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A NOTE ON A RECENT RESULT: ON THE LOCATION OF THE ZEROS OF POLYNOMIALS (LACUNARY TYPE)

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Dedicated to Professor J. A. Adepoju on his 75th birthday.

Abstract. In this paper, we give some corrections and comments about a result which is contained in a published paper in [1].

1. Introduction

For a class of polynomials P_n, defined by

$$P_n = \left\{ p(z) : p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j, \ \ a_0 \neq 0, \ \ 1 \leq \mu \leq n \right\}.$$

The following result was stated and proved by Tripathi et al. [1].

Theorem 1.1. *Let* $p \in \mathbb{P}_n$ *be a* n^{th} *degree polynomial with complex coefficients. If* $\operatorname{Re} a_j = \alpha_j$, $\operatorname{Im} a_j = \beta_j$, $0 \leq j \leq n$ *such that for some* $t \geq 1, \lambda, \delta$ *and* $1 \leq k \leq n$, *either,* $t^{n-k+2}\alpha_n \geq t^{n-k}\alpha_{n-2} \geq \cdots \geq t^2\alpha_{k+2} \geq \alpha_k \leq \alpha_{k-2} \leq \cdots \leq \alpha_\mu - \lambda$ *and* $t^{n-k+1}\alpha_{n-1} \geq t^{n-k-1}\alpha_{n-3} \geq \cdots \geq t\alpha_{k+1} \geq \alpha_{k-1} \leq \alpha_{k-3} \leq \cdots \leq \alpha_{\mu+1} - \lambda$, *if* n *and* μ *is odd, or* $t^{n-k+2}\alpha_n \geq t^{n-k}\alpha_{n-2} \geq \cdots \geq t^2\alpha_{k+2} \geq \alpha_k \leq \alpha_{k-2} \leq \cdots \leq \alpha_\mu - \lambda$ *and* $t^{n-k+1}\alpha_{n-1} \geq t^{n-k-1}\alpha_{n-3} \geq \cdots \geq t\alpha_{k+1} \geq \alpha_{k-1} \leq \alpha_{k-3} \leq \cdots \leq \alpha_{\mu+1} - \lambda$, *if* n *and* μ *is even and* $\beta_n \leq \beta_{n-1} \leq \cdots \leq \delta + \beta_k \geq$

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$\beta_{k-1} \geq \cdots \geq \beta_{\mu}$, then the n zeros of $p(z)$ lies in the disc

$$\left|z+\frac{a_{n-1}}{a_n}\right| \leq \frac{1}{|a_n|}\Big[\alpha_n+\alpha_{n-1}+(t-1)\sum_{j=k+1}^{n}(\alpha_j+|\alpha_j|)+2(\lambda+|\lambda|) - 2(\alpha_k+\alpha_{k-1})+(\alpha_\mu+|\alpha_\mu|)+(\alpha_{\mu+1}+|\alpha_{\mu+1}|)+|\alpha_0| + 3\delta+|\delta|+4\beta_k-\beta_{\mu+2}-(\beta_\mu-|\beta_\mu|)-\beta_{n-1}-\beta_n+|\beta_0|\Big].$$

In this note, we shall give some comments on the errors and omissions in Theorem 1.1. Thereafter, give the correct statement, and proof of Theorem 1.1.

Some comments on Theorem 1.1.

Going through the proof of Theorem 1.1. (see [1]), we notice the following errors and omissions, which lead to the incorrect of the result and proof.

(i) The authors did not place any condition on the parameters λ and δ in the statement of Theorem 1.1, if λ is taking to be very large, $\alpha_\mu - \lambda$ and $\alpha_{\mu+1}-\lambda$ cannot be larger than $\alpha_{\mu+2}$ and $\alpha_{\mu+3}$ respectively. Thus, Theorem 1.1 is not possible for large values of λ. For the parameters to be well defined, λ and δ must satisfy the following conditions: $\lambda \geq 0$, $\delta \geq 0$.

(ii) The condition $0 \leq j \leq n$ is misleading because $j \neq 0$, and ν in the proof of Theorem 1.1. (see page 558 of [1] line 8) is meaningless.

(iii) In the statement of Theorem 1.1, there is no need of keeping the indexed coefficients since it leads to no further generalization. In the course of the proof of Theorem 1.1, the indexed coefficients are redundants.

(iv) The claims in the proof of Theorem 1.1 (see page 559 of [1] lines 8 upto the end of the proof) is incorrect, because there is no way all the zeros of f(z) for which $|z| > 1$ lie in the disk given (see line 10 on page 556). This could be noticed with the right choice of a_0, $a_n$and a_j for $j = \mu, \mu+1, ..., n-1$, where $1 \leq \mu \leq n$.

2. Main results

The following theorem is the revision of the Theorem 1.1 for the errors and omissions in the statements.

Theorem 2.1. *Let $p(z) = a_0 + \sum_{j=\mu}^{n} a_j z^j$ for some $1 \leq \mu \leq n$, where $Re(a_j) = \alpha_j$ and $Im(a_j) = \beta_j$ for $1 \leq j \leq n$. Suppose that for some real $\lambda \geq 0$, $\delta \geq 0$ and $1 \leq k \leq n$, either, $\alpha_n \geq \alpha_{n-2} \geq ... \geq \alpha_{k+2} \geq \alpha_k \leq \alpha_{k-2} \leq$*

... $\le \alpha_\mu + \lambda$ and $\alpha_{n-1} \ge \alpha_{n-3} \ge ... \ge \alpha_{k+1} \ge \alpha_{k-1} \le \alpha_{k-3} \le ... \le \alpha_{\mu+1} + \lambda$, if n and μ are odd, or $\alpha_n \ge \alpha_{n-2} \ge ... \ge \alpha_{k+2} \ge \alpha_k \le \alpha_{k-2} \le ... \le \alpha_\mu + \lambda$ and $\alpha_{n-1} \ge \alpha_{n-3} \ge ... \ge \alpha_{k+1} \ge \alpha_{k-1} \le \alpha_{k-3} \le ... \le \alpha_{\mu+1} + \lambda$, if n and μ are even and $\beta_n \le \beta_{n-1} \le ... \le \delta + \beta_k \ge \beta_{k-1} \ge ... \ge \beta_\mu$. Then the number of zeros of $p(z)$ lies in the disk

$$\begin{aligned}\left|z + \frac{a_{n-1}}{a_n}\right| \le \frac{1}{|a_n|}\Big[&\alpha_n + \alpha_{n-1} + (t-1)\sum_{j=k+1}^{n}(\alpha_j + |\alpha_j|) + 2(\lambda + |\lambda|)\\ &- 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) + (\alpha_{\mu+1} + |\alpha_{\mu+1}|) + |\alpha_0|\\ &+ 3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} - (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|\Big],\end{aligned}$$

for some $t \ge 1$.

Proof. Consider the polynomial :

$$\begin{aligned}F(z) &= (1-z^2)(a_0 + \sum_{j=\mu}^{n} a_j z^j)\\ &= -a_n z^{n+2} - a_{n-1}z^{n+1} + \sum_{j=\mu+2}^{n}(a_j - a_{j-2})z^j\\ &\quad + a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1-z^2)a_0\\ &= -(a_n z + a_{n-1})z^{n+1} + \sum_{j=\mu+2}^{n}\{(\alpha_j - \alpha_{j-2})z^j + i(\beta_j - \beta_{j-2})z^j\}\\ &\quad + a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1-z^2)a_0\\ &= -(a_n z + a_{n-1})z^{n+1} + \sum_{j=k+1}^{n}(t\alpha_j - \alpha_{j-2})z^j - (t-1)\sum_{j=k+1}^{n}\alpha_j z^j\\ &\quad + \sum_{j=\mu+4}^{k}(\alpha_j - \alpha_{j-2})z^j + (\alpha_{\mu+3} - \alpha_{\mu+1} - \lambda)z^{\mu+3} + \lambda z^{\mu+3}\\ &\quad + (\alpha_{\mu+2} - \alpha_\mu - \lambda)z^{\mu+2} + \lambda z^{\mu+2}\\ &\quad + i\sum_{j=k+2}^{n}(\beta_j - \beta_{j-2})z^j + i(\beta_{k+1} - (\delta + \beta_k))z^{k+1}\\ &\quad + i((\delta + \beta_k) - \beta_{k-1})z^{k+1}\\ &\quad + i(\delta + \beta_k - \beta_{k-2})z^k - i\delta z^k + i\sum_{j=\mu+2}^{k-1}(\beta_j - \beta_{j-2})z^j\end{aligned}$$

$$+ a_{\mu+1}z^{\mu+1} + a_\mu z^\mu + (1-z^2)a_0,$$

for some $t \geq 1$. Then, we have

$$\begin{aligned}|F(z)| \geq |a_n z + a_{n-1}||z|^{n+1} - |z|^n\Big[&\sum_{j=k+1}^{n} \frac{|t\alpha_j - \alpha_{j-2}|}{|z|^{n-j}} \\ &+ |t-1| \sum_{j=k+1}^{n} \frac{|\alpha_j|}{|z|^{n-j}} + \sum_{j=\mu+4}^{k} \frac{|\alpha_j - \alpha_{j-2}|}{|z|^{n-j}} \\ &+ \frac{|\alpha_{\mu+3} - \alpha_{\mu+1} - \lambda|}{|z|^{n-\mu-3}} + \frac{|\lambda|}{|z|^{n-\mu-3}} \\ &+ \frac{|\alpha_{\mu+2} - \alpha_\mu - \lambda|}{|z|^{n-\mu-2}} + \frac{|\lambda|}{|z|^{n-\mu-2}} \\ &+ \sum_{j=k+2}^{n} \frac{|\beta_j - \beta_{j-2}|}{|z|^{n-j}} + \frac{|\beta_{k+1} - (\beta_k + \delta)|}{|z|^{n-k-1}} \\ &+ \frac{|(\delta + \beta_k) - \beta_{k-1}|}{|z|^{n-k-1}} + \frac{|\delta + \beta_k - \beta_{k-2}|}{|z|^{n-k}} \\ &+ \frac{|\delta|}{|z|^{n-k}} + \sum_{j=\mu+2}^{k-1} \frac{|\beta_j - \beta_{j-2}|}{|z|^{n-j}} \\ &+ \frac{|\alpha_{\mu+1}| + |\beta_{\mu+1}|}{|z|^{n-\mu-1}} + \frac{|\alpha_\mu| + |\beta_\mu|}{|z|^{n-\mu}} \\ &+ \frac{|\alpha_0| + |\beta_0|}{|z|^n} + \frac{|\alpha_0| + |\beta_0|}{|z|^{n-2}}\Big],\end{aligned}$$

for some $t \geq 1$.

Now, let $|z| > 1$, so that $\frac{1}{|z|^{n-j}} < 1$ for $1 \leq \mu \leq j \leq n$. Then, we have

$$\begin{aligned}|F(z)| > |a_n z + a_{n-1}||z|^{n+1} - |z|^n\Big[&\sum_{j=k+1}^{n} |t\alpha_j - \alpha_{j-2}| \\ &+ (t-1) \sum_{j=k+1}^{n} |\alpha_j| + \sum_{j=\mu+4}^{k} |a_j - a_{j-2}| \\ &+ |\alpha_{\mu+3} - \alpha_{\mu+1} - \lambda| + |\lambda| \\ &+ |\alpha_{\mu+2} - \alpha_\mu - \lambda| + |\lambda|\end{aligned}$$

$$
\begin{aligned}
&+ \sum_{j=k+2}^{n} |\beta_j - \beta_{j-2}| + |\beta_{k+1} - (\beta_k + \delta)| \\
&+ |\delta + \beta_k - \beta_{k-2}| + |\delta| + \sum_{j=\mu+2}^{k-1} |\beta_j - \beta_{j-2}| \\
&+ |\alpha_{\mu+1}| + |\beta_{\mu+1}| + |\alpha_\mu| + |\beta_\mu| + |\alpha_0| + |\beta_0|\Big] \\
> 0,
\end{aligned}
$$

for some $t \geq 1$.

Next, it is easy to see that,

$$
\begin{aligned}
|a_n z + a_{n-1}| > \alpha_n + \alpha_{n-1} + (t-1) \sum_{j=k+1}^{n} (\alpha_j + |\alpha_j|) \\
+ 2(\lambda + |\lambda|) - 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) \\
+ (\alpha_{\mu+1} + |\alpha_{\mu+1}|) + |\alpha_0| + 3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} \\
- (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|,
\end{aligned}
$$

for some $t \geq 1$. It implies that

$$
\begin{aligned}
\left|z + \frac{a_{n-1}}{a_n}\right| > \frac{1}{|a_n|}\Big[\alpha_n + \alpha_{n-1} + (t-1) \sum_{j=k+1}^{n} (\alpha_j + |\alpha_j|) \\
+ 2(\lambda + |\lambda|) - 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) \\
+ (\alpha_{\mu+1} + |\alpha_{\mu+1}|) + |\alpha_0| + 3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} \\
- (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|\Big],
\end{aligned}
$$

then all the zeros of $F(z)$ whose modulus is greater than 1 lie in

$$
\begin{aligned}
\left|z + \frac{a_{n-1}}{a_n}\right| \leq \frac{1}{|a_n|}\Big[\alpha_n + \alpha_{n-1} + (t-1) \sum_{j=k+1}^{n} (\alpha_j + |\alpha_j|) \\
+ 2(\lambda + |\lambda|) - 2(\alpha_k + \alpha_{k-1}) + (\alpha_\mu + |\alpha_\mu|) \\
+ (\alpha_{\mu+1} + |\alpha_{\mu+1}|) + |\alpha_0| + 3\delta + |\delta| + 4\beta_k - \beta_{\mu+2} \\
- (\beta_\mu - |\beta_\mu|) - \beta_{n-1} - \beta_n + |\beta_0|\Big],
\end{aligned}
$$

for some $t \geq 1$. But observe that all the zeros of $p(z)$ are also the zeros of $F(z)$. Hence it follows that all the zeros of $F(z)$ and hence of $p(z)$ lie in the union of the disks $|z| < 1$ and

$$\Big|z+\frac{a_{n-1}}{a_n}\Big| \le \frac{1}{|a_n|}\Big[\alpha_n+\alpha_{n-1}+(t-1)\sum_{j=k+1}^{n}(\alpha_j+|\alpha_j|) + 2(\lambda+|\lambda|)-2(\alpha_k+\alpha_{k-1})+(\alpha_\mu+|\alpha_\mu|) + (\alpha_{\mu+1}+|\alpha_{\mu+1}|)+|\alpha_0|+3\delta+|\delta|+4\beta_k-\beta_{\mu+2} - (\beta_\mu-|\beta_\mu|)-(\beta_\mu-|\beta_\mu|)-\beta_{n-1}-\beta_n+|\beta_0|\Big],$$

for some $t \ge 1$. This completes the proof. □

Taking $t=1$ in Theorem 2.1, we get the following result.

Corollary 2.2. *Let $p(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$ for some $1\le\mu\le n$, where $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$ for $1\le j\le n$. Suppose that for some real $\lambda\ge0$, $\delta\ge0$ and $1\le k\le n$, either, $\alpha_n\ge\alpha_{n-2}\ge...\ge\alpha_{k+2}\ge\alpha_k\le\alpha_{k-2}\le ...\le\alpha_\mu+\lambda$ and $\alpha_{n-1}\ge\alpha_{n-3}\ge...\ge\alpha_{k+1}\ge\alpha_{k-1}\le\alpha_{k-3}\le...\le\alpha_{\mu+1}+\lambda$, if n and μ are odd, or $\alpha_n\ge\alpha_{n-2}\ge...\ge\alpha_{k+2}\ge\alpha_k\le\alpha_{k-2}\le...\le\alpha_\mu+\lambda$ and $\alpha_{n-1}\ge\alpha_{n-3}\ge...\ge\alpha_{k+1}\ge\alpha_{k-1}\le\alpha_{k-3}\le...\le\alpha_{\mu+1}+\lambda$, if n and μ are even and $\beta_n\le\beta_{n-1}\le...\le\delta+\beta_k\ge\beta_{k-1}\ge...\ge\beta_\mu$. Then the number of zeros of p(z) lies in the disk*

$$\Big|z+\frac{a_{n-1}}{a_n}\Big| \le \frac{1}{|a_n|}\Big[\alpha_n+\alpha_{n-1}-2(\alpha_k+\alpha_{k-1})+2(\lambda+|\lambda|) + (\alpha_\mu+|\alpha_\mu|)+(\alpha_{\mu+1}+|\alpha_{\mu+1}|)+|\alpha_0|+3\delta+|\delta| + 4\beta_k-\beta_{\mu+2}-(\beta_\mu-|\beta_\mu|)-\beta_{n-1}-\beta_n+|\beta_0|\Big].$$

Taking $\lambda=0$ in Theorem 2.1, we get the following result.

Corollary 2.3. *Let $p(z)=a_0+\sum_{j=\mu}^{n}a_jz^j$ for some $1\le\mu\le n$, where $Re(a_j)=\alpha_j$ and $Im(a_j)=\beta_j$ for $1\le j\le n$. Suppose that for some real $\delta\ge0$ and $1\le k\le n$, either, $\alpha_n\ge\alpha_{n-2}\ge...\ge\alpha_{k+2}\ge\alpha_k\le\alpha_{k-2}\le...\le\alpha_\mu$ and $\alpha_{n-1}\ge\alpha_{n-3}\ge...\ge\alpha_{k+1}\ge\alpha_{k-1}\le\alpha_{k-3}\le...\le\alpha_{\mu+1}$, if n and μ are odd, or $\alpha_n\ge\alpha_{n-2}\ge...\ge\alpha_{k+2}\ge\alpha_k\le\alpha_{k-2}\le...\le\alpha_\mu$ and $\alpha_{n-1}\ge\alpha_{n-3}\ge...\ge\alpha_{k+1}\ge\alpha_{k-1}\le\alpha_{k-3}\le...\le\alpha_{\mu+1}$, if n and μ are even and $\beta_n\le\beta_{n-1}\le...\le\delta+\beta_k\ge\beta_{k-1}\ge...\ge\beta_\mu$. Then the number of zeros of p(z) lies in the disk*

$$\Big|z+\frac{a_{n-1}}{a_n}\Big| \le \frac{1}{|a_n|}\Big[\alpha_n+\alpha_{n-1}+(t-1)\sum_{j=k+1}^{n}(\alpha_j+|\alpha_j|) - 2(\alpha_k+\alpha_{k-1})+(\alpha_\mu+|\alpha_\mu|)+(\alpha_{\mu+1}+|\alpha_{\mu+1}|) + |\alpha_0|+3\delta+|\delta|+4\beta_k-\beta_{\mu+2} - (\beta_\mu-|\beta_\mu|)-\beta_{n-1}-\beta_n+|\beta_0|\Big],$$

for some $t \geq 1$.

References

[1] D. Tripathi, H.E. Hyun and S. Hans, *On location of the zeros of polynomials (Lacunary type)*, Nonlinear Funct. Anal. Appl., **24(3)** (2019), 555–564, doi.org/10.22771/nfaa.2019.24.03.09.