}_X Y - h(X,Y)\phi\widetilde{N} \\
(\widetilde{\nabla}_X \phi)Y &= \widetilde{\nabla}_X fX + h(X, fY)\widetilde{N} + u(Y)\widetilde{\nabla}_X \widetilde{N} + (\widetilde{\nabla}_X u(Y))\widetilde{N} - f(\widetilde{\nabla}_X Y) \\
&\quad - u(\widetilde{\nabla}_X Y)\widetilde{N} + h(X,Y)U \\
(\widetilde{\nabla}_X \phi)Y &= (\widetilde{\nabla}_X f)X + f(\widetilde{\nabla}_X X) - u(Y)A_{\widetilde{N}}X + h(X,Y)U + h(X, fY)\widetilde{N} \\
&\quad - f(\widetilde{\nabla}_X Y) + (\widetilde{\nabla}_X u(Y))\widetilde{N} - u(\widetilde{\nabla}_X Y)\widetilde{N} \\
(\widetilde{\nabla}_X \phi)Y &= (\widetilde{\nabla}_X f)X - u(Y)A_{\widetilde{N}}X + h(X,Y)U + h(X, fY)\widetilde{N} + (\widetilde{\nabla}_X u(Y))\widetilde{N} \\
&\quad - u(\widetilde{\nabla}_X Y)\widetilde{N} \\
(\widetilde{\nabla}_X \phi)Y &= (\widetilde{\nabla}_X f)X - u(Y)A_{\widetilde{N}}X + h(X,Y)U + h(X, fY)\widetilde{N} + (\widetilde{\nabla}_X u(Y) \\
&\quad + h(X, u(Y))\widetilde{N} - u(\widetilde{\nabla}_X Y))\widetilde{N} \\
(\widetilde{\nabla}_X \phi)Y &= (\widetilde{\nabla}_X f)X - u(Y)A_{\widetilde{N}}X + h(X,Y)U + ((\widetilde{\nabla}_X u)Y + h(X, fY))\widetilde{N}.
\end{aligned}$$

Also, we have

$$(\widetilde{\nabla}_X \eta)Y = \widetilde{\nabla}_X \eta(Y) - \eta(\widetilde{\nabla}_X Y).$$

Using (2.8), (2.11) and (2.13), we have

$$(\widetilde{\nabla}_X\eta)Y=\widetilde{\nabla}_X(v(Y))-\eta(\widetilde{\nabla}_XY),$$
$$(\widetilde{\nabla}_X\eta)Y=\widetilde{\nabla}_X(v(Y))+h(X,v(Y))\widetilde{N}-\eta(\widetilde{\nabla}_XY+h(X,Y)\widetilde{N}),$$
$$(\widetilde{\nabla}_X\eta)Y=\widetilde{\nabla}_X(v(Y)-\eta(\widetilde{\nabla}_XY)-h(X,Y)\eta(\widetilde{N}),$$
$$(\widetilde{\nabla}_X\eta)Y=\widetilde{\nabla}_Xv(Y)-v(\widetilde{\nabla}_XY)-h(X,Y)\eta(\widetilde{N}),$$
$$(\widetilde{\nabla}_X\eta)Y=(\widetilde{\nabla}_Xv)Y-\lambda h(X,Y).$$

Further, consider using (2.13) and using (2.10), we have

$$\widetilde{\nabla}_X\xi=\widetilde{\nabla}_X\xi+h(X,\xi)\widetilde{N},$$
$$\widetilde{\nabla}_X\xi=\widetilde{\nabla}_X(V+\lambda\widetilde{N})+h(X,V+\lambda\widetilde{N})\widetilde{N},$$
$$\widetilde{\nabla}_X\xi=\widetilde{\nabla}_XV+\widetilde{\nabla}_X(\lambda\widetilde{N})+h(X,V)\widetilde{N}+\lambda h(X,\widetilde{N})\widetilde{N},$$
$$\widetilde{\nabla}_X\xi=\widetilde{\nabla}_XV+\lambda(\widetilde{\nabla}_X\widetilde{N})+(X\lambda)\widetilde{N}+h(X,V)\widetilde{N},$$
$$\widetilde{\nabla}_X\xi=\widetilde{\nabla}_XV-\lambda A_{\widetilde{N}}X+(h(X,V)+X\lambda)\widetilde{N},$$

for all $X,Y\in TM$. □

Theorem 3.2. *Let M be a non-invariant hypersurface with (f,g,u,v,λ)-structure of (ϵ,δ)-trans Sasakian manifold $\widetilde{M}$. Then we have*

$$h(X,\xi)=\epsilon\alpha f^2X-\epsilon\alpha u(X)U-\delta\beta fX+f(\widetilde{\nabla}_X\xi),\tag{3.4}$$
$$u(\widetilde{\nabla}_X\xi)=-\epsilon\alpha u(fX)+\delta\beta u(X),\tag{3.5}$$

for all $X,Y\in TM$.

Proof. Consider

$$(\widetilde{\nabla}_X\phi)\xi=\widetilde{\nabla}_X\phi\xi-\phi(\widetilde{\nabla}_X\xi),$$
$$(\widetilde{\nabla}_X\phi)\xi=-\phi(\widetilde{\nabla}_X\xi).\tag{3.6}$$

Using equations (2.2), (2.7), (2.8) and (2.9) in above, we have

$$(\widetilde{\nabla}_X\phi)\xi=-\phi(-\epsilon\alpha(\phi X)-\delta\beta\phi^2X),$$
$$(\widetilde{\nabla}_X\phi)\xi=\phi(\epsilon\alpha(fX+u(X)\widetilde{N}))+\delta\beta\phi(-X+\eta(X)\xi),$$
$$(\widetilde{\nabla}_X\phi)\xi=\epsilon\alpha f^2X+\epsilon\alpha u(Xf)\widetilde{N}-\epsilon\alpha u(X)U-\delta\beta fX-\delta\beta u(X)\widetilde{N}.\tag{3.7}$$

Using equation (2.13) in (3.6), we get

$$(\widetilde{\nabla}_X\phi)\xi=-\phi(\widetilde{\nabla}_X\xi)-h(X,\xi)\phi\widetilde{N}.$$

Using equation (2.8) and (2.9) in above, we get

$$(\widetilde{\nabla}_X\phi)\xi=-f(\widetilde{\nabla}_X\xi)-u(\widetilde{\nabla}_X\xi)\widetilde{N}+h(X,\xi)U.\tag{3.8}$$

Comparing equation (3.7) and (3.8), we have

$$\begin{aligned}&- f(\widetilde{\nabla}_X\xi) - u(\widetilde{\nabla}_X\xi)\widetilde{N} + h(X,\xi)U \\ &\quad = \epsilon\alpha f^2X + \epsilon\alpha u(Xf)\widetilde{N} - \epsilon\alpha u(X)U - \delta\beta fX - \delta\beta u(X)\widetilde{N}.\end{aligned}$$

Equating tangential and normal parts on both sides, we have

$$h(X,\xi)U = \epsilon\alpha f^2X - \epsilon\alpha u(X)U - \delta\beta fX + f(\widetilde{\nabla}_X\xi)$$

and

$$u(\widetilde{\nabla}_X\xi) = -\epsilon\alpha u(Xf) + \delta\beta u(X),$$

for all $X, Y \in TM$. □

Theorem 3.3. *Let M be a non-invariant hypersurface with (f,g,u,v,λ)-structure of (ϵ,δ)-trans Sasakian manifold $\widetilde{M}$. Then, we have*

$$\begin{aligned}(\nabla_X f)Y = {}& u(Y)A_{\widetilde{N}}X - h(X,Y)U + \alpha g(X,Y)V \\ &- \epsilon\alpha v(Y)X + \beta g(fX,Y)V - \delta\beta v(Y)fX\end{aligned} \tag{3.9}$$

and

$$(\widetilde{\nabla}_X u)Y = \lambda\alpha g(X,Y) + \lambda\beta g(fX,Y) - \delta\beta v(Y)u(X) - h(X,fY), \tag{3.10}$$

for all $X, Y \in TM$.

Proof. Consider covariant differentiation, then we have

$$(\widetilde{\nabla}_X\phi)Y = \widetilde{\nabla}_X\phi Y - \phi(\widetilde{\nabla}_X Y). \tag{3.11}$$

Using equation (2.8) in (2.13), we have

$$(\widetilde{\nabla}_X\phi)Y = \widetilde{\nabla}_X fY + \widetilde{\nabla}_X(u(Y)\widetilde{N}) - \phi\widetilde{\nabla}_X Y - h(X,Y)\phi\widetilde{N}.$$

Using (2.8), (2.9) and (2.13), we have

$$\begin{aligned}(\widetilde{\nabla}_X\phi)Y = {}& \widetilde{\nabla}_X fY + h(X,fY)\widetilde{N} + u(Y)(\widetilde{\nabla}_X\widetilde{N}) \\ &+ (\widetilde{\nabla}_X u(Y))\widetilde{N} - f\widetilde{\nabla}_X Y - u(\widetilde{\nabla}_X Y)\widetilde{N} + h(X,Y)U.\end{aligned}$$

Using (2.13) and (2.14) in above, we have

$$(\widetilde{\nabla}_X\phi)Y = (\widetilde{\nabla}_X f)Y - u(Y)A_{\widetilde{N}}X + h(X,Y)U + ((\widetilde{\nabla}_X u)Y + h(X,fY))\widetilde{N}. \tag{3.12}$$

Now, using (2.8), (2.10) and (2.11) in (2.6), we have

$$\begin{aligned}(\widetilde{\nabla}_X\phi)Y = {}& \alpha g(X,Y)V + \lambda\alpha g(X,Y)\widetilde{N} - \epsilon\alpha v(Y)X + \beta g(fX,Y)V \\ &+ \lambda\beta g(fX,Y)\widetilde{N} - \delta\beta v(Y)fX - \delta\beta v(Y)u(X)\widetilde{N}.\end{aligned} \tag{3.13}$$

Comparing (3.12) and (3.13), we have

$$\begin{aligned}(\nabla_X f)Y - u(Y)A_{\widetilde{N}}X + h(X,Y)U + ((\widetilde{\nabla}_X u)Y + h(X,fY))\widetilde{N} \\ = \alpha g(X,Y)V + \lambda\alpha g(X,Y)\widetilde{N} - \epsilon\alpha v(Y)X + \beta g(fX,Y)V \\ + \lambda\beta g(fX,Y)\widetilde{N} - \delta\beta v(Y)fX - \delta\beta v(Y)u(X)\widetilde{N}.\end{aligned}$$

Equating tangential and normal part, we have

$$\begin{aligned}(\nabla_X f)Y = u(Y)A_{\widetilde{N}}X - h(X,Y)U + \alpha g(X,Y)V - \epsilon\alpha v(Y)X \\ + \beta g(fX,Y)V - \delta\beta v(Y)fX\end{aligned}$$

and

$$(\widetilde{\nabla}_X u)Y = \lambda\alpha g(X,Y) + \lambda\beta g(fX,Y) - \delta\beta v(Y)u(X) - h(X,fY),$$

for all $X,Y \in TM$. □

Theorem 3.4. *Let M be a non-invariant hypersurface with (f,g,u,v,λ)-structure of (ϵ,δ)-trans Sasakian manifold $\widetilde{M}$. Then we have*

$$\widetilde{\nabla}_X V = \lambda A_{\widetilde{N}}X - \epsilon\alpha fX + \delta\beta X - \delta\beta v(X)V \tag{3.14}$$

and

$$h(X,V) = -\epsilon\alpha u(X) - \lambda\delta\beta v(X) - X\lambda, \tag{3.15}$$

for all $X,Y \in TM$.

Proof. Using equation (2.1), (2.8) and (2.11) in (2.7), we have

$$\widetilde{\nabla}_X\xi = -\epsilon\alpha fX - \epsilon\alpha u(X)\widetilde{N} + \delta\beta X - \delta\beta v(X)V - \lambda\delta\beta v(X)\widetilde{N}. \tag{3.16}$$

Comparing equation (3.16) and (3.3) we have

$$\begin{aligned}\widetilde{\nabla}_X V - \lambda A_{\widetilde{N}}X + (h(X,V) + X\lambda)\widetilde{N} \\ = -\epsilon\alpha fX - \epsilon\alpha u(X)\widetilde{N} + \delta\beta X - \delta\beta v(X)V - \lambda\delta\beta v(X)\widetilde{N}.\end{aligned}$$

Equating tangential and normal part, we have

$$\widetilde{\nabla}_X V = \lambda A_{\widetilde{N}}X - \epsilon\alpha fX + \delta\beta X - \delta\beta v(X)V$$

and

$$h(X,V) = -\epsilon\alpha u(X) - \lambda\delta\beta v(X) - X\lambda,$$

for all $X,Y \in TM$. □

Theorem 3.5. *Let M be a non-invariant hypersurface with (f, g, u, v, λ)-structure of (ϵ, δ)-trans Sasakian manifold $\widetilde{M}$. Then, we have*

$$(\widetilde{\nabla}_X \phi)Y = \alpha g(X,Y)V - \epsilon v(Y)X + \beta g(fX,Y)V - \delta\beta v(Y)fX + (\lambda\alpha g(X,Y) + \lambda\beta g(fX,Y) - \delta\beta v(Y)u(X))\widetilde{N} \tag{3.17}$$

for all $X, Y \in TM$.

Proof. Using (3.9) and (3.10) in (3.13), we have

$$\begin{aligned}(\widetilde{\nabla}_X \phi)Y &= u(Y)A_{\widetilde{N}}X - h(X,Y)U + \alpha g(X,Y)V - \epsilon\alpha v(Y)X + \beta g(fX,Y)V \\ &\quad - \delta\beta v(Y)fX - u(Y)A_{\widetilde{N}}X + h(X,Y)U + (\lambda\alpha g(X,Y) + \lambda\beta g(fX,Y) \\ &\quad - \delta\beta v(Y)u(X) - h(X,fY) + h(X,fY))\widetilde{N}, \\ (\widetilde{\nabla}_X \phi)Y &= \alpha g(X,Y)V - \epsilon\alpha v(Y)X + \beta g(fX,Y)V - \delta\beta v(Y)fX + (\lambda\alpha g(X,Y) \\ &\quad + \lambda\beta g(fX,Y) - \delta\beta v(Y)u(X))\widetilde{N}, \\ (\widetilde{\nabla}_X \phi)Y &= \alpha\{g(X,Y)V - \epsilon v(Y)X\} + \beta\{g(fX,Y)V - \delta\beta v(Y)fX\} \\ &\quad + (\lambda\alpha g(X,Y) + \lambda\beta g(fX,Y) - \delta\beta v(Y)u(X))\widetilde{N},\end{aligned}$$

for all $X, Y \in TM$. □

Theorem 3.6. *Let M be a totally umbilical non-invariant hypersurface with (f, g, u, v, λ)-structure of (ϵ, δ)-trans Sasakian manifold $\widetilde{M}$. Then it is totally geodesic if and only if*

$$\epsilon\alpha u(X) + \lambda\delta\beta v(X) + X\lambda = 0, \tag{3.18}$$

for all $X, Y \in TM$.

Proof. Using equation (2.1), (2.8) and (2.11) in (2.7), we have

$$\widetilde{\nabla}_X \xi = -\epsilon\alpha fX - \epsilon\alpha u(X)\widetilde{N} + \delta\beta X - \delta\beta v(X)V - \lambda\delta\beta v(X)\widetilde{N}.$$

Using (3.3) in above equation, we have

$$\begin{aligned}\widetilde{\nabla}_X V - \lambda A_{\widetilde{N}}X + (h(X,V) + X\lambda)\widetilde{N} &= -\epsilon\alpha fX - \epsilon\alpha u(X)\widetilde{N} + \delta\beta X \\ &\quad - \delta\beta v(X)V - \lambda\delta\beta v(X)\widetilde{N}.\end{aligned}$$

Equating normal part, we have

$$h(X,V) = -\epsilon\alpha u(X) - \lambda\delta\beta v(X) - X\lambda. \tag{3.19}$$

If M is totally umbilical, then $A_{\widetilde{N}} = \zeta I$, where ζ is Kahlerian metric

$$\begin{aligned}&h(X,Y) = g(A_{\widetilde{N}}X,Y) = g(\zeta X,Y) = \zeta g(X,Y) = \zeta v(X), \\ &h(X,V) = \zeta g(X,V) = \zeta v(X).\end{aligned} \tag{3.20}$$

Then, from (3.19) and (3.20), we, have

$$\epsilon\alpha u(X) + \lambda\delta\beta v(X) + X\lambda + \zeta v(X) = 0. \tag{3.21}$$

If M is totally geodesic, that is, $\zeta = 0$, then from (3.21), we have

$$\epsilon\alpha u(X) + \lambda\delta\beta v(X) + X\lambda = 0,$$

for all $X, Y \in TM$. □

Theorem 3.7. *Let M be a non-invariant hypersurface with (f, g, u, v, λ)-structure of (ϵ, δ)-trans Sasakian manifold $\widetilde{M}$. If U is parallel, then we have*

$$\epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) = 0, \tag{3.22}$$

for all $X, Y \in TM$.

Proof. Consider covariant differentiation, then we have

$$(\widetilde{\nabla}_X\phi)\widetilde{N} = \widetilde{\nabla}_X\phi\widetilde{N} - \phi(\widetilde{\nabla}_X\widetilde{N}). \tag{3.23}$$

Using equation (2.8), (2.9), (2.13) and (2.14) in above, we have

$$(\widetilde{\nabla}_X\phi)\widetilde{N} = \nabla_X\phi\widetilde{N} + h(X, \phi\widetilde{N})\widetilde{N} - f(\widetilde{\nabla}_X\widetilde{N}) - u(\widetilde{\nabla}_X\widetilde{N})\widetilde{N},$$
$$(\widetilde{\nabla}_X\phi)\widetilde{N} = -\nabla_X U + f(A_{\widetilde{N}}X). \tag{3.24}$$

From (2.6), we have

$$(\widetilde{\nabla}_X\phi)\widetilde{N} = \alpha\{g(X, \widetilde{N})\xi - \epsilon\lambda X\} + \beta\{g(\phi X, \widetilde{N})\xi - \delta\lambda\phi X\},$$
$$(\widetilde{\nabla}_X\phi)\widetilde{N} = -\epsilon\alpha\lambda X - \beta\delta\lambda(fX) - \beta\delta\lambda u(X)\widetilde{N}. \tag{3.25}$$

From (3.25) and (3.26), we have

$$-\nabla_X U + f(A_{\widetilde{N}}X) = -\epsilon\alpha\lambda X - \beta\delta\lambda(fX) - \beta\delta\lambda u(X)\widetilde{N},$$
$$\nabla_X U = \epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) + \beta\delta\lambda u(X)\widetilde{N}.$$

If U is parallel, then $\nabla_X U = 0$, so from above equation, we have

$$\epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) + \beta\delta\lambda u(X)\widetilde{N} = 0.$$

Now, equating tangential part, we have

$$\epsilon\alpha\lambda X + f(A_{\widetilde{N}}X) + \beta\delta\lambda(fX) = 0,$$

for all $X, Y \in TM$. □

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