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APPROXIMATION OF FIXED POINTS AND THE SOLUTION OF A NONLINEAR INTEGRAL EQUATION

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Abstract. In this article, we define Picard's three-step iteration process for the approximation of fixed points of Zamfirescu operators in an arbitrary Banach space. We prove a convergence result for Zamfirescu operator using the proposed iteration process. Further, we prove that Picard's three-step iteration process is almost T-stable and converges faster than all the known and leading iteration processes. To support our results, we furnish an illustrative numerical example. Finally, we apply the proposed iteration process to approximate the solution of a mixed Volterra-Fredholm functional nonlinear integral equation.

1. Introduction and Preliminaries

Throughout this paper, we assume that $\mathbb{Z}_+$ is the set of nonnegative integers. We consider that C is a nonempty subset of a Banach space X and $F(T)$, the set of fixed points of the mapping T defined on C.

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A mapping $T : C \to C$ is said to be:

(1) a contraction if there exists a constant $\delta \in [0, 1)$ such that
$$\|Tx - Ty\| \leq \delta\|x - y\|, \ \forall\ x, y \in C; \tag{1.1}$$

(2) a Kannan map [18] if there exists a constant $b \in (0, \frac{1}{2})$ such that
$$\|Tx - Ty\| \leq b(\|x - Tx\| + \|y - Ty\|), \ \forall\ x, y \in C; \tag{1.2}$$

(3) a Chatterjea map [8] if there exists a constant $c \in (0, \frac{1}{2})$ such that
$$\|Tx - Ty\| \leq c(\|x - Ty\| + \|y - Tx\|), \ \forall x, y \in C. \tag{1.3}$$

It is known that conditions (1.1), (1.2) and (1.3) are independent (see [28]).

Definition 1.1. ([36]) An operator $T : C \to C$ is said to be a Zamfirescu operator or Z-operator if it satisfies at least one of the conditions (1.1), (1.2) and (1.3).

Fixed point theory plays an important role in mathematics and it provides useful tools to solve many linear and nonlinear problems that have many applications in different fields like Engineering, Differential equations, Integral equations, Economics, Chemistry, Game theory, etc. (e.g., see [5, 20]). However, when the existence of a fixed point of some operators is accomplished, then to find the fixed point is not an easy task, that's why we use iteration processes for computing them. A large number of researchers introduced and studied iteration processes to compute fixed points for different mappings (see [3, 4]). In several cases, there can be more than one iteration process to reckon fixed points of a particular mapping. In such cases, the speed of iteration processes does matter, the better speed of iteration processes to approximate fixed point save time.

The following definitions about the speed of convergence of iteration processes are due to Berinde [6].

Definition 1.2. Let $\{\alpha_n\}$ and $\{\beta_n\}$ be two sequences of real numbers that converge to α and β, respectively. Assume that
$$\ell = \lim_{n\to\infty} \frac{|\alpha_n - \alpha|}{|\beta_n - \beta|}.$$

(i) If $\ell = 0$, then $\{\alpha_n\}$ converges to α faster than $\{\beta_n\}$ to β.

(ii) If $0 < \ell < \infty$, then $\{\alpha_n\}$ and $\{\beta_n\}$ have the same rate of convergence.

Definition 1.3. Suppose that $\{x_n\}$ and $\{y_n\}$ are two fixed point iteration processes both converging to the same point p with the following error estimates (best ones available):
$$\begin{aligned} |x_n - p| &\leq \alpha_n, \\ |y_n - p| &\leq \beta_n. \end{aligned}$$

If $\lim_{n\to\infty} \frac{\alpha_n}{\beta_n} = 0$, then $\{x_n\}$ converges faster than $\{y_n\}$ and $\{y_n\}$ slower than $\{x_n\}$.

In 1967, Ostrowski [25] coined the concept of stability for fixed point iteration processes and proved that Picard's iteration process is stable with respect to contractions. In 1987, Harder [14] in his Thesis extended the work due to Ostrowski for more general iteration processes and contractive conditions.

In the process of approximating fixed points, we consider an approximate sequence $\{t_n\}$ instead of the theoretical sequence $\{x_n\}$ because of rounding errors and numerical approximation of functions. The following definition of stability is due to Ostrowski [25].

Definition 1.4. ([25]) Let T be a self mapping on a subset C of a Banach space X with a fixed point p and $\{t_n\}$ be an arbitrary sequence in C. Then an iteration procedure $x_{n+1} = f(T, x_n)$ for some function f, converging to a fixed point p, is said to be T-stable or stable with respect to T, if for $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$, $n \in \mathbb{Z}_+$, we have $\lim_{n\to\infty} \epsilon_n = 0$ if and only if $\lim_{n\to\infty} t_n = p$.

In 1998, Osilike [24] introduced the concept of almost stability of iterative processes which is a weaker class of stability due to Ostrowski [25] and defined as follows:

Definition 1.5. Let T be a self mapping on a Banach space X with a fixed point p. Assume that $x_0 \in X$ and $x_{n+1} = f(T, x_n)$, $n \in \mathbb{Z}_+$ is an iterative process for some function f. Let $\{t_n\}$ be an approximate sequence of the sequence $\{x_n\}$ in X and define $\epsilon_n = \|t_{n+1} - f(T, t_n)\|$. Then the iterative process $x_{n+1} = f(T, x_n)$ is called almost T-stable or almost stable with respect to T if $\sum_{n=0}^{\infty} \epsilon_n < \infty$, then $\lim_{n\to\infty} t_n = p$.

Also, Osilike proved the stability of the Ishikawa process for a class of pseudo-contractive operators. It can be easily seen that every stable iterative method is also almost stable with respect to the mapping T but the converse is not true in general. It is also known that some iterative methods are neither T-stable nor almost T-stable, for more details, one can refer [24].

Lemma 1.6. ([7]) *Let $\{\epsilon_n\}$ and $\{u_n\}$ be any two sequences of nonnegative real numbers satisfying $u_{n+1} \le \delta u_n + \epsilon_n$, $n \in \mathbb{Z}_+$, where $0 \le \delta < 1$. If $\lim_{n\to\infty} \epsilon_n = 0$, then $\lim_{n\to\infty} u_n = 0$.*

Banach's contraction principle assures the existence and uniqueness of a fixed point of a contraction which can be approximated by Picard's iteration process [27]. In the following iteration processes, the sequence $\{x_n\}$ is generated by an arbitrary point $x_0 \in C$ for the mapping $T : C \to C$ and defined as follows:

$$x_{n+1} = Tx_n,\ n \in \mathbb{Z}_+. \tag{1.4}$$

It is well known that Picard's iteration process may not converge to a fixed point of nonexpansive mappings.

Therefore, in 1953, Mann [21] introduced the following iteration process to approximate fixed points of nonexpansive mappings:

$$x_{n+1} = (1 - a_n)x_n + a_nTx_n,\ n \in \mathbb{Z}_+. \tag{1.5}$$

It is also known that Mann iteration process fails to converge to a fixed point of pseudo-contractive mapping.

So in 1974, Ishikawa [15] introduced a two-step Mann iteration process to approximate fixed points of pseudo-contractive mappings which is defined by

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_nTy_n, \\ y_n = (1 - b_n)x_n + b_nTx_n,\ n \in \mathbb{Z}_+. \end{cases} \tag{1.6}$$

In 2000, Noor [23] introduced the following three-step iteration process for the solution of general variational inequalities:

$$\begin{cases} x_{n+1} = (1 - a_n)x_n + a_nTy_n, \\ y_n = (1 - b_n)x_n + b_nTz_n, \\ z_n = (1 - c_n)x_n + c_nTx_n,\ n \in \mathbb{Z}_+. \end{cases} \tag{1.7}$$

He also studied the convergence criteria of this process.

After that, in 2007, Agrawal et al. [2] introduced the following two-step iteration process, called S iteration process, for nearly asymptotically nonexpansive mappings:

$$\begin{cases} x_{n+1} = (1 - a_n)Tx_n + a_nTy_n, \\ y_n = (1 - b_n)x_n + b_nTx_n,\ n \in \mathbb{Z}_+. \end{cases} \tag{1.8}$$

They claimed that this process converges at the same rate of convergence as Picard's iteration process and faster than Mann process for contractions.

In 2011, Phuengrattana and Suantai [26] introduced the following iteration process for continuous functions, called SP iteration process:

$$\begin{cases} x_{n+1} = (1 - a_n)y_n + a_nTy_n, \\ y_n = (1 - b_n)z_n + b_nTz_n, \\ z_n = (1 - c_n)x_n + c_nTx_n,\ n \in \mathbb{Z}_+. \end{cases} \tag{1.9}$$

They showed numerically that SP iteration process converges faster than Mann, Ishikawa and Noor iteration processes for the class of continuous and nondecreasing functions.

In the same year, Sahu [29] introduced the normal-S iteration process for nonexpansive mappings in the following manner:

$$x_{n+1} = T((1 - a_n)x_n + a_nTx_n)\ , n \in \mathbb{Z}_+. \tag{1.10}$$

After that, in 2012, Chugh et al. [10] introduced a new three-step iteration process, called CR process, for a certain class of quasi-contractive mappings in Banach spaces:

$$\begin{cases} x_{n+1} = (1-a_n)y_n + a_n T y_n, \\ y_n = (1-b_n)Tx_n + b_n T z_n, \\ z_n = (1-c_n)x_n + c_n T x_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.11}$$

They showed that the CR iteration process is faster than Picard, Mann, Ishikawa, S, Noor and SP iteration processes. They also pointed out that for increasing functions CR process is best while for decreasing function SP process is best.

In 2013, Khan [19] introduced the following new iteration process, known as Picard-Mann hybrid iteration process, for the class of non-expansive mappings:

$$\begin{cases} x_{n+1} = T y_n, \\ y_n = (1-a_n)x_n + a_n T x_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.12}$$

He mentioned that Picard-Mann hybrid iteration process is independent of all Picard, Mann and Ishikawa iteration processes.

In the same year, Karahan and Ozdemir [17] introduced the following three-step iteration process, called S^* iteration process, for non-expansive mappings in Banach spaces:

$$\begin{cases} x_{n+1} = (1-a_n)Tx_n + a_n T y_n, \\ y_n = (1-b_n)Tx_n + b_n T z_n, \\ z_n = (1-c_n)x_n + c_n T x_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.13}$$

They showed numerically that this iteration process converges faster than Picard, Mann, Ishikawa and S iteration processes.

After that, in 2014, Abbas and Nazir [1] introduced a new three-step iteration process for non-expansive mappings in uniformly convex Banach spaces:

$$\begin{cases} x_{n+1} = (1-a_n)Ty_n + a_n T z_n, \\ y_n = (1-b_n)Tx_n + b_n T z_n, \\ z_n = (1-c_n)x_n + c_n T x_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.14}$$

They showed numerically that this process converges faster than Picard, Mann and S iteration processes for contractions.

In 2014, Thakur et al. [34] introduced the following iteration process for nonexpansive mappings:

$$\begin{cases} x_{n+1} = (1-a_n)Tx_n + a_n T y_n, \\ y_n = (1-b_n)z_n + b_n T z_n, \\ z_n = (1-c_n)x_n + c_n T x_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.15}$$

They claimed that this process converges faster than Picard, Mann, Ishikawa, Noor, S and Abbas and Nazir iteration processes for contractions.

In 2014, Gursoy and Karakaya [13] introduced the following iteration process for contractions, called Picard-S iteration process and defined as follows:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-a_n)Tx_n + a_nTz_n, \\ z_n = (1-b_n)x_n + b_nTx_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.16}$$

They showed numerically that Picard-S iteration process converges faster than Picard, Mann, Ishikawa, Noor, SP, S and some other iteration processes for contractions.

In 2014, Kadioglu and Yildirm [16] and Çeliker [9] introduced independently the following same iteration process, called modified SP (MSP) process, for non-expansive mappings:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = (1-a_n)z_n + a_nTz_n, \\ z_n = (1-b_n)x_n + b_nTx_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.17}$$

After that, in 2016, Thakur et al. [32] introduced the following iteration process for Suzuki's generalized non-expansive mappings in uniformly convex Banach space, referred as Thakur-new iteration process:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = T((1-a_n)x_n + a_nz_n), \\ z_n = (1-b_n)x_n + b_nTx_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.18}$$

In 2016, Sahu et al. [30] and Thakur et al. [33] introduced independently the same iteration process indicated below for non-expansive mappings in uniformly convex Banach spaces:

$$\begin{cases} x_{n+1} = (1-a_n)Tz_n + a_nTy_n, \\ y_n = (1-b_n)z_n + b_nTz_n, \\ z_n = (1-c_n)x_n + c_nTx_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.19}$$

They claimed that this process converges faster than all the known iteration processes for contractions.

In process, Sintunavarat and Pitea [31] introduced a new three-step iteration process for Berinde mappings in Banach space, which we call Varat iteration process:

$$\begin{cases} x_{n+1} = (1-a_n)Tz_n + a_nTy_n, \\ z_n = (1-b_n)x_n + b_ny_n, \\ y_n = (1-c_n)x_n + c_nTx_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.20}$$

Recently, Mogbademu [22] raised the question that "Is it possible to define an iteration process which converges faster than Picard and Khan iteration processes?" As an answer, Mogbademu [22] introduced the following iteration process, called Picard hybrid process:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = Tx_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.21}$$

He claimed that Picard hybrid iteration process converges faster than the Picard-Mann and Picard's iteration processes for contractions.

Quite recently, Ullah and Arshad [35] introduced the following new iteration process, called M^* process, for Suzuki's generalized nonexpansive mappings in uniformly convex Banach spaces:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = T((1-a_n)x_n + a_n Tz_n), \\ z_n = (1-b_n)x_n + b_n Tx_n, \ n \in \mathbb{Z}_+, \end{cases} \tag{1.22}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are control sequences in $(0,1)$.

Inspired by the above, we raise the following two questions:

Question 1. Is it possible to define an iteration process which converges faster than all iteration processes as defined above in Banach spaces?

Question 2. Is it possible to define an iteration process which converges faster than Picard and Picard hybrid iteration processes in metric spaces?

As an answer to above questions, we introduce Picard's three-step iteration process which is defined as follows:

Let X be a metric or Banach space and T a self mapping on X. The sequence $\{x_n\}$ with initial guess $x_0 \in X$ is defined in the following manner:

$$\begin{cases} x_{n+1} = Ty_n, \\ y_n = Tz_n, \\ z_n = Tx_n, \ n \in \mathbb{Z}_+. \end{cases} \tag{1.23}$$

Remark 1.7. Since the rate of convergence of iteration processes depends on the control sequences $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ in $(0,1)$ and the iteration process (1.23) is free from control sequences, thus the rate of convergence of the iteration process (1.23) depends only on the initial point $x_0 \in C$.

Motivated by the above, we prove that the iteration process (1.23) converges strongly to a fixed point of a Zamfirescu operator in an arbitrary Banach space. Further, we prove that the proposed iteration process is almost T-stable and converges faster than some leading iteration processes. To support our results we present an illustrative numerical example. Finally, we approximate the solution of a mixed Volterra-Fredholm functional nonlinear integral equation via the proposed iteration process (1.23).

2. Main results

In this section, we prove our main results for Zamfirescu operator using the iteration process (1.23) in an arbitrary Banach space.

Theorem 2.1. *Let C be a nonempty and closed subset of a Banach space X and $T : C \to C$ be a Zamfirescu operator. Then the sequence $\{x_n\}$ defined by the iteration process* (1.23) *converges to a unique fixed point of T.*

Proof. Suppose $p \in F(T)$ and $x \in C$. Since T is a Zamfirescu operator, it follows that

$$\|Tx - p\| \leq \delta\|x - p\|, \; 0 < \delta < 1.$$

Now using the iteration process (1.23), we have

$$\begin{aligned} \|x_{n+1} - p\| &\leq \|Ty_n - p\| \leq \delta\|y_n - p\| \\ &= \delta\|Tz_n - p\| \leq \delta^2\|z_n - p\| \\ &= \delta^2\|Tx_n - p\| \leq \delta^3\|x_n - p\| \\ &\vdots \\ &\leq \delta^{3(n+1)}\|x_0 - p\|. \end{aligned} \tag{2.1}$$

Since $\delta \in [0, 1)$, it follows that $\{x_n\}$ converges to p. □

The following theorem shows the almost stability of the iteration process (1.23).

Theorem 2.2. *Let C be a nonempty closed subset of a Banach space X and $T : C \to C$ be a Zamfirescu operator. Let $\{x_n\}$ be a sequence defined by the iteration process* (1.23)*. Then the iteration process* (1.23) *is almost T-stable.*

Proof. Let $\{t_n\}$ be an arbitrary sequence in C. Let $x_{n+1} = f(T, x_n)$ be a sequence defined by (1.23) which converges to a unique fixed point p (by Theorem 2.1) and $\epsilon_n = \|t_{n+1} - f(T, t_n)\|, \; n \in \mathbb{Z}_+$. Now, we will prove that $\sum_{n=0}^{\infty} \epsilon_n < \infty$ implies $\lim\limits_{n\to\infty} t_n = p$.

Assume $\sum_{n=0}^{\infty} \epsilon_n < \infty$. By the iteration process (1.23), we have

$$\begin{aligned} \|t_{n+1} - p\| &\leq \|t_{n+1} - f(T, t_n)\| + \|f(T, t_n) - p\| \\ &= \epsilon_n + \|T(T(Tt_n)) - p\| \\ &\leq \epsilon_n + \delta\|T(Tt_n) - p\| \\ &\leq \epsilon_n + \delta^2\|Tt_n - p\| \\ &\leq \epsilon_n + \delta^3\|t_n - p\|. \end{aligned}$$

Define $u_n = \|t_n - p\|$, then

$$u_{n+1} \leq \delta^3 u_n + \epsilon_n.$$

Now by Lemma 1.6, we have $\lim_{n\to\infty} u_n = 0$, *i.e.* $\lim_{n\to\infty} t_n = p$. Thus the iteration process (1.23) is almost T-stable. □

Theorem 2.3. *Let C be a nonempty, closed and convex subset of a Banach space X and $T : C \to C$ be a Zamfirescu operator. Consider the sequences $\{x_{1,n}\}$ defined by Picard* (1.4)*, $\{x_{2,n}\}$ by Mann* (1.5)*, $\{x_{3,n}\}$ by Ishikawa* (1.6)*, $\{x_{4,n}\}$ by Noor* (1.7)*, $\{x_{5,n}\}$ by S* (1.8)*, $\{x_{6,n}\}$ by SP* (1.9)*, $\{x_{7,n}\}$ by normal S* (1.10)*, $\{x_{8,n}\}$ by CR* (1.11)*, $\{x_{9,n}\}$ by Picard- Mann hybrid* (1.12)*, $\{x_{10,n}\}$ by S^** (1.13)*, $\{x_{11,n}\}$ by Abbas and Nazir* (1.14)*, $\{x_{12,n}\}$ by Thakur* (1.15)*, $\{x_{13,n}\}$ by Picard-S* (1.16)*, $\{x_{14,n}\}$ by modified SP* (1.17)*, $\{x_{15,n}\}$ by Thakur new* (1.18)*, $\{x_{16,n}\}$ by Sahu, Thakur* (1.19)*, $\{x_{17,n}\}$ by Sintunavarat and Pitea* (1.20)*, $\{x_{18,n}\}$ by Picard hybrid* (1.21)*, $\{x_{19,n}\}$ by M^** (1.22) *and $\{x_n\}$ by Picard's three-step* (1.23) *iteration processes, and assume that they all converge to the same point p. Then the iteration process* (1.23) *converges to a fixed point p of T faster than all the iteration processes* (1.4)-(1.22)*.*

Proof. As proved by Khan ([19], Proposition 1),

$$\|x_{1,n} - p\| \leq \delta^{n+1}\|x_{1,0} - p\| = \alpha_{1,n},\ n \in \mathbb{Z}_+.$$

Further, we proved in the inequality (2.1) that

$$\|x_{n+1} - p\| \leq \delta^{3(n+1)}\|x_0 - p\| = \alpha_n,\ n \in \mathbb{Z}_+.$$

Then,

$$\begin{aligned}\frac{\alpha_n}{\alpha_{1,n}} &= \frac{\delta^{3(n+1)}\|x_0 - p\|}{\delta^{n+1}\|x_{1,0} - p\|}\\ &= \delta^{2(n+1)}\frac{\|x_0 - p\|}{\|x_{1,0} - p\|}.\end{aligned}$$

Since $0 \leq \delta < 1$, we have $\frac{\alpha_n}{\alpha_{1,n}} \to 0$ as $n \to \infty$. Thus, $\{x_n\}$ converges to p faster than $\{x_{1,n}\}$. Now, as proved by Khan ([19], Proposition 1),

$$\|x_{9,n} - p\| \leq [\delta(1 - (1 - \delta)a_n)]^{n+1}\|x_{9,0} - p\| = \alpha_{9,n},\ n \in \mathbb{Z}_+.$$

Then,

$$\begin{aligned}\frac{\alpha_n}{\alpha_{9,n}} &= \frac{\delta^{3(n+1)}\|x_0 - p\|}{[\delta(1 - (1 - \delta)a_n)]^{n+1}\|x_{9,0} - p\|}\\ &= \delta^{n+1}\left(\frac{\delta}{1 - (1 - \delta)a_n}\right)^{n+1}\frac{\|x_0 - p\|}{\|x_{9,0} - p\|}\\ &\to 0,\ as\ n \to \infty.\end{aligned}$$

Hence, $\{x_n\}$ converges to p faster than $\{x_{9,n}\}$.

In similar way, as proved by Sahu ([30], Theorem 3.1)

$$\begin{aligned}\|x_{16,n} - p\| &\leq [\delta(1-(1-\delta^2)a_nb_nc_n)]^{n+1}\|x_{16,0} - p\| \\ &= \alpha_{16,n},\ n \in \mathbb{Z}_+.\end{aligned}$$

Then,

$$\begin{aligned}\frac{\alpha_n}{\alpha_{16,n}} &= \frac{\delta^{3(n+1)}\|x_0 - p\|}{[\delta(1-(1-\delta^2)a_nb_nc_n)]^{n+1}\|x_{16,0} - p\|} \\ &= \delta^{n+1}\left(\frac{\delta}{1-(1-\delta^2)a_nb_nc_n}\right)^{n+1}\frac{\|x_0 - p\|}{\|x_{16,0} - p\|} \\ &\to 0\ as\ n \to \infty.\end{aligned}$$

Thus, $\{x_n\}$ converges to p faster than $\{x_{16,n}\}$.

As proved by Mogbademu ([22], Inequality (2.1))

$$\|x_{18,n} - p\| \leq \delta^{2(n+1)}\|x_{18,0} - p\| = \alpha_{18,n},\ n \in \mathbb{Z}_+.$$

Then,

$$\begin{aligned}\frac{\alpha_n}{\alpha_{18,n}} &= \frac{\delta^{3(n+1)}\|x_0 - p\|}{\delta^{2(n+1)}\|x_{18,0} - p\|} \\ &= \delta^{n+1}\frac{\|x_0 - p\|}{\|x_{18,0} - p\|} \\ &\to 0.\end{aligned}$$

Thus, the sequence $\{x_n\}$ converges to p faster than the sequence $\{x_{18,n}\}$.

Similarly, we can prove that iteration process (1.23) converges to p faster than all the other iteration processes. □

Now we furnish the following example in support of the above theorem.

Example 2.4. Let $X = \mathbb{R}$ be a Banach space with usual norm and $C = [0, \infty)$. Let $T : C \to C$ be a self mapping defined by

$$Tx = \sqrt{x+2},$$

for all $x \in C$. We can easily verify that T is a Zamfirescu operator and has a fixed point $p = 2$. Now, we choose $a_n = 0.85$, $b_n = 0.65$ and $c_n = 0.45$ with the initial guess $x_0 = 15$.

With the help of Matlab program 2015a, we verify that the iteration process (1.23) converges to $p = 2$ faster than all the iteration processes (1.4)-(1.22).

Iter.	Iter. (1.23)	Picard	Mann	Ishikawa	Noor	S	SP
1	15.000000	15.000000	15.000000	15.000000	15.000000	15.000000	15.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
5	2.000000	2.028620	2.152998	2.047442	2.041246	2.003777	2.002213
9	2.000000	2.000112	2.002619	2.000213	2.000154	2.000002	2.000001
10	2.000000	2.000028	2.000949	2.000055	2.000038	2.000000	2.000000
11	2.000000	2.000007	2.000344	2.000014	2.000009	2.000000	2.000000
12	2.000000	2.000002	2.000125	2.000004	2.000002	2.000000	2.000000
13	2.000000	2.000000	2.000045	2.000001	2.000001	2.000000	2.000000
14	2.000000	2.000000	2.000016	2.000000	2.000000	2.000000	2.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
18	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000

Table 1. A comparison table of the iteration processes.

Iter.	Normal-S	CR	Pica.-Mann	S^*	Abbas	Thakur	Picard-S
1	15.000000	15.000000	15.000000	15.000000	15.000000	15.000000	15.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
7	2.000005	2.000001	2.000005	2.000052	2.000087	2.000015	2.000000
8	2.000000	2.000000	2.000000	2.000007	2.000013	2.000002	2.000000
9	2.000000	2.000000	2.000000	2.000001	2.000002	2.000000	2.000000
10	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000

Table 2. A comparison table of the iteration processes.

Iter.	MSP	Thakur New	Sahu,Thakur	Varat	Picard hybrid	M^*
1	15.000000	15.000000	15.000000	15.000000	15.000000	15.000000
⋮	⋮	⋮	⋮	⋮	⋮	⋮
6	2.000002	2.000001	2.000076	2.001163	2.000007	2.000000
7	2.000000	2.000000	2.000007	2.000198	2.000000	2.000000
8	2.000000	2.000000	2.000001	2.000034	2.000000	2.000000
9	2.000000	2.000000	2.000000	2.000006	2.000000	2.000000
10	2.000000	2.000000	2.000000	2.000001	2.000000	2.000000
11	2.000000	2.000000	2.000000	2.000000	2.000000	2.000000

Table 3. A comparison table of the iteration processes.

The Figures 1, 2 and 3 show the convergence behavior of the iteration process (1.23) with iteration processes (1.4)-(1.22).

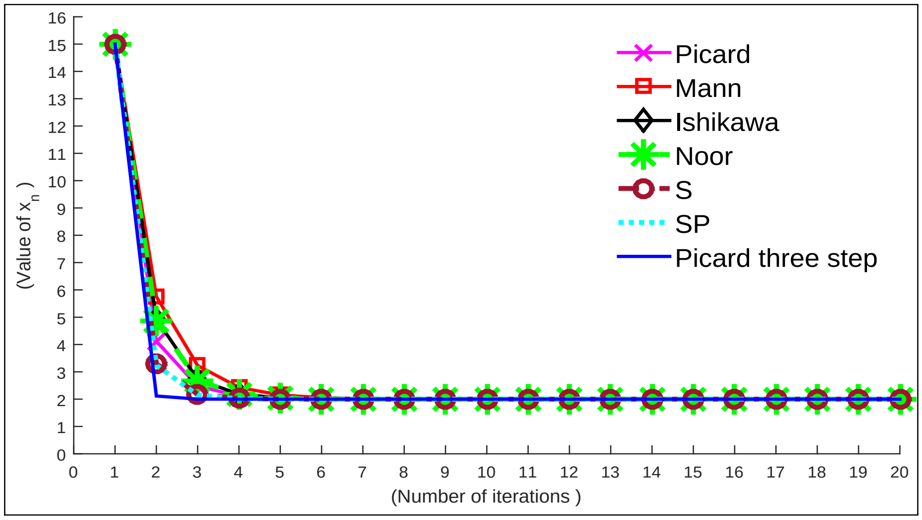

Figure 1. Convergence behavior of the iteration process (1.23) with various iteration processes.

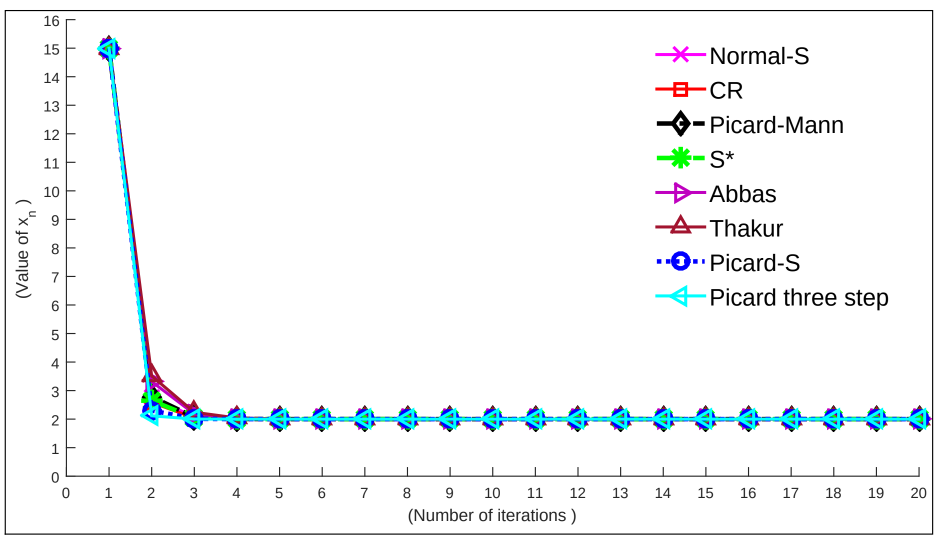

Figure 2. Convergence behavior of the iteration process (1.23) with various iteration processes.

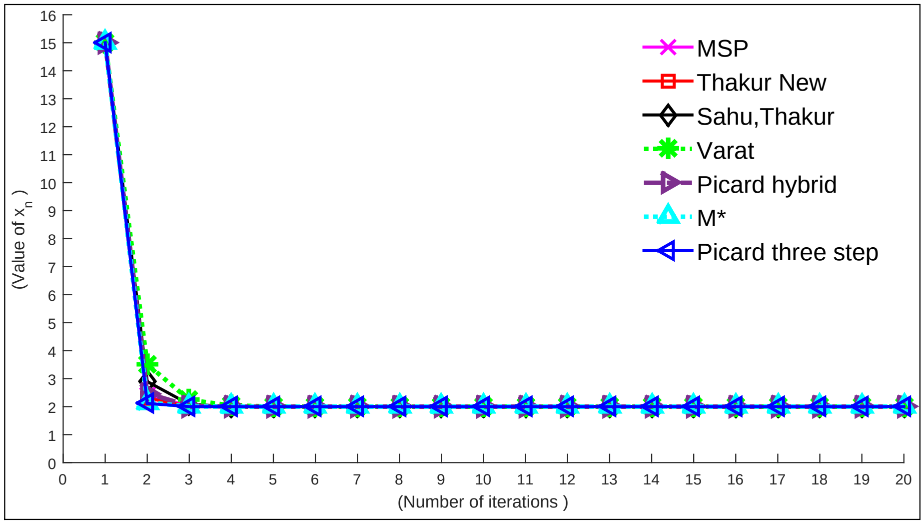

FIGURE 3. Convergence behavior of the iteration process (1.23) with various iteration processes.

3. APPLICATION

In this section, we approximate the solution of a mixed Volterra-Fredholm functional nonlinear integral equation using the iteration process (1.23).

Consider the following mixed Volterra-Fredholm functional nonlinear integral equation (see [11]).

$$x(t) = F\left(t, x(t), \int_{a_1}^{t_1} ... \int_{a_n}^{t_n} K(t,s,x(s))ds, \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} H(t,s,x(s))ds\right), \tag{3.1}$$

where $[a_1,b_1] \times \cdots \times [a_n,b_n]$ is an interval in $\mathbb{R}^n$, $t = (t_1,t_2,\cdots,t_n)$, $s = (s_1,s_2,\cdots,s_n) \in [a_1,b_1] \times \cdots \times [a_n,b_n]$, $K, H : [a_1,b_1] \times \cdots \times [a_n,b_n] \times [a_1,b_1] \times \cdots \times [a_n,b_n] \times \mathbb{R} \to \mathbb{R}$ are continuous functions and $F : [a_1,b_1] \times \cdots \times [a_n,b_n] \times \mathbb{R}^3 \to \mathbb{R}$.

Assume that the following conditions are satisfied:

(C_1) $K, H \in C([a_1,b_1] \times \cdots \times [a_n,b_n] \times [a_1,b_1] \times \cdots \times [a_n,b_n] \times \mathbb{R})$;
(C_2) $F \in C([a_1,b_1] \times \cdots \times [a_n,b_n] \times \mathbb{R}^3)$;
(C_3) there exist nonnegative constants α, β, γ such that

$$|F(t,u_1,u_2,u_3) - F(t,v_1,v_2,v_3)| \le \alpha|u_1 - v_1| + \beta|u_2 - v_2| + \gamma|u_3 - v_3|,$$

for all $t \in [a_1,b_1] \times \cdots \times [a_n,b_n]$, $u_i, v_i \in \mathbb{R}$, $i = 1,\ 2,\ 3$;

(C_4) there exist nonnegative constants L_K and L_H such that

$$|K(t,s,u) - K(t,s,v)| \leq L_K|u-v|,$$

$$|H(t,s,u) - H(t,s,v)| \leq L_H|u-v|,$$

for all $t, s \in [a_1, b_1] \times \cdots \times [a_n, b_n]$, and $u,\ v\ \in \mathbb{R}$;

(C_5) $\alpha + (\beta L_K + \gamma L_H)(b_1 - a_1)\cdots(b_n - a_n) < 1$.

By the solution of problem (3.1), we mean a function $x_* \in C([a_1, b_1] \times \cdots \times [a_n, b_n])$.

Crăciun and Şerban [11] proved the following existence result for the problem (3.1).

Theorem 3.1. *Assume that conditions* $(C_1) - (C_5)$ *are satisfied. Then the problem* (3.1) *has a unique solution* $x_* \in C([a_1, b_1] \times \cdots \times [a_n, b_n])$.

Now, we prove the following main result using iteration process (1.23).

Theorem 3.2. *Let* $X = C([a_1, b_1] \times \cdots \times [a_n, b_n], \|.\|)$ *be a Banach space with Chebyshev's norm. Let* $\{x_n\}$ *be a sequence defined by iteration process* (1.23) *for the operator* $T : X \to X$ *defined by*

$$Tx(t) = F\left(t, x(t), \int_{a_1}^{t_1} ... \int_{a_n}^{t_n} K(t,s,x(s))ds, \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} H(t,s,x(s))ds\right), \tag{3.2}$$

where F, K and H are defined as above. Assume that conditions $(C_1)-(C_5)$ *are satisfied. Then the iteration process* (1.23) *converges to the unique solution, say,* $x_* \in C([a_1, b_1] \times \cdots \times [a_n, b_n])$ *of the problem* (3.1).

Proof. In view of Theorem 3.1, we assume that x_* is the fixed point of T.

Now we show that $x_n \to x_*$ as $n \to \infty$.

Using iteration process (1.23), equation (3.2) and conditions $(C_1) - (C_4)$, we obtain

$$\begin{aligned}
&\|z_n - x_*\| \\
&= \|Tx_n - x_*\| = |Tx_n(t) - Tx_*(t)| \\
&= \Bigg| F\left(t, x_n(t), \int_{a_1}^{t_1} ... \int_{a_n}^{t_n} K(t,s,x_n(s))ds, \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} H(t,s,x_n(s))ds\right) \\
&\quad - F\left(t, x_*(t), \int_{a_1}^{t_1} ... \int_{a_n}^{t_n} K(t,s,x_*(s))ds, \int_{a_1}^{b_1} ... \int_{a_n}^{b_n} H(t,s,x_*(s))ds\right) \Bigg|
\end{aligned}$$

$$
\begin{aligned}
&\le \alpha\big|x_n(t)-x_*(t)\big| \\
&\quad + \beta\left|\int_{a_1}^{t_1}\dots\int_{a_n}^{t_n} K(t,s,x_n(s))ds - \int_{a_1}^{t_1}\dots\int_{a_n}^{t_n} K(t,s,x_*(s))ds\right| \\
&\quad + \gamma\left|\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n} H(t,s,x_n(s))ds - \int_{a_1}^{b_1}\dots\int_{a_n}^{b_n} H(t,s,x_*(s))ds\right| \\
&\le \alpha\big|x_n(t)-x_*(t)\big| \\
&\quad + \beta\int_{a_1}^{t_1}\dots\int_{a_n}^{t_n} \Big|K(t,s,x_n(s)) - K(t,s,x_*(s))\Big|ds \\
&\quad + \gamma\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n} \Big|H(t,s,x_n(s)) - H(t,s,x_*(s))\Big|ds \\
&\le \alpha\big|x_n(t)-x_*(t)\big| + \beta\int_{a_1}^{t_1}\dots\int_{a_n}^{t_n} L_K\Big|x_n(s)-x_*(s)\Big|ds \\
&\quad + \gamma\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n} L_H\Big|x_n(s)-x_*(s)\Big|ds \\
&\le \alpha\big\|x_n-x_*\big\| + \beta\int_{a_1}^{t_1}\dots\int_{a_n}^{t_n} L_K\big\|x_n-x_*\big\|ds \\
&\quad + \gamma\int_{a_1}^{b_1}\dots\int_{a_n}^{b_n} L_H\big\|x_n-x_*\big\|ds \\
&= \alpha\|x_n-x_*\| + \beta L_K(t_1-a_1)\dots(t_n-a_n)\|x_n-x_*\| \\
&\quad + \gamma L_H(b_1-a_1)\dots(b_n-a_n)\|x_n-x_*\|.
\end{aligned}
$$

It implies that

$$\|z_n - x_*\| \le [\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\dots(b_n-a_n)]\|x_n - x_*\|. \tag{3.3}$$

Similarly, we can obtain that

$$\|y_n - x_*\| \le [\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\dots(b_n-a_n)]\|z_n - x_*\| \tag{3.4}$$

and

$$\|x_{n+1} - x_*\| \le [\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\dots(b_n-a_n)]\|y_n - x_*\|. \tag{3.5}$$

Combining equations (3.3), (3.4) and (3.5), we get

$$\|x_{n+1} - x_*\| \le [\alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\dots(b_n-a_n)]^3\|x_n - x_*\|. \tag{3.6}$$

By using condition (C_5) and defining

$$\delta := \alpha + (\beta L_K + \gamma L_H)(b_1-a_1)\dots(b_n-a_n) < 1,$$

equation (3.6) becomes

$$\|x_{n+1} - x_*\| \leq \delta^3 \|x_n - x_*\|.$$

Inductively, we get

$$\|x_{n+1} - x_*\| \leq \delta^{3(n+1)} \|x_0 - x_*\|.$$

Thus, $\lim_{n\to\infty} \|x_n - x_*\| = 0$. This completes the proof. □

Remark 3.3. Theorem 3.2 generalizes and improves the results of Crăciun and Şerban [11], Gursoy [12] and several relevant results in the literature.

References

[1] M. Abbas and T. Nazir, *A new faster iteration process applied to constrained minimization and feasibility problems,* Mat. Vesn., **66**(2) (2014), 223–234.

[2] R.P. Agrawal, D. O'Regan and D.R. Sahu, *Iterative construction of fixed points of nearly asymptotically nonexpansive mappings,* J. Nonlinear Convex Anal., **8**(1) (2007), 61–79.

[3] F. Ali, J. Ali and J.J. Nieto, *Some observations on generalized non-expansive mappings with an application,* Comp. Appl. Math., **39**, 74 (2020).

[4] F. Ali and J. Ali, Convergence, stability, and data dependence of a new iterative algorithm with an application, Comp. Appl. Math., **39**, 267 (2020).

[5] F. Ali, J. Ali and I. Uddin, *A novel approach for the solution of BVPs via Greens function and fixed point iterative method,* J. Appl. Math. Comput., **66**(1-2) (2021), 167-181.

[6] V. Berinde, *Picard iteration converges faster than Mann iteration for a class of quasi-contractive operators,* Fixed Point Theory Appl., **2004**(2) (2004), 97–105.

[7] V. Berinde, *On the stability of some fixed point procedures,* Bull. Stiint. Univ. Baia Mare, Ser. b fasc. Mat.-Inform, **18**(1) (2002), 7–14.

[8] S.K. Chatterjea, *Fixed point theorem,* C. R. Acad. Bulgare Sci., **25** (1972), 727–730.

[9] F. Ö. Çeliker, *Convergence analysis for a modified SP iterative method,* Sci. World J. **2014** (2014), 1–5.

[10] R. Chugh, V. Kumar and S. Kumar, *Strong Convergence of a new three-step iterative scheme in Banach spaces,* Amer. J. Comp. Math., **2** (2012), 345–357.

[11] C. Crăciun and M.A. Şerban, *A nonlinear integral equation via Picard operators,* Fixed Point Theory, **12** (2011), 57–70.

[12] F. Gursoy, *A Picard-S iterative method for approximating fixed point of weak-contractions,* Filomat, **30** (2016), 2829–2845.

[13] F. Gursoy and V. Karakaya, *A Picard-S hybrid type iteration method for solving a differential equation with retarded argument,* arXiv:1403.2546v2 (2014).

[14] A.M. Harder, *Fixed point theory and stability results for fixed point iteration procedures,* PhD thesis, University of Missouri-Rolla, Missouri, 1987.

[15] S. Ishikawa, *Fixed points by a new iteration method,* Proc. Amer. Math. Soc., **44** (1974), 147–150.

[16] N. Kadioglu and I. Yildirm, *Approximating fixed points of nonexpansive mappings by a faster iteration process,* arXiv:1402.6530v1 (2014).

[17] I. Karahan and M. Ozdemir, *A general iterative method for approximation of fixed points and their applications*, Adv. Fixed Point Theory, **3** (2013), 510–526.

[18] R. Kannan, *Some results on fixed points,* Bull. Calcutta Math. Soc., **10** (1968), 71–76.
[19] S.H. Khan, *A Picard-Mann hybrid iterative process,* Fixed Point Theory Appl., **2013** (2013), 1–10.
[20] Z. Liu, F. Hou, J.S. Ume and S.M. Kang, *Positive solutions and convergence of Mann iterative schemes for a fourth order neutral delay difference equation,* Adv. Differ. Equ. **2015** (2015).
[21] W.R. Mann, *Mean value methods in iteration,* Proc. Amer. Math. Soc., **4** (1953), 506–510.
[22] A.L. Mogbademu, *New iteration process for a general class of contractive mappings,* Acutm. Math. Ut. Ee., **20**(2) (2016), 117–122.
[23] M.A. Noor, *New approximation schemes for general variational inequalities,* J. Math. Anal. Appl., **251**(1) (2000), 217–229.
[24] M.O. Osilike, *Stability of the Mann and Ishikawa iteration procedures for ϕ-strong pseudocontractions and nonlinear equations of the ϕ-strongly accretive type,* J. Math. Anal. Appl., **227** (1998), 319–334.
[25] A.M. Ostrowski, *The round-off stability of iterations,* Z. Angew. Math. Mech., **47**(1) (1967), 77–81.
[26] W. Phuengrattana and S. Suantai, *On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval,* J. Comp. Appl. Math., **235** (2011), 3006–3014.
[27] E. Picard, *Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives,* J. Math. Pures Appl., **6** (1890), 145–210.
[28] B.E. Rhoades, *A comparison of various definitions of contractive mappings,* Trans. Amer. Math. Soc., **226** (1977), 257–290.
[29] D.R. Sahu, *Applications of the S-iteration process to constrained minimization problems and split feasibility problems,* Fixed Point Theory, **12** (2011), 187–204.
[30] V.K. Sahu, H.K. Pathak and R. Tiwari, *Convergence theorems for new iteration scheme and comparison results,* Aligarh Bull. Math., **35**(1) (2016), 19–42.
[31] W. Sintunavarat and A. Pitea, *On a new iteration scheme for numerical reckoning fixed points of Berinde mappings with convergence analysis*, J. Nonl. Sci. Appl., **9**(5) (2016), 2553–2562.
[32] B.S. Thakur, D. Thakur and M. Postolache, *A new iterative scheme for numerical reckoning fixed points of Suzuki's generalized non-expansive mappings,* Appl. Math. Comput., **275** (2016), 147–155.
[33] B.S. Thakur, D. Thakur and M. Postolache, *New iteration scheme for approximating fixed point of nonexpansive mappings,* Filomat, **30**(10) (2016), 2711–2720.
[34] D. Thakur, B.S. Thakur and M., Postolache, *New iteration scheme for numerical reckoning fixed points of nonexpansive mappings,* J. Inequal. Appl., **2014** (2014).
[35] K. Ullah and M. Arshad, *New iteration process and numerical reckoning fixed points in Banach space,* U.P.B. Sci. Bull., Series A, **79**(4) (2017), 113–122.
[36] T. Zamfirescu, *Fixed point theorems in metric spaces,* Arch. Math. (Basel), **23** (1972), 292–298.