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SOLUTIONS OF A CLASS OF COUPLED SYSTEMS OF FUZZY DELAY DIFFERENTIAL EQUATIONS

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Abstract. The purpose of this paper is to introduce and study a class of coupled systems of fuzzy delay differential equations involving fuzzy initial values and fuzzy source functions of triangular type. We assume that these initial values and source functions are triangular fuzzy functions and define solutions of the coupled systems as a triangular fuzzy function matrix consisting of real functional matrices. The method of triangular fuzzy function, fractional steps and fuzzy terms separation are used to solve the problems. Furthermore, we prove existence and uniqueness of solution for the considered systems, and then a solution algorithm is proposed. Finally, we present an example to illustrate our main results and give some work that can be done later.

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1. Introduction

In an ecosystem, it is out of the question for only one object to exist, so we need to consider that many species exist and interact with each other (that is, coupled systems). Moreover, owing to a lot of objective reasons in real life, such as the aging of production facilities, time delay and ambiguity should be taken into account. Thus, in this paper, motivated by the work of Fatullayev et al. [7] and Gasilov and Amrahov [9], we come up with the following coupled system of fuzzy delay differential equations (FDDCS):

$$\begin{cases} x^{'}(t) = -m_1(t)x(t) + n_1(t)y(t-\tau) + \widetilde{F_1}(t), \ t > 0, \\ y^{'}(t) = -m_2(t)y(t) + n_2(t)x(t-\tau) + \widetilde{F_2}(t), \ t > 0, \\ x(t) = \widetilde{\Phi_1}(t), \ -\tau \le t \le 0, \\ y(t) = \widetilde{\Phi_2}(t), \ -\tau \le t \le 0, \end{cases} \tag{1.1}$$

and explore existence and uniqueness result of solution for the FDDCS (1.1), where τ is the value of time delay, and $\widetilde{\Phi_i}(t)$ is triangular fuzzy function (TFF) defined on $[-\tau, 0]$, $\widetilde{F_i}(t)$ is also TFF on $(0, \infty)$ and, $m_i(t)$ and $n_i(t)$ are continuous crisp functions for $i = 1, 2$.

If $\tau = 0$, the FDDCS (1.1) can be rewritten as the fuzzy delay differential coupled system as follows:

$$\begin{cases} x^{'}(t) = -m_1(t)x(t) + n_1(t)y(t) + \widetilde{F_1}(t), \ t > 0, \\ y^{'}(t) = -m_2(t)y(t) + n_2(t)x(t) + \widetilde{F_2}(t), \ t > 0, \\ x(0) = \widetilde{\Phi_1}(0), \ y(0) = \widetilde{\Phi_2}(0). \end{cases} \tag{1.2}$$

We note that (1.2) is new and not reported in the literature, and the system (1.1) is an extension of the linear inhomogeneous fuzzy delay differential equations (FDDEs) studied by Fatullayev et al. [7]:

$$\begin{cases} x'(t) = n(t)x(t) + m(t)x(t-\tau) + \widetilde{F}(t), \quad t > 0, \\ x(t) = \widetilde{\Phi}(t), \ -\tau \le t \le 0, \end{cases}$$

which solution was shown as a fuzzy set of real functions via presenting a method, and Fatullayev et al. [7] also proved existence and uniqueness of solution for FDDE involving TFFs. Furthermore, Fatullayev et al. [7] put forward that the proposed method can be extended to the system of FDDEs by using the research results of Gasilov and Amrahov [9].

In this paper, we shall generalize the method proposed in [7] to discuss existence and uniqueness of solution for the FDDCS (1.1).

As we all know, there are not only a single species in biological ecosystems. That is to say, multiple species exist and there must be competition among them. Thus, it is indispensable to take coupling into account (see [6, 30]).

Coupling relationship refers to the interaction and mutual influence between two or more objects. The coupled systems of differential equations in broader domains, such as ordinary differential equations and functional differential-difference equations, have been studied by multitudinous researchers.

Recently, Wang et al. [25] considered existence of extreme solutions to a class of coupled causal differential equations as follows:

$$\begin{cases} x^{'}(t) = Q_1(x, y)(t),\ t \in I, \\ y^{'}(t) = Q_2(y, x)(t),\ t \in I, \\ g_i(x(0), x(\Upsilon), y(0), y(\Upsilon)) = 0,\ \text{for } i = 1, 2, \end{cases}$$

and sufficient conditions under which the equations have extreme solutions were obtained. Gasilov and Amrahov [9] investigated the following system of linear differential equation:

$$\begin{cases} x^{'}(t) = a_{11}\ x(t) + a_{12}\ y(t) + F_1, \quad t > 0, \\ y^{'}(t) = a_{21}\ y(t) + a_{22}\ x(t) + F_2, \quad t > 0, \\ x(0) = A, \quad y(0) = B, \end{cases}$$

where a_{ij} is a given real number for $i, j = 1, 2$, $A = [a_1, a_2]$ and $B = [b_1, b_2]$ are given intervals, $F_i = \langle f_{ia}, f_{ib} \rangle$ is given convex bunche of function for $i = 1, 2$, and also expressed the fuzzy terms in the problem as interval values. Assuming these real functions to be linearly distributed between two given real functions as for each forcing term, Gasilov and Amrahov [9] presented a new approach to nonhomogeneous systems of interval differential equations and established an existence and uniqueness theorem. For more detail work, one can refer to [3, 4, 17, 21] and the references cited therein.

On the other hand, in the actual production process, time delay usually occurs on account of a variety of reasons in science and engineering. Thereby, in recent years, the scholars have been devoted themselves to solve the differential equations with time delay (DDEs). Weng [26] present an efficient algorithm based on Schauder's fixed point theorem and researched existence of positive T-periodic solutions for the following problem:

$$y^{'}(t) = p(t)y(t) + q(t)y^k(t - \tau(t)), \quad t \geq 0.$$

Of course, DDEs may be linear [5] or non-linear [28]. By using an appropriate fixed point theorem, Miraoui and Repovs [20] obtained several new sufficient conditions which ensure existence, exponential stability, and uniqueness of (μ, ν)-pap solutions for a class of DDEs. Furthermore, time delay can also be combined with coupling, which is of great significance for studying biological problems in particular.

In [24], the following biological mathematical model of mRNA was mentioned, which is a kind of coupled delay differential equations (CDDEs):

$$\begin{cases} \dfrac{dm}{dt} = -\mu_m m(t) + H(p(t-T)), \\ \dfrac{dp}{dt} = m(t) - \mu_p p(t). \end{cases}$$

More papers on CDDEs are available to readers. See, for instance, [2, 13, 22, 29] and the references therein.

Besides coupling and time delay, ambiguity is also the one we need to think about. As Rouvray [23] pointed out, "all scientific pronouncements have some inherent uncertainty about them and cannot be assumed to be strictly valid". In fact, many practical problems arising in physics, biology, engineering, signal processing, finance and other fields are often uncertainty. As one of the more restrictive uncertainty models, the fuzzy set theory has been further developed and a wide number of applications to dynamical conducts arising in many mathematical or computer models of some deterministic real-world phenomena affected by uncertainty and has attracted the attention of several researchers. That is to say, it is interesting and very important to explore fuzzy differential equations (FDEs).

In recent years, research on FDEs has been continuously investigated. For instance, Gomes et al. [11] focused on FDEs and explained the basics of various approaches of FDEs. Liu [15] provided a numerical method to solve a FDE via differential inclusions and showed that the solution of a FDE via differential inclusions is proved to be equal to that of the master equation of fuzzy dynamics. Other recent related work of considering (implicit) FDEs, see, for example, [12, 16, 27] and references cited therein.

The remainder of this paper is organized as follows. The necessary preliminaries about the fuzzy theory that we are going to use are listed in Section 2. In Section 3, we introduce concept of solution of the FDDCS (1.1) and describe how to get it. Ultimately, a solution algorithm is proposed. In Section 4, we present an example to illustrate the correctness of the solution algorithm. Finally, in Section 5, we summarize the research results of this paper, and put forward the content that one can study in the future.

2. Preliminaries

We define a fuzzy set $\widetilde{A}$ as a pair of the universal set U and the membership function (MF) $\mu : U \to [0, 1]$. The MF of a fuzzy set $\widetilde{A}$ can be denoted as $\mu_{\widetilde{A}}$. For each $x \in U$, the numerical value $\mu_{\widetilde{A}}(x)$ is called the membership degree

(MD) of x in $\widetilde{A}$. The crisp set $supp(\widetilde{A}) = \{x \in U \mid \mu_{\widetilde{A}}(x) > 0\}$ is called the support of $\widetilde{A}$.

Afterwards, let U be the set of real numbers $\mathbb{R}$, and a, c and b be real numbers which meet $a \leq c \leq b$. Then the set $\widetilde{u}$ with MF

$$\mu(x) = \begin{cases} \frac{x-a}{c-a}, & a < x < c, \\ 1, & x = c, \\ \frac{b-x}{b-c}, & c < x < b, \\ 0, & \text{otherwise} \end{cases}$$

is called a triangular fuzzy number (TFN) and we denote it as $\widetilde{u} = (a, c, b)$. On the grounds of the geometric interpretation, the number c is called the vertex of $\widetilde{u}$, we denote $\underline{u} = a$ and $\overline{u} = b$ to represent the left and the right end-points of $\widetilde{u}$, respectively. Frequently, we express $\widetilde{u} = (a, c, b)$ as $\widetilde{u} = u_{cr} + \widetilde{u}_{un}$. Here, $u_{cr} = c$ is the crisp part and $\widetilde{u}_{un} = (a - c, 0, b - c)$ is the uncertain part of $\widetilde{u}$.

It is also useful to represent the fuzzy sets though their α-*cuts*. For each $\alpha \in (0, 1]$, the crisp set $A_\alpha = \{x \in U \mid \mu_{\widetilde{A}}(x) \geq \alpha\}$ is called the α-*cut* of $\widetilde{A}$. For $\alpha = 0$, the $A_0 = closure(supp(\widetilde{A}))$.

For the TFN $\widetilde{u} = (a, c, b)$, the α-*cuts* are intervals $u_\alpha = [\underline{u_\alpha}, \overline{u_\alpha}]$, where $\underline{u_\alpha} = a + \alpha(c - a)$ and $\overline{u_\alpha} = b + \alpha(c - b)$. These formulas can be rewritten as $\underline{u_\alpha} = c + (1 - \alpha)(a - c)$ and $\overline{u_\alpha} = c + (1 - \alpha)(b - c)$. Hence, $u_\alpha = [\underline{u_\alpha}, \overline{u_\alpha}] = c + (1 - \alpha)[a - c, b - c]$. From here we can see that an α-*cut* is homothetic to $[a, b]$ (which is the 0-*cut*) with center c and with ratio $(1 - \alpha)$.

There are different notions about the fuzzy functions. In this study, we use the concept of the fuzzy function which was proposed by Gasilov et al. [10], namely, fuzzy function is a bunch of fuzzy real functions. As a value $\widetilde{F}(t)$ of a fuzzy bunch $\widetilde{F}$ at time t, we understand the fuzzy set, which elements are the values of the real functions at t, with the higher MD of the corresponding functions. Mathematically, $\mu_{\widetilde{F}(t)}(x) = \alpha$ if and only if

$$\exists y(\cdot) : \left(\mu_{\widetilde{F}}(y) = \alpha \wedge y(t) = x\right) \wedge \forall z(\cdot) : \left(\mu_{\widetilde{F}}(z) > \alpha \rightarrow z(t) \neq x\right),$$

where "$\wedge$" and "$\rightarrow$" are the logical conjunction and implication symbols, respectively.

Definition 2.1. ([10]) Let U be a set of continuous functions defined on an interval I, and $F_a(\cdot), F_c(\cdot), F_b(\cdot) \in U$. We call the fuzzy subset $\widetilde{F}$ of U, determined by the MF as follows:

$$\mu_{\widetilde{F}(y(\cdot))} = \begin{cases} \alpha, & y = F_a + \alpha(F_c - F_a) \quad \text{and} \quad 0 < \alpha \leq 1, \\ \alpha, & y = F_b + \alpha(F_c - F_b) \quad \text{and} \quad 0 < \alpha \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

as TFF and denote it as $\widetilde{F} = \langle F_a, F_c, F_b \rangle$.

According to this definition, a TFF is a fuzzy set (or, fuzzy bunch) of real functions. Among them only two functions have the MD α: the functions $y_1 = F_a + \alpha(F_c - F_a)$ and $y_2 = F_b + \alpha(F_c - F_b)$.

In order to intuitively show a group of TFFs, in the next moment we give the following example based on the corresponding work of Gasilov et al. [10].

Example 2.2. In Figure 1, we depict a group of TFFs as

$$\begin{cases} \widetilde{F_1} = \langle F_{1a}, F_{1c}, F_{1b} \rangle, \\ \widetilde{F_2} = \langle F_{2a}, F_{2c}, F_{2b} \rangle, \end{cases}$$

where

(1) $F_{1a}(t) = -t^2 + 6\,t - 5$ (MD is 0, the black curve that is at bottom on $[0, 1]$ and at upper on $[1, 2]$);
(2) $F_{1c}(t) = t^2 - 3\,t + 2$ (MD is 1, the black dashed line);
(3) $F_{1b}(t) = t^2 - 6\,t + 5$ (MD is 0, the black curve that is at upper on $[0, 1]$ and at bottom on $[1, 2]$);
(4) $F_{2a}(t) = -3\,t^2 + 6\,t - 3$ (MD is 0, the blue curve that is at bottom on $[0, 2]$);
(5) $F_{2c}(t) = t^2 - 2\,t + 1$ (MD is 1, the blue dashed line);
(6) $F_{2b}(t) = 3\,t^2 - 6\,t + 3$ (MD is 0, the blue curve that is at upper on $[0, 2]$).

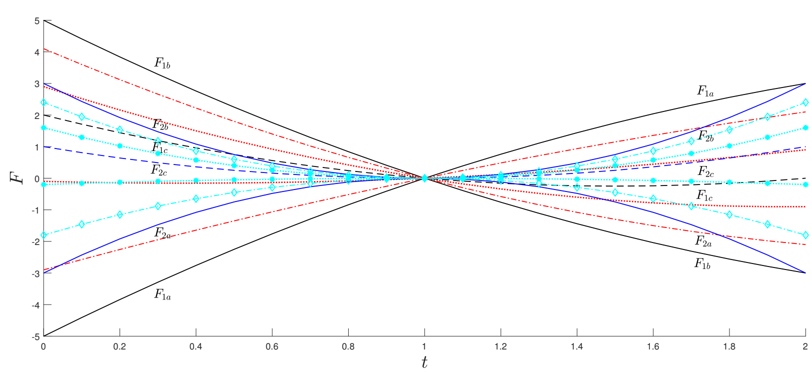

FIGURE 1. A group of TFFs

The TFF $\widetilde{F_1}$ with MDs 0.7 and 0.3 are depicted by red dotted and dashed-dotted lines, respectively. And the TFF $\widetilde{F_2}$ with MDs 0.7 and 0.3 are described by cyan dotted and dashed-dotted lines which are marked, respectively.

The value of a TFF at a time $t \in I$ can be expressed by the following formula:

$$\widetilde{F}(t) = \left(\min\left\{F_a(t), F_c(t), F_b(t)\right\}, F_c(t), \max\left\{F_a(t), F_c(t), F_b(t)\right\}\right).$$

We can easily find out that this value is a TFN, and a TFF $\widetilde{F} = \langle F_a, F_b, F_c \rangle$ is not a fuzzy number-valued function. In fact, it is a fuzzy subset of the universe of continuous functions. Each element of this fuzzy subset is a real function with a certain MD.

If $F_a(t) \le F_c(t) \le F_b(t)$ for all $t \in I$, then the TFF $\widetilde{F} = \langle F_a, F_c, F_b \rangle$ is regular TFF on I, and we have $\widetilde{F}(t) = (F_a(t), F_c(t), F_b(t))$. Further, when a TFF $\widetilde{F}$ is not regular, we call it as non-regular TFF. It means that for a non-regular TFF, in general, $\widetilde{F}(t) \neq (F_a(t), F_c(t), F_b(t))$. And the graphs of functions F_a, F_c and F_b can interchanged as t goes. In Figure 1, the TFFs $\widetilde{F_1}$ and $\widetilde{F_2}$ are non-regular and regular on $[0, 2]$, respectively. Without loss of generality, no matter regular TFFs or non-regular TFFs, main results and the algorithm in this paper are all valid. In the sequel, we assume that TTFs are regular.

3. Main results

As for the FDDCS (1.1), we can solve it by the method of steps.

At first, we deal with the FDDCS (1.1) on interval $[0, \tau]$. Since $x(t-\tau) = \widetilde{\Phi_1}(t-\tau), y(t-\tau) = \widetilde{\Phi_2}(t-\tau)$ for $t \in [0, \tau]$, the FDDCS (1.1) is equivalent to the fuzzy coupled initial value problem with delays (FCDIVP) as follows:

$$\begin{cases} x'(t) = -m_1(t)x(t) + \widetilde{G_1}(t), \\ y'(t) = -m_2(t)y(t) + \widetilde{G_2}(t), \\ x(0) = \widetilde{\Phi_1}(0), \\ y(0) = \widetilde{\Phi_2}(0), \end{cases} \tag{3.1}$$

where

$$\widetilde{G_1}(t) = n_1(t)\widetilde{\Phi_2}(t-\tau) + \widetilde{F_1}(t), \quad \widetilde{G_2}(t) = n_2(t)\widetilde{\Phi_1}(t-\tau) + \widetilde{F_2}(t).$$

Then, we transform the FCDIVP (3.1) into matrix form:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t) + \widetilde{G}(t), \\ Z(0) = \widetilde{\Phi}(0), \end{cases} \tag{3.2}$$

where,

$$Z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} -m_1(t) & 0 \\ 0 & -m_2(t) \end{pmatrix},$$

$$\widetilde{G}(t) = \begin{pmatrix} \widetilde{G_1}(t) \\ \widetilde{G_2}(t) \end{pmatrix}, \quad \widetilde{\Phi}(0) = \begin{pmatrix} \widetilde{\Phi_1}(0) \\ \widetilde{\Phi_2}(0) \end{pmatrix}.$$

Definition 3.1. For the problem (3.2), the fuzzy set $\widetilde{Z}$ with MF

$$\mu_{\widetilde{Z}}\left(Z(\cdot)\right) = \min\left\{\mu_{\widetilde{\Phi}}\left(\phi(0)\right), \mu_{\widetilde{G}}\left(Z^{'}(t) - A(t)\cdot Z(t)\right)\right\} \tag{3.3}$$

is called to be a solution of the problem (3.2), which is also a solution to the FCDIVP (3.1), where $\phi(0) = Z(0)$.

However, how to comprehend the formula (3.3)?

Let $Z(t)$ be a functional matrix. We determine $\phi(0) = Z(0)$ for $t \in [0, \tau]$ and calculate $\mu_1 \triangleq \mu_{\widetilde{\Phi}}(\phi(0))$. After that, we compute $g(t) = Z^{'}(t) - A(t)\cdot Z(t)$ on interval $[0, \tau]$ and determine $\mu_2 \triangleq \mu_{\widetilde{G}}(g(t))$. Finally, we calculate the MD μ as $\mu = \min\{\mu_1, \mu_2\}$. We assign the number μ as the MD of $Z(t)$ and define the set of all functional matrices such as $Z(t)$ with their MDs μ as the fuzzy solution $\widetilde{Z}$.

According to Definition 3.1, the solution $\widetilde{Z}$ is a fuzzy bunch of real functional matrices, which consists of functional matrices such as $Z(t)$. If a functional matrix $Z(t)$ satisfies

$$\begin{cases} Z^{'}(t) = A(t)\cdot Z(t) + g(t), \\ Z(0) = \phi(0), \end{cases}$$

for some functional matrices $g \in supp(\widetilde{G})$ and $\phi(0) \in supp(\widetilde{\Phi}(0))$, then it has a positive MD.

Let us represent $\widetilde{G_1} = g_{1cr} + \widetilde{g_1}$(crisp part + uncertainty), where $g_{1cr} = G_{1c}$ and $\widetilde{g_1} = \langle g_{1a}, 0, g_{1b}\rangle = \langle G_{1a} - G_{1c}, 0, G_{1b} - G_{1c}\rangle$. Similarly, $\widetilde{G_2} = g_{2cr} + \widetilde{g_2}$, where $g_{2cr} = G_{2c}$ and $\widetilde{g_2} = \langle g_{2a}, 0, g_{2b}\rangle$.

Assume that $\widetilde{\Phi_1}$ and $\widetilde{\Phi_2}$ are regular TFFs. Then $\widetilde{\Phi_1}(0)$ and $\widetilde{\Phi_2}(0)$ are TFNs. Further, $\widetilde{\Phi_1}(0) = \phi_{1cr}(0) + \widetilde{\phi_1}(0)$, here $\phi_{1cr}(0) = \Phi_{1c}(0)$ and $\widetilde{\phi_1}(0) = (\phi_{1a}(0), 0, \phi_{1b}(0)) = (\Phi_{1a}(0) - \Phi_{1c}(0), 0, \Phi_{1b}(0) - \Phi_{1c}(0))$. In the same way, $\widetilde{\Phi_2}(0) = \phi_{2cr}(0) + \widetilde{\phi_2}(0)$, here $\phi_{2cr}(0) = \Phi_{2c}(0)$ and $\widetilde{\phi_2}(0) = (\phi_{2a}(0), 0, \phi_{2b}(0))$. Thus, we can obtain

$$\widetilde{G}(t) = g_{cr}(t) + \widetilde{g}(t), \quad \widetilde{\Phi}(0) = \phi_{cr}(0) + \widetilde{\phi}(0),$$

where

$$\widetilde{G}(t) = \begin{pmatrix} \widetilde{G_1}(t) \\ \widetilde{G_2}(t) \end{pmatrix}, \quad g_{cr}(t) = \begin{pmatrix} g_{1cr}(t) \\ g_{2cr}(t) \end{pmatrix}, \quad \widetilde{g}(t) = \begin{pmatrix} \widetilde{g_1}(t) \\ \widetilde{g_2}(t) \end{pmatrix};$$

$$\widetilde{\Phi}(0) = \begin{pmatrix} \widetilde{\Phi_1}(0) \\ \widetilde{\Phi_2}(0) \end{pmatrix}, \quad \phi_{cr}(0) = \begin{pmatrix} \phi_{1cr}(0) \\ \phi_{2cr}(0) \end{pmatrix}, \quad \widetilde{\phi}(0) = \begin{pmatrix} \widetilde{\phi_1}(0) \\ \widetilde{\phi_2}(0) \end{pmatrix}.$$

On account of the problem (3.2) is linear, we can solve it by using the superposition principle. This follows that in order to get the solution of the problem (3.2), we can consider the following subproblems, separately:

(1) The associated inhomogeneous crisp problem:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t) + g_{cr}(t), \\ Z(0) = \phi_{cr}(0). \end{cases} \tag{3.4}$$

(2) The homogeneous problem with initial TFFs:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t), \\ Z(0) = \widetilde{\phi}(0). \end{cases} \tag{3.5}$$

(3) The problem with fuzzy source functions and zero initial functions:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t) + \widetilde{g}(t), \\ Z(0) = \mathbf{0}. \end{cases} \tag{3.6}$$

By solving the above three subproblems (3.4)-(3.6), we can obtain the solution of the FDDCS (1.1) on interval $[0, \tau]$. Similarly, at the next step, the solution on interval $[\tau, 2\tau]$ can be found. Therefore, we can conclude that the solution of the FDDCS (1.1) exists for $t \in [0, \infty)$ using the method of steps.

In the sequel, we make clear how to solve each of these three subproblems (3.4)-(3.6).

3.1. **The associated inhomogeneous crisp problem.** Consider the problem (3.4), it has a unique solution, on condition that $m_i(t)$, $g_{icr}(t)$ and $\phi_{icr}(t)$ are continuous functions for $i = 1, 2$.

Using the integrating factor matrix $U(s) = e^{-\int_0^s A(t)\,dt}$, the solution is

$$Z_{cr}(t) = \frac{1}{U(t)} \left(\phi_{cr}(0) + \int_0^t U(s) g_{cr}(s)\, ds \right). \tag{3.7}$$

3.2. **The homogeneous problem with initial TFFs.** This subsection explains how to solve the problem (3.5). Now, we give the following theorem.

Theorem 3.2. *Consider the problem* (3.5)*, where* $\widetilde{\phi_1}(0) = (\phi_{1a}(0), 0, \phi_{1b}(0))$, $\widetilde{\phi_2}(0) = (\phi_{2a}(0), 0, \phi_{2b}(0))$. *And* $m_1(t), m_2(t)$ *are continuous functions. If* $Z_a(t) = \begin{pmatrix} x_a(t) \\ y_a(t) \end{pmatrix}$ *and* $Z_b(t) = \begin{pmatrix} x_b(t) \\ y_b(t) \end{pmatrix}$ *are solutions of the problem*

$$\begin{cases} Z'(t) = A(t) \cdot Z(t), \\ Z(0) = \phi(0), \end{cases} \tag{3.8}$$

for $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_{1a}(0) \\ \phi_{2a}(0) \end{pmatrix}$ *and* $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_{1b}(0) \\ \phi_{2b}(0) \end{pmatrix}$*, respectively, then the problem* (3.5) *has a unique solution* $\widetilde{Z_\phi}$*, which is a TFF matrix given by*

$$\widetilde{Z_\phi} = \langle Z_a, 0, Z_b \rangle. \tag{3.9}$$

Proof. First and foremost, on the basis of Definition 3.1, we know that each $Z(t)$ with nonzero MD from the bunch $\widetilde{Z_\phi}$ is a solution of the problem (3.8) for some $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix}$ from $\begin{pmatrix} \widetilde{\phi_1}(0) \\ \widetilde{\phi_2}(0) \end{pmatrix}$ with ease.

Secondly, according to Definition 2.1, the bunch $\widetilde{\phi_1} = \langle \phi_{1a}, 0, \phi_{1b} \rangle$ consists of functions $k\phi_{1a}$ and $k\phi_{1b}$ $([0,1] \ni k = 1 - \alpha)$ owing to $\phi_{1c} = 0$. And the bunch $\widetilde{\phi_2} = \langle \phi_{2a}, 0, \phi_{2b} \rangle$ is similar to the former.

In addition, if a matrix Z(t) is a solution of the problem (3.8), then $kZ(t)$ is a solution of the same equation with $\begin{pmatrix} k\phi_1(0) \\ k\phi_2(0) \end{pmatrix}$ taken instead of $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix}$.

Based on the above analysis we can get the conclusion that the bunch $\widetilde{Z_\phi}$ consists of kZ_a and kZ_b. Therefore, the bunch $\widetilde{Z_\phi}$ is a TFF matrix determined to be $\widetilde{Z_\phi} = \langle Z_a, 0, Z_b \rangle$. □

As a matter of fact, we can get the solution $Z_a(t)$, and the solution $Z_b(t)$ can be solved by the same way. It is well known that the solution $Z_a(t)$ can be expressed as $Z_a(t) = e^{\int_0^t A(s)\,ds}\phi_{01}(0)$, where $\phi_{01}(0) = \begin{pmatrix} \phi_{1a}(0) \\ \phi_{2a}(0) \end{pmatrix}$.

The value of the TFF matrix $\widetilde{Z_\phi}$ (3.9) at a time t can be written as the formula $\widetilde{Z_\phi}(t) = (\min\{Z_a(t), 0, Z_b(t)\}, 0, \max\{Z_a(t), 0, Z_b(t)\})$. Note that this value is a matrix of TFNs.

3.3. **The problem with fuzzy source functions and zero initial functions.** With respect to the problem (3.6), the theorem is given to deal with it as follows.

Theorem 3.3. *For the problem* (3.6)*, where* $\widetilde{g_1} = \langle g_{1a}, 0, g_{1b} \rangle, \widetilde{g_2} = \langle g_{2a}, 0, g_{2b} \rangle$*, and* $m_1(t), m_2(t)$ *are continuous functions. If* $Z_u(t) = \begin{pmatrix} x_u(t) \\ y_u(t) \end{pmatrix}$ *and* $Z_v(t) = \begin{pmatrix} x_v(t) \\ y_v(t) \end{pmatrix}$ *are solutions of the problem*

$$\begin{cases} Z'(t) = A(t) \cdot Z(t) + g(t), \\ Z(0) = \boldsymbol{0}, \end{cases} \tag{3.10}$$

for $\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} g_{1a}(t) \\ g_{2a}(t) \end{pmatrix}$ *and* $\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} g_{1b}(t) \\ g_{2b}(t) \end{pmatrix}$ *in several, then the problem* (3.6) *has a unique solution* $\widetilde{Z_g}$*, which is a TFF matrix given by*

$$\widetilde{Z_g} = \langle Z_u, 0, Z_v \rangle . \tag{3.11}$$

Proof. The proof can be done by the same way as in Theorem 3.2. □

Actually, we can obtain the solution $Z_u(t)$, and the solution $Z_v(t)$ can be worked out by the same method. The solution $Z_u(t)$, as everyone knows, can be written as

$$Z_u(t) = \frac{1}{U(t)} \left(\int_0^t U(s) g_1(s)\, ds \right),$$

where $U(s) = e^{-\int_0^s A(t)\,dt}$ and $g_1(s) = \begin{pmatrix} g_{1a}(s) \\ g_{2a}(s) \end{pmatrix}$. Therefore, we can also express the value of the TFF matrix $\widetilde{Z_g}$ (3.11) at a time t by

$$\widetilde{Z_g}(t) = (\min\{Z_u(t), 0, Z_v(t)\}, 0, \max\{Z_u(t), 0, Z_v(t)\}),$$

which is a matrix of TFNs.

3.4. **A solution algorithm.** From what has been discussed above, one can get the following solution algorithm for solving the FDDCS (1.1).

Algorithm 3.4.

Step 1 Using the method of steps and solving the FDDCS (1.1) on interval $[0, \tau]$, transform the FDDCS (1.1) into the FCDIVP (3.1). Further, change the FCDIVP (3.1) to the problem (3.2).

Step 2 Represent the source functions and initial values as

$$\widetilde{G_1} = g_{1cr} + \langle g_{1a}, 0, g_{1b} \rangle, \quad \widetilde{G_2} = g_{2cr} + \langle g_{2a}, 0, g_{2b} \rangle,$$

$$\widetilde{\Phi_1}(0) = \phi_{1cr}(0) + (\phi_{1a}(0), 0, \phi_{1b}(0)),\ \widetilde{\Phi_2}(0) = \phi_{2cr}(0) + (\phi_{2a}(0), 0, \phi_{2b}(0)).$$

Step 3 Find the solution $Z_{cr}(t)$ of the problem (3.4).

Step 4 Solve the problem (3.8) and denote the solutions by $Z_a(t)$ and $Z_b(t)$, corresponding to $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_{1a}(0) \\ \phi_{2a}(0) \end{pmatrix}$ and $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_{1b}(0) \\ \phi_{2b}(0) \end{pmatrix}$, respectively, and let

$$\widetilde{Z_\phi}(t) = (\min\{Z_a(t), 0, Z_b(t)\}, 0, \max\{Z_a(t), 0, Z_b(t)\}).$$

Step 5 Seek the solutions $Z_u(t)$ and $Z_v(t)$ of the problem (3.10), in regard to $\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} g_{1a}(t) \\ g_{2a}(t) \end{pmatrix}$ and $\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} g_{1b}(t) \\ g_{2b}(t) \end{pmatrix}$, respectively, and define

$$\widetilde{Z_g}(t) = (\min\{Z_u(t), 0, Z_v(t)\}, 0, \max\{Z_u(t), 0, Z_v(t)\}).$$

Step 6 Construct the unique solution of the FDDCS (1.1) on interval $[0,\tau]$ as follows:

$$\widetilde{Z}(t) = Z_{cr}(t) + \widetilde{Z_\phi}(t) + \widetilde{Z_g}(t).$$

Moreover, the unique solution on interval $[\tau, 2\tau]$, $\cdots$ can be obtained similarly. In consequence, we get a unique solution of the FDDCS (1.1) which exists at $t \geq 0$.

4. An example

In this section, to verify the correctness of the solution algorithm, we present the following example.

Example 4.1. Let us solve the problem:

$$\begin{cases} x^{'}(t) = -\cos t\, x(t) + y\left(t - \frac{\pi}{2}\right) + \widetilde{F_1}(t), \quad t > 0, \\ y^{'}(t) = -\sin t\, y(t) + x\left(t - \frac{\pi}{2}\right) + \widetilde{F_2}(t), \quad t > 0, \\ x(t) = \widetilde{\Phi_1}(t), \ -\frac{\pi}{2} \leq t \leq 0, \\ y(t) = \widetilde{\Phi_2}(t), \ -\frac{\pi}{2} \leq t \leq 0, \end{cases} \tag{4.1}$$

where

$$\widetilde{F_1} = f_{1cr} + \langle f_{1a}, 0, f_{1b}\rangle, \quad \widetilde{F_2} = f_{2cr} + \langle f_{2a}, 0, f_{2b}\rangle,$$
$$\widetilde{\Phi_1} = \phi_{1cr} + \langle \phi_{1a}, 0, \phi_{1b}\rangle, \quad \widetilde{\Phi_2} = \phi_{2cr} + \langle \phi_{2a}, 0, \phi_{2b}\rangle$$

with

$$\begin{aligned} &f_{1cr} = (\sin t + 1)\cos t, \quad f_{2cr} = (-\cos t + 1)\sin t, \\ &f_{1a} = -0.25\ \sin t\ \cos t,\ f_{1b} = 0.25\ \sin t\ \cos t, \\ &f_{2a} = 0.25\ \sin t\ \cos t, \quad f_{2b} = -0.25\ \sin t\ \cos t, \\ &\phi_{1cr} = -\sin t\ \cos t, \quad \phi_{2cr} = \sin t\ \cos t, \\ &\phi_{1a} = -0.15\ \cos t + 0.25\ \sin t\ \cos t,\ \phi_{1b} = 0.15\ \cos t - 0.25\ \sin t\ \cos t, \\ &\phi_{2a} = -0.15\ \sin t - 0.25\ \sin t\ \cos t,\ \phi_{2b} = 0.15\ \sin t + 0.25\ \sin t\ \cos t. \end{aligned}$$

At first, we deal with the problem (4.1) on interval $\left[0, \dfrac{\pi}{2}\right]$. The problem (4.1) can be transformed as the problem:

$$\begin{cases} x^{'}(t) = -\cos t\, x(t) + \widetilde{G_1}(t), \\ y^{'}(t) = -\sin t\, y(t) + \widetilde{G_2}(t), \\ x(0) = \widetilde{\Phi_1}(0), \quad y(0) = \widetilde{\Phi_2}(0). \end{cases} \tag{4.2}$$

Furthermore, we can rewrite the problem (4.2) in matrix form:

$$\begin{cases} Z^{'}(t) = A(t) \cdot Z(t) + \widetilde{G}(t), \\ Z(0) = \widetilde{\Phi}(0), \end{cases} \tag{4.3}$$

where

$$Z(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}, \quad A(t) = \begin{pmatrix} -\cos t & 0 \\ 0 & -\sin t \end{pmatrix},$$

$$\widetilde{G}(t) = \begin{pmatrix} \widetilde{G_1}(t) \\ \widetilde{G_2}(t) \end{pmatrix} = \begin{pmatrix} \widetilde{\Phi_2}\left(t - \frac{\pi}{2}\right) + \widetilde{F_1}(t) \\ \widetilde{\Phi_1}\left(t - \frac{\pi}{2}\right) + \widetilde{F_2}(t) \end{pmatrix}, \quad \widetilde{\Phi}(0) = \begin{pmatrix} \widetilde{\Phi_1}(0) \\ \widetilde{\Phi_2}(0) \end{pmatrix}.$$

Then, after a series of calculations,we can obtain

$$\widetilde{G_1} = g_{1cr} + \langle g_{1a}, 0, g_{1b} \rangle, \quad \widetilde{G_2} = g_{2cr} + \langle g_{2a}, 0, g_{2b} \rangle,$$
$$\widetilde{\Phi_1}(0) = \phi_{1cr}(0) + \widetilde{\phi_1}(0), \quad \widetilde{\Phi_2}(0) = \phi_{2cr}(0) + \widetilde{\phi_2}(0),$$

where

$$g_{1cr} = \cos t, \quad g_{2cr} = \sin t, \quad g_{1a} = 0.15 \cos t, \quad g_{1b} = -0.15 \cos t,$$
$$g_{2a} = -0.15 \sin t, \quad g_{2b} = 0.15 \sin t, \quad \phi_{1cr}(0) = 0, \quad \phi_{2cr}(0) = 0,$$
$$\widetilde{\phi_1}(0) = (-0.15, 0, 0.15), \quad \widetilde{\phi_2}(0) = (0, 0, 0).$$

(i) We notice the associated inhomogenous crisp problem:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t) + g_{cr}(t), \\ Z(0) = \phi_{cr}(0), \end{cases} \tag{4.4}$$

where

$$g_{cr}(t) = \begin{pmatrix} g_{1cr}(t) \\ g_{2cr}(t) \end{pmatrix}, \quad \phi_{cr}(0) = \begin{pmatrix} \phi_{1cr}(0) \\ \phi_{2cr}(0) \end{pmatrix}$$

and find the crisp solution $Z_{cr}(t) = \begin{pmatrix} x_{cr}(t) \\ y_{cr}(t) \end{pmatrix} = \begin{pmatrix} 1 - e^{-\sin t} \\ 1 - e^{\cos t - 1} \end{pmatrix}$, which is illustrated in Figure 2.

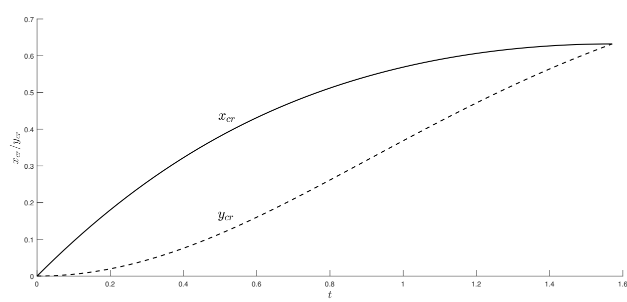

FIGURE 2. The crisp solutions.

(ii) Consider the following crisp problem:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t), \\ Z(0) = \phi(0), \end{cases} \tag{4.5}$$

for $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_{1a}(0) \\ \phi_{2a}(0) \end{pmatrix}$ and $\begin{pmatrix} \phi_1(0) \\ \phi_2(0) \end{pmatrix} = \begin{pmatrix} \phi_{1b}(0) \\ \phi_{2b}(0) \end{pmatrix}$, and find the solutions $Z_a(t) = \begin{pmatrix} x_a(t) \\ y_a(t) \end{pmatrix} = \begin{pmatrix} -0.15\, e^{-\sin t} \\ 0 \end{pmatrix}$ and $Z_b(t) = \begin{pmatrix} x_b(t) \\ y_b(t) \end{pmatrix} = \begin{pmatrix} 0.15\, e^{-\sin t} \\ 0 \end{pmatrix}$, respectively. Then the solution to the following problem:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t), \\ Z(0) = \widetilde{\phi}(0), \end{cases} \tag{4.6}$$

is the TFF matrix $\widetilde{Z_\phi} = \langle Z_a, 0, Z_b \rangle$, which is graphed in Figure 3, and we know that

$$\widetilde{Z_\phi}(t) = (\min\{Z_a(t), 0, Z_b(t)\}, 0, \max\{Z_a(t), 0, Z_b(t)\}) \,.$$

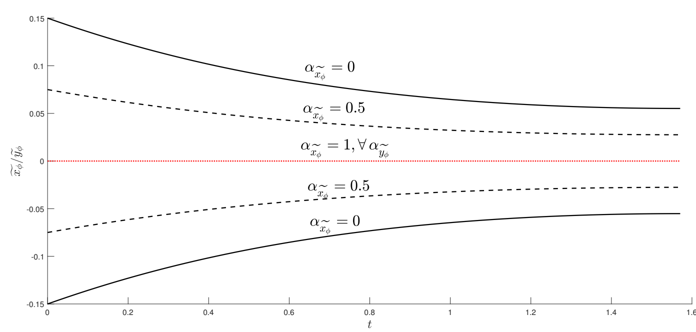

FIGURE 3. Uncertainty of the solutions due to initial functions.

(iii) Deal with the crisp problem as follows:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t) + g(t), \\ Z(0) = \mathbf{0}, \end{cases} \tag{4.7}$$

for $\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} g_{1a}(t) \\ g_{2a}(t) \end{pmatrix}$ and $\begin{pmatrix} g_1(t) \\ g_2(t) \end{pmatrix} = \begin{pmatrix} g_{1b}(t) \\ g_{2b}(t) \end{pmatrix}$, and find the solutions $Z_u(t) = \begin{pmatrix} x_u(t) \\ y_u(t) \end{pmatrix} = \begin{pmatrix} 0.15\,(1 - e^{-\sin t}) \\ -0.15\,(1 - e^{\cos t - 1}) \end{pmatrix}$ and $Z_v(t) = \begin{pmatrix} x_v(t) \\ y_v(t) \end{pmatrix} = \begin{pmatrix} -0.15\,(1 - e^{-\sin t}) \\ 0.15\,(1 - e^{\cos t - 1}) \end{pmatrix}$, in several. Then it follows that the solution to the third subproblem of the problem (4.3) as hereunder mentioned:

$$\begin{cases} Z'(t) = A(t) \cdot Z(t) + \widetilde{g}(t), \\ Z(0) = \mathbf{0}, \end{cases} \tag{4.8}$$

is the TFF matrix $\widetilde{Z_g} = \langle Z_u, 0, Z_v \rangle$, which is depicted in Figure 4, and one knows that

$$\widetilde{Z_g}(t) = \left(\min\{Z_u(t), 0, Z_v(t)\}, 0, \max\{Z_u(t), 0, Z_v(t)\}\right).$$

In general, by solving the problems (4.4), (4.6) and (4.8), it follows from Algorithm 3.4 that we can get the unique solution of the problem (4.1) on interval $\left[0, \dfrac{\pi}{2}\right]$ as

$$\widetilde{Z}(t) = Z_{cr}(t) + \widetilde{Z_\phi}(t) + \widetilde{Z_g}(t),$$

which is illustrated in Figure 5.

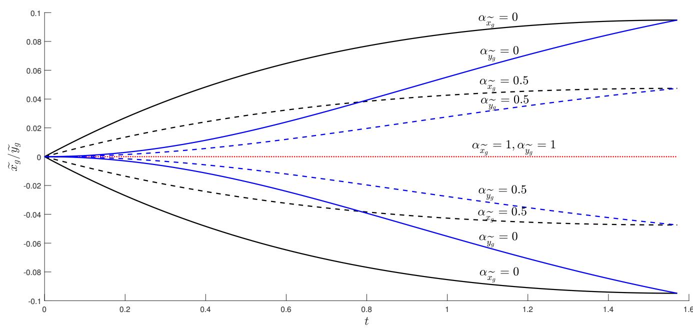

FIGURE 4. Uncertainty of the solutions owing to source functions.

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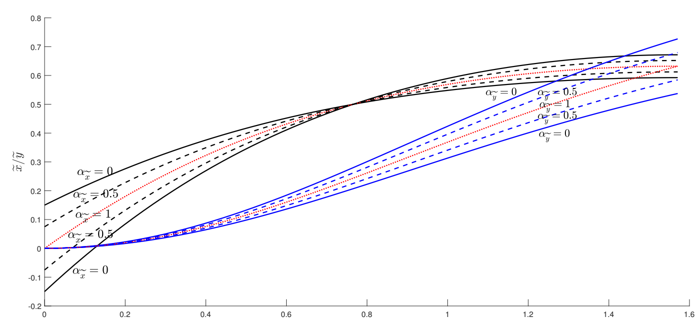

FIGURE 5. The fuzzy solutions obtained by the proposed method.

Furthermore, the unique solution of the problem (4.1) on interval $\left[\dfrac{\pi}{2}, \pi\right]$, $\cdots$ can be gotten analogously. Hence, we know that the unique solution of the problem (4.1) exists at $t \geq 0$.

5. Concluding remarks

In this paper, inspired by Fatullayev et al. [7] and Gasilov and Amrahov [9], the coupled system of fuzzy delay differential equations (FDDCS) (1.1) with fuzzy initial values and fuzzy source functions were solved by the methods of separating steps and fuzzy items separation. In terms of Theorems 3.2 and 3.3, we obtained existence and uniqueness of solution to the FDDCS (1.1). We notice that the solution is a triangular fuzzy function matrix which is made up of real functional matrices. Moreover, an algorithm for solution procedure and an example for confirmation were given.

Currently, more and more scholars have probed into the relevant differentiable set-value problems [1], impulsive problems [8, 14] and differential inclusion problems [18, 19]. As a result, for further study, we can introduce the set-valued delay, the impulse or differential inclusion into the FDDCS (1.1), and extend the approaches brought forward in this paper to deal with the new problems.

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