

Nonlinear Functional Analysis and Applications

Vol. 26, No. 4 (2021), pp. 781-792

ISSN: 1229-1595(print), 2466-0973(online)

<https://doi.org/10.22771/nfaa.2021.26.04.08>

<http://nfaa.kyungnam.ac.kr/journal-nfaa>

Copyright © 2021 Kyungnam University Press



ON OPTIMAL SOLUTIONS OF WELL-POSED PROBLEMS AND VARIATIONAL INEQUALITIES

Tirth Ram¹, Jong Kyu Kim² and Ravdeep Kour³

¹Department of Mathematics, University of Jammu
Jammu, Jammu & Kashmir 180006, India
e-mail: tir1ram2@yahoo.com

²Department of Mathematics Education, Kyungnam University
Changwon, Gyeongnam 51767, Korea
e-mail: jongkyuk@kyungnam.ac.kr

³Department of Mathematics, University of Jammu
Jammu, Jammu & Kashmir 180006, India
e-mail: ravdeepkour2011@gmail.com

Abstract. In this paper, we study well-posed problems and variational inequalities in locally convex Hausdorff topological vector spaces. The necessary and sufficient conditions are obtained for the existence of solutions of variational inequality problems and quasi variational inequalities even when the underlying set K is not convex. In certain cases, solutions obtained are not unique. Moreover, counter examples are also presented for the authenticity of the main results.

1. INTRODUCTION

The theory of variational inequalities has become a rich source of inspiration in both mathematical and engineering sciences, which has begun with the works of Fichera [8] and Hartmann and Stampacchia [11]. Variational inequalities and its various generalizations have been very effective and quite powerful tool in studying the existence of solutions of constrained problems arising in

⁰Received May 15, 2021. Revised August 10, 2021. Accepted August 15, 2021.

⁰2010 Mathematics Subject Classification: 49J40, 49K40, 52A07.

⁰Keywords: Variational inequality, well-posed problem, global minimizer, optimal solutions.

⁰Corresponding author: J. K. Kim(jongkyuk@kyungnam.ac.kr).

mechanics, optimization and control, operator research, general equilibrium problems in economics and transportation, and so on. It also provides us unified and general framework for studying boundary value problems arising out of contact problems in electrostatics, fluid flow through porous media etc.

The study of variational inequality problems has been carried out by several mathematicians see Anh et al. [1], Baiocchi and Capelo [4], Crank [6], Glowinski, Lions and Tremoliers [9], Kikuchi and Oden [15], Kim [16], Kim et al. [17, 18, 19, 20], Kinderlehrer and Stampacchia [21], Noor [27].

The work is also generalized in several directions like quasi variational inequality [3], vector variational inequality [2, 19], generalized vector variational inequalities [26, 34], variational-like inequalities for multivalued maps [29], Stampacchia variational inequality with weak convex mappings [5], and so on.

There are several notion of well-posedness related to optimization problems, see for example, [7], [13], [14], [24], [25], [28], [30], [31], [33] and the references therein. These notions of well-posedness can be divided into three classes, namely, Tykhonov type [32], Levitin-Polyak type [23] and Hadamard type [10]. Generally speaking, in the study of Tykhonov well-posedness of a problem one induces the notion of approximate sequence for the solution requires some convergence of such sequences to a solution of the problem, Levitin-Polyak well-posedness of a problem means the convergence of the approximating solution sequence to the problem with some constraints, while Hadamard well-posedness of a problem means the continuous dependence of the solutions on the data or the parameter of the problem, see for example, [7], [13], [14], [24], [25], [28], [30], [31], [33] and the reference therein.

In this paper, we study well-posed problems in locally convex Hausdorff topological vector spaces. The necessary and sufficient conditions are obtained for the existence of optimal solutions of variational and quasi variational inequality problems by using Cauchy-Schwartz inequality and Theorem 5.1.1 in [22].

The rest of this article is structured as follows. In section 2, we recall some preliminary material on functional analysis and then introduce notion of well-posedness, variational inequality problem (VIP) and quasi variational inequality problem (QVIP). In section 3, we study well-posed problem for locally convex Hausdorff topological vector space and and gave some examples to illustrate the result. The necessary and sufficient conditions for the variational inequality problem(VIP) and quasi variational inequality problem (QVIP) to get optimal solution are established in section 4.

2. FORMULATIONS AND PRELIMINARIES

Let (X, τ) be a locally convex topological vector space over field $\Phi (= \mathbb{R})$. A subset B of X is called balanced if $aB \subseteq B$ whenever $|a| \leq 1, a \in \Phi$. A subset B of X is known as absorbing if for every $x \in X$, there exists $a > 0$ such that $x \in aB$.

Let B be a convex, balanced, absorbing set in X . The Minkowski functional (gauge) $p_B : X \rightarrow R$ is defined as $p_B(x) = \inf \{a > 0 : x \in aB\}$. Since gauge of a convex balanced absorbing set in topological vector space is a seminorm, it follows that p_B is a seminorm.

Consider $\mathcal{P} = \{p_B : B \text{ is convex balanced absorbing set}\}$. Then \mathcal{P} is a family of seminorms on X .

For each $p \in \mathcal{P}$ and $U(x, \epsilon, p) = \{y \in X : p(y - x) < \epsilon\}$, let

$$\mathcal{S} = \{U(x, \epsilon, p) : x \in X, \epsilon > 0, p \in \mathcal{P}\}.$$

Then the topology generated by \mathcal{S} as a subbase is denoted by $\tau_{\mathcal{P}}$. It is well known that if τ is a locally convex topology on X then $\tau = \tau_{\mathcal{P}}$.

Let $I : X \rightarrow (-\infty, \infty]$ be a proper extended real valued function defined on a topological space X . Consider the problem to minimize $I(x)$ subject to $x \in X$ is denoted by (X, I) . If $I(x_0) \leq I(x)$, for all $x \in X$, then x_0 is called a global minimizer of (X, I) . The set of all global minimizer is denoted by $\arg \min(X, I)$. For $\arg \min(X, I) = \{x_0\}$, we say that (X, I) is well-posed if $I(x_n) \rightarrow I(x_0)$ then $x_n \rightarrow x_0$. In other words, the problem (X, I) is Tykhonov well-posed if and only if I has a unique global minimum point on X towards which every minimizing sequence converges.

Now we shall work under the following settings, unless otherwise specified:

Let X be a topological vector space with its topological dual X^* and K be a nonempty convex subset of X . The value of $l \in X^*$ at x is denoted by $\langle l, x \rangle$. Let $F : K \rightarrow X^*$ be a mapping. Then the variational inequality problem (in short, VIP) is to find $x \in K$ such that

$$\langle F(x), y - x \rangle \geq 0, \text{ for all } y \in K \tag{2.1}$$

or equivalently

$$\langle F(x), x \rangle \leq \langle F(x), y \rangle, \text{ for all } y \in K.$$

Suppose $K : X \rightarrow 2^X$, where 2^X is the power set of X , be a set-valued map such that $K(x)$ is nonempty. Then the quasi-variational inequality problem (in short, QVIP) consists in finding a vector $x \in K(x)$ such that

$$\langle F(x), y - x \rangle \geq 0, \text{ for all } y \in K(x). \tag{2.2}$$

Now we need following results to prove the main results of the paper.

Theorem 2.1. ([22], Theorem 5.3.1) *Let X be a vector space over Φ , let $L \subset X$ be a real hyperplane, and let $K \subset X$ be convex. Then the following are equivalent:*

- (i) K lies strictly on one side of L .
- (ii) $K \cap L = \phi$.

Theorem 2.2. ([22], Theorem 13.1.1) *Let $(X, \langle \cdot, \cdot \rangle)$ be an inner product space over Φ . Then*

$$|\langle x, y \rangle| \leq \|x\| \|y\|, \quad (x, y \in X).$$

Moreover,

$$|\langle x, y \rangle| = \|x\| \|y\|, \quad (x, y \in X)$$

if and only if x, y are linear independent.

3. WELL-POSED PROBLEM FOR LOCALLY CONVEX HAUSDORFF T.V.S

In this section we shall give a formulation for well-posed problem in a locally convex Hausdorff topological vector space.

Theorem 3.1. *Let (X, τ) be a locally convex Hausdorff topological vector space. Then $I : X \rightarrow (-\infty, \infty)$ is well-posed if and only if*

- (i) *there exists $x_0 \in X$ such that $I(x_0) \leq I(x)$, for all $x \in X$,*
- (ii) *for every $\epsilon > 0$, there exists $\delta > 0$ with $|I(x) - I(x_0)| < \delta$ such that $p(x - x_0) < \epsilon$, for all $p \in \mathcal{P}$ and $x \in X$.*

Proof. Since (X, τ) is a locally convex Hausdorff topological vector space, there exists a family of seminorms \mathcal{P} on X such that $\tau = \tau_{\mathcal{P}}$. Suppose that $I : X \rightarrow (-\infty, \infty)$ is well-posed. We first prove that the condition (ii) is true. If the condition (ii) is false, then there exists $\epsilon > 0$ such that for every $\delta > 0$, $|I(x) - I(x_0)| < \delta$ but $p(x - x_0) \geq \epsilon$, for some $p \in \mathcal{P}$ and $x \in X$. Choose $\delta = \frac{1}{n}$. Then we can find $x_n \in X$ such that $|I(x_n) - I(x_0)| < \frac{1}{n}$ and $p(x_n - x_0) \geq \epsilon$, for some seminorm p . This means that $I(x_n) \rightarrow I(x_0)$ as $n \rightarrow \infty$. But $x_n \not\rightarrow x_0$. Hence the condition (ii) must be true.

Next we prove that the condition (i) is true. If the condition (i) is false, then there exists $y \in X$ such that $I(y) < I(x_0)$ so that $|I(y) - I(x_0)| < \delta$, for all $\delta > 0$. Hence by the condition (ii) $y \in U(x_0, \epsilon, p)$, for every $p \in \mathcal{P}$, $\epsilon > 0$ and hence y belongs to every neighborhood of x_0 , which is a contradiction, since (X, τ) is a Hausdorff space. Therefore $I(x_0) \leq I(x)$, for all $x \in X$.

Conversely, suppose that (i) and (ii) are true. From condition (ii), for every $\epsilon > 0$, there exists $\delta > 0$ with $|I(x_n) - I(x_0)| < \delta$ such that $p(x_n - x_0) < \epsilon$. This implies that whenever $\{x_n\}$ is a sequence and $I(x_n) \rightarrow I(x_0)$, we have $x_n \rightarrow x_0$ in (X, τ_p) . That is, $\arg \min (X, I) = \{x_0\}$. Hence I is well-posed. \square

Example 3.2. Let $X = l^p(\mathbb{N})$, the space of all p^{th} summable sequence of real numbers. Define $I : X \rightarrow \mathbb{R}$ by $I(x) = \left(\sum_{k=1}^{\infty} |x_k|^p \right)^{\frac{1}{p}}$. For each $n \in \mathbb{N}$, define $p_n : l^p(\mathbb{N}) \rightarrow \mathbb{R}$ by $p_n(x) = \left(\sum_{k=1}^n |x_k|^p \right)^{\frac{1}{p}}$. Then, each p_n is a seminorm on X . Let $\mathcal{P} = \{p_n : n \in \mathbb{N}\}$. Then $(X, \tau_{\mathcal{P}})$ is a locally convex topological vector space. Let $x_0(k) = 0$, for every $k \in \mathbb{N}$. Then $x_0 \in l^p$ and $I(x_0) = 0$. Now for each $n \in \mathbb{N}$,

$$p_n(x^k) \leq I(x^k). \tag{3.1}$$

Suppose $I(x^k) \rightarrow 0$ in $(X, \tau_{\mathcal{P}})$. Then from (3.1) as $p_n(x^k) \rightarrow 0$ as $k \rightarrow \infty$. This is true for every $n \in \mathbb{N}$. Hence $x^k \rightarrow 0$ as $k \rightarrow \infty$. Also $I(x_0) \leq I(x)$, for every $x \in l^p(\mathbb{N})$. Thus I is well-posed.

Example 3.3. Let $X = C_b(\mathbb{R})$, the space of all bounded continuous real valued functions defined on \mathbb{R} . Let $I : X \rightarrow \mathbb{R}$ be a function defined as $I(x) = \inf\{|x(t)| : t \in \mathbb{R}\}$. For $m \in \mathbb{N}$, set $p_m(x) = \sup_{|t| \leq m} |x(t)|$. Then p_m is a seminorm. If we take $\mathcal{P} = \{p_m : m \in \mathbb{N}\}$, then \mathcal{P} is a family of seminorms. Then for $n \in \mathbb{N}$, let $x_n : \mathbb{R} \rightarrow \mathbb{R}$ be defined by

$$x_n(t) = \begin{cases} |t|, & \text{when } |t| \leq n, \\ n, & \text{when } |t| > n. \end{cases}$$

Then $x_n \in C_b(\mathbb{R})$ for every $n \in \mathbb{N}$. Define $x_0 : \mathbb{R} \rightarrow \mathbb{R}$ by $x_0(t) = 0$, for every $t \in \mathbb{R}$. Now $I(x_0) = \inf\{|x_0(t)| : t \in \mathbb{R}\} = 0$ and $I(x_n) = 0$, for every $n \in \mathbb{N}$. Consider $|I(x_n) - I(x_0)| = 0$ so that $I(x_n) \rightarrow I(x_0)$. But $p_1(x_n - x_0) = \sup_{|t| \leq 1} \{|x_n(t) - x_0(t)|\} \geq 1$, for every $n \in \mathbb{N}$. This shows that $x_n \not\rightarrow x_0$ in $(X, \tau_{\mathcal{P}})$. Hence I is not well-posed.

4. OPTIMAL SOLUTION OF VARIATIONAL INEQUALITY PROBLEMS

In this section, we shall obtain the necessary and sufficient conditions for the variational inequality problem and quasi variational inequality problem to have optimal solutions.

Theorem 4.1. *Let H be a real Hilbert space and let K be a nonempty convex subset of H . For $x \in K$, let $F : K \rightarrow H^*$ be defined by $F(x) = f_x$, where $f_x : H \rightarrow \mathbb{R}$ is given by $f_x(y) = \langle x, y \rangle$, for all $y \in H$. Then*

- (i) *if $0 \in K$, then for all $y \in K$, VIP-(2.1) has a solution if and only if $x = 0$.*

- (ii) if $0 \notin K$, then for all $y \in K$, VIP-(2.1) has a solution if and only if K contains a vector of smallest norm and $K \subset H \setminus \ker F(x)$.

Proof. (i) Suppose VIP-(2.1) has a solution $x \in K$. Then

$$\langle F(x), x \rangle \leq \langle F(x), y \rangle, \text{ for all } y \in K. \quad (4.1)$$

For $y = 0$, we have

$$\|x\|^2 \leq |\langle x, y \rangle| \leq \|x\| \|y\|,$$

which implies that $x = 0$.

Conversely, if $x = 0$, then clearly (4.1) is satisfied. Therefore variational inequality problem has a solution.

(ii) Let $0 \notin K$. Suppose the VIP-(2.1) has a solution for all $y \in K$. We prove that K contains a vector of smallest norm. Now there exists $x \in K$ such that

$$\langle F(x), x \rangle \leq \langle F(x), y \rangle, \text{ for all } y \in K. \quad (4.2)$$

Since $\langle F(x), x \rangle = \|x\|^2$, we can see that $\langle F(x), x \rangle = 0$ if and only if $x = 0$. Therefore from (4.2), we have

$$\|x\|^2 \leq |\langle x, y \rangle| \leq \|x\| \|y\|, \text{ for all } y \in K,$$

which implies that $\|x\| \leq \|y\|$ for all $y \in K$, because $x \neq 0$ by hypothesis. Thus $x \in K$ is a vector of smallest norm.

Next we prove that $K \cap \ker F(x) = \phi$. In this case, there exists $z \in K \cap \ker F(x)$. Thus $\langle F(x), z \rangle = 0$ so that $\langle F(x), x \rangle \leq \langle F(x), z \rangle$, which is a contradiction that VIP-(2.1) has a solution. Hence $K \cap \ker F(x) = \phi$.

Conversely, suppose K contains a vector $x \in K$ of smallest norm and $K \cap \ker F(x) = \phi$, that is, $K \subset H \setminus \ker F(x)$. But $\ker F(x)$ is a maximal subspace of H , it is easy to see that $y_1, y_2 \in K$ are linearly dependent. For, if y_1 and y_2 are linearly independent, then $\text{span}\{y_1 \cup \ker F(x)\}$ is a maximal subspace of H which contradicts the maximality of $\ker F(x)$. This proves that all vectors of K are linearly dependent. In view of Theorem 2.2, if x, y are linearly dependent then the equality holds in Cauchy-Schwartz inequality, that is,

$$|\langle x, y \rangle| = \|x\| \|y\|, \text{ for all } y \in K.$$

Since $0 \notin K$ and K is convex, so $F(K)$ is convex. Now $\ker F(x)$ is a hyperplane and K is a convex set such that $K \cap \ker F(x) = \phi$. In view of Theorem 2.2, K lies strictly on one side of the $\ker F(x)$. Since $0 \notin K$, so $\langle F(x), x \rangle > 0$. Therefore $\langle F(x), y \rangle > 0$, for all $y \in K$.

Now $\|x\| \leq \|y\|$ implies that

$$\|x\|^2 \leq \|x\| \|y\| = |\langle x, y \rangle| = \langle x, y \rangle,$$

that is, $\langle F(x), x \rangle \leq \langle F(x), y \rangle$, for all $y \in K$. Thus x is a solution of VIP(2.1). This completes the proof. \square

Example 4.2. Let $X = \mathbb{R}^3$ and K be the region bounded by the lines $x = 1, z = 0; y = 0, z = 0; x = 3, z = 0; 2x - y = 2, z = 0$. Then K is a convex set and $\|K\| = [1, 5]$. Define $F : K \rightarrow (\mathbb{R}^3)^*$ by

$$\langle F(x), y \rangle = x \cdot y.$$

Let $x = (1, 0, 0) \in K$. Then

$$\langle F(x), y \rangle = x \cdot y = (1, 0, 0) \cdot (a, b, c) = a.$$

Clearly $\ker F(x) = YZ$ plane and $\ker F(x) \cap K = \phi$. Now $\langle F(x), x \rangle = x \cdot x = \|x\|^2 = 1$ and $\langle F(x), y \rangle = a$, but $1 \leq a \leq 3$. Thus $\langle F(x), x \rangle \leq \langle F(x), y \rangle$, for all $y \in K$. Therefore x is a solution of VIP-(2.1).

In the next theorem, we shall make use of the following notations:

Let H be a Hilbert space. For $x^* \in H^*$ and $K \subset H$, let

$$K^- = \{y \in K : \langle x^*, y \rangle < 0\},$$

$$K^+ = \{y \in K : \langle x^*, y \rangle > 0\}$$

and

$$K^0 = \{y \in K : \langle x^*, y \rangle = 0\}.$$

Clearly that $K = K^- \cup K^0 \cup K^+$.

Now we shall consider the main result of this paper.

Theorem 4.3. *Suppose K is a subset of a Hilbert space H . Let $F : K \rightarrow H^*$ be a mapping. Consider the variational inequality problem to find $x \in K$ such that*

$$\langle F(x), y - x \rangle \geq 0, \text{ for all } y \in K. \tag{4.3}$$

Then we have the following statements:

- (a) *Suppose $K^- \neq \phi$. Then VIP-(4.3) has a solution $x \in K$ if and only if x is a vector of greatest norm in K^- .*
- (b) *Suppose $K^- = \phi$ and $K^0 = \phi$. Then VIP-(4.3) has a solution $x \in K$ if and only if x is a vector of smallest norm in K .*
- (c) *Suppose $K^- = \phi$ and $K^0 \neq \phi$. Then VIP-(4.3) has a solution x if and only if $x \in K^0$.*

Proof. If K is a finite set or $F(x) = 0$, for $x \in K$, then the proof is trivial. Suppose $F(x) \neq 0$ and the cardinality of K is an infinite set. For the sake of convenience, we denote $F(x)$ by x^* then the inequality (4.3) can be written as

$$\langle x^*, x \rangle \leq \langle x^*, y \rangle, \text{ for all } y \in K. \tag{4.4}$$

Now by using Riesz-Representation theorem [22], there exists $z \in H$ such that $\langle x^*, y \rangle = \langle y, z \rangle$, for every $y \in H$. We claim that $z \neq 0$. If $z = 0$, then we get $x^* = 0$, which is not true. Next we see that $\langle x^*, z \rangle \neq 0$. This can be seen from the relation $\langle x^*, z \rangle = \|z\|^2$. Let $u, v \in H \setminus \ker x^*$. we prove that u, v are linearly dependent. For $M = \text{span}(\{v\} \cup \ker x^*)$ is a proper subspace of H which properly contains $\ker x^*$. This contradicts the fact that $\ker x^*$ is a maximal subspace of H . Hence u, v are linearly dependent.

Now we shall begin with the proof of the main theorem.

(a) In this case, there exists $x \in K$ such that $\langle x^*, x \rangle < 0$. If x_0 is a solution of variational inequality problem, then $\langle x^*, x_0 \rangle \leq \langle x^*, y \rangle$, for all $y \in K$ and hence

$$\langle x^*, x_0 \rangle \leq \langle x^*, y' \rangle, \text{ for all } y' \in K^-. \quad (4.5)$$

As proved above, $z, x_0, y' \in H \setminus \ker x^*$ are linearly dependent for every $y' \in K^-$. In view of Theorem 2.2, equality holds in Cauchy-Schwartz inequality $|\langle x, y \rangle| \leq \|x\| \|y\|$ for $x, y \in H$ if and only if x and y are linearly dependent. Hence from (4.5), we have

$$\langle x_0, z \rangle \leq \langle y', z \rangle, \text{ for all } y' \in K^-$$

or

$$-\langle y', z \rangle \leq -\langle x_0, z \rangle, \text{ for all } y' \in K^-$$

or

$$|\langle y', z \rangle| \leq |\langle x_0, z \rangle|, \text{ for all } y' \in K^-$$

or

$$\|y'\| \|z\| \leq \|x_0\| \|z\|, \text{ for all } y' \in K^-.$$

This yields that x_0 is a vector of greatest norm in K^- .

Conversely, suppose that $x_0 \in K$ is an element of greatest norm in K^- . Then we have

$$\|y'\| \|z\| \leq \|x_0\| \|z\|, \text{ for all } y' \in K^-.$$

By reversing the arguments, we have $\langle x^*, x_0 \rangle \leq \langle x^*, y' \rangle$, for all $y' \in K^-$. Since for $y \in K^0 \cup K^+$ we have $\langle x^*, y \rangle \geq 0$. Thus $\langle x^*, x_0 \rangle \leq \langle x^*, y \rangle$, for all $y \in K$. Thus x_0 is a solution of VIP-(4.3).

(b) In this case $\langle x^*, y \rangle > 0$, for all $y \in K$. Let $x \in K$ be a solution of VIP-(4.3). Then $\langle x^*, x \rangle \leq \langle x^*, y \rangle$, for all $y \in K$ which implies that $|\langle x, z \rangle| \leq |\langle y, z \rangle|$, for all $y \in K$. Hence $\|x\| \|z\| \leq \|y\| \|z\|$, for all $z \in K$. Thus $\|x\| \leq \|y\|$, for all $y \in K$ so that x is a vector of smallest norm in K .

The converse follows by reversing the arguments.

(c) Suppose $K^- = \phi$ and $K^0 \neq \phi$. Then $\langle x^*, y \rangle \geq 0$, for all $y \in K$ and $\langle x^*, y_0 \rangle = 0$, for some $y_0 \in K$. Now if $x \in K$ is a solution of VIP-(4.3) then $\langle x^*, x \rangle \leq \langle x^*, y \rangle$, for all $y \in K$. This implies that $0 \leq \langle x^*, x \rangle \leq \langle x^*, y_0 \rangle = 0$.

Hence $\langle x^*, x \rangle = 0$. Thus $x \in K^0$.
 Conversely, suppose $x \in K^0$. Then we show that x is a solution of VIP-(4.3).
 Now $\langle x^*, x \rangle = 0$ and $\langle x^*, y \rangle \geq 0$, for all $y \in K$. Therefore we can conclude that $\langle x^*, x \rangle \leq \langle x^*, y \rangle$, for all $y \in K$. This proves that x is a solution of VIP-(4.3). \square

Corollary 4.4. *Suppose K is a closed convex subset of H and $x^*(K) \subseteq \mathbb{R}^+$. Then VIP-(4.1) has a unique solution.*

Proof. By Theorem 4.3(b), $x \in K$ is a solution of variational inequality problem VIP-(4.3) if and only if x is a vector of smallest norm in K . But K has a unique vector of smallest norm. Thus x is a unique solution of VIP-(4.1). \square

Example 4.5. Let $H = \mathbb{R}^2$ and K be the region defined by $-2 \leq x \leq 2$; $1 \leq y \leq 2$. For $x = (x_1, x_2) \in K$, define $F : K \rightarrow (\mathbb{R}^2)^*$ by

$$\langle F(x), (y_1, y_2) \rangle = x_1 y_2, \quad \text{for } (y_1, y_2) \in \mathbb{R}^2.$$

Then, clearly $\ker F(x) = \{(y_1, y_2) \in \mathbb{R}^2 : y_1 = 0\} = Y$ - axis. For $x = (-2, 2)$, $K^- = [-2, 2] \times [1, 2] \setminus \{(x_1, y_1) \in K : x_1 = 0\}$, which is nonempty. Now $x = (-2, 2)$ is an element in K^- of greatest norm. In view of Theorem 4.3(a), $x = (-2, 2)$ is a solution of VIP-(4.1).

Example 4.6. Let $H = \mathbb{R}^3$ and K be the region described by $1 \leq x \leq 2$; $0 \leq y \leq 1$; $0 \leq z \leq 1$. For $x = (x_1, x_2, x_3) \in K$, define $F : K \rightarrow (\mathbb{R}^3)^*$ by

$$\langle F(x), (y_1, y_2, y_3) \rangle = x_1 y_1, \quad \text{for } (y_1, y_2, y_3) \in \mathbb{R}^3.$$

Then $F(x)$ is a bounded linear functional on \mathbb{R}^3 . If $(y_1, y_2, y_3) \in \mathbb{R}^3$ and $\langle F(x), y \rangle = x_1 y_1 = 0$, then $y_1 = 0$ because $x_1 \neq 0$ for $(x_1, x_2, x_3) \in K$. That is, $\ker F(x)$ is YZ plane. Clearly $K \cap \ker F(x) = \phi$. Now $K^0 = \phi$, $K^- = \phi$ and $x = (1, 0, 0)$ is an element of smallest norm in K . In view of Theorem 4.3(b), $x = (1, 0, 0)$ is a solution of VIP-(4.1).

Example 4.7. Let $H = \mathbb{R}^2$ and K be the region defined by

$$1 \leq x \leq 2; \quad 0 \leq y \leq 2.$$

For $x = (x_1, x_2) \in K$, define $F : K \rightarrow (\mathbb{R}^2)^*$ by

$$\langle F(x), (y_1, y_2) \rangle = x_1 y_2, \quad \text{for } (y_1, y_2) \in \mathbb{R}^2.$$

Then $F(x) \in (\mathbb{R}^2)^*$ and $\ker F(x) = \{(x, y) : y = 0\} = X$ -axis. If we take $x = (2, 0) \in K$, then $K^0 = \{(y_1, y_2) : 1 \leq y_1 \leq 2, y_2 = 0\}$. Also $K^- = \phi$. Therefore in view of Theorem 4.3(c), every element of K^0 is a solution of VIP-(4.1). Thus VIP-(4.1) can have infinitely many solutions.

In the following theorem, we characterize the existence of the solutions of quasi variational inequality problem.

For the set-valued map $K : H \rightarrow 2^H$ and for the mapping $F : H \rightarrow H^*$, the following notations will be used:

$$K_x^- = \{y \in K_x : \langle F(x), y \rangle < 0\},$$

$$K_x^+ = \{y \in K_x : \langle F(x), y \rangle > 0\}$$

and

$$K_x^0 = \{y \in K_x : \langle F(x), y \rangle = 0\}.$$

Then, clearly $K_x = K_x^- \cup K_x^+ \cup K_x^0$.

Theorem 4.8. *Let H be a Hilbert space and let $K : H \rightarrow 2^H$ be a set-valued mapping. Consider the quasi-variational inequality problem (for short, QVIP): Find $x \in K_x$ such that*

$$\langle F(x), y - x \rangle \geq 0, \text{ for all } y \in K_x. \quad (4.6)$$

Then we have the following statements:

- (a) *Suppose $K_x^- \neq \phi$. Then QVIP-(4.6) has a solution $x \in K_x$ if and only if x is a vector of greatest norm in K_x^- .*
- (b) *Suppose $K_x^- = \phi$ and $K_x^0 = \phi$. Then QVIP-(4.6) has a solution $x \in K_x$ if and only if x is a vector of smallest norm in K_x .*
- (c) *Suppose $K_x^- = \phi$ and $K_x^0 \neq \phi$. Then QVIP-(4.6) has a solution x if and only if $x \in K_x^0$.*

Proof. If K_x is a finite for $x \in H$ or $F(x) = 0$, then the proof of this theorem is trivial. Therefore we suppose that K_x is an infinite set and $F(x) \neq 0$. The proof of the theorem is obtained on the same line as in the proof of the Theorem 4.1. \square

Conclusion: A criterion for well-posed problem in locally convex Hausdorff topological vector space setting is obtained. The characterizations for the existence of solutions of variational inequality problem and quasi variational inequality problem are obtained even when the underlying set K is not convex. It has been shown that in some cases there are infinitely many solutions of (VIP) and (QVIP).

Acknowledgments: This work was supported by the Basic Science Research Program through the National Research Foundation(NRF) Grant funded by Ministry of Education of the republic of Korea (2018R1D1A1B07045427).

REFERENCES

- [1] P.N. Anh, H.T.C. Thach and J.K. Kim, *Proximal-like subgradient methods for solving multi-valued variational inequalities*, *Nonlinear Funct. Anal. Appl.*, **25**(3) (2020), 437–451, <https://doi.org/10.22771/nfaa.2020.25.03.03>
- [2] Q.H. Ansari, E. Kobis and J.C. Yao, *Vector Variational Inequalities and Vector Optimization* International Publishing AG, Switzerland, (2018).
- [3] A.S. Antipin, M. Jacimovic and N. Mijajlovic, *Extragradient method for solving quasi-variational inequalities*, *Optimization*, **67**(1) (2018), 103–112.
- [4] C. Baiocchi and A. Capelo, *Variational and Quasi Variational Inequalities*, J. Wiley and Sons, New York, London, (1984).
- [5] M. Balaj, *Stampacchia variational inequality with weak convex mappings*, *A Journal of Mathematical Programming and Operations Research*, **67** (2018), 1571–1577.
- [6] J. Crank, *Free and Moving Boundary Value Problems*, Clarendon press, Oxford, U.K, (1984).
- [7] A.L. Dontchev and T. Zolezzi, *Well Posed Optimization Problems, Lecture Notes in Mathematics*, Springer Verlag, Berlin, Germany, 1993.
- [8] G. Fichera, *Problemi elastostatici con vincoli unilaterali il problema di signorini con ambigue condizioni al contorno*, *Atti. Acad. Naz. Lincei, Mem. cl. Acta Mat. Nature Sez. La*, **7** (1963-1964), 91–140.
- [9] R. Glowinski, J. Lions and R. Tremolieres, *Numerical Analysis of Variational Inequalities*, North Holland, New York, (1981).
- [10] J. Hadamard, *Sur les problemes aux derivees partielles et leur signification physique*, *Princeton Univ. Bull.*, **13** (1902), 49–52.
- [11] G. Hartmann and G. Stampacchia, *On some nonlinear elliptic differential functional equations*, *Acta Math.*, **115** (1966), 271–310.
- [12] R. Hu, *Equivalence results of well-posedness for split variational-hemivariational inequalities*, *North Holland, J. Nonlinear Convex Anal.* **20** (2019), 447–459.
- [13] X.X. Huang, *Extended and strongly extended well-posedness of set valued optimization problems*, *Math Methods Oper. Res.* **53** (2001), 101–116.
- [14] X.X. Huang and X.Q. Yang, *Generalized Levitin-Polyak well-posedness in constrained optimization*, *SIAM J. Optim.*, **17** (2006), 243–258.
- [15] N. Kikuchi and J.T. Oden, *Contact Problems in Elasticity*, SIAM, (1987).
- [16] J.K. Kim, *Sensitivity analysis for general nonlinear nonvex set-valued variational inequalities in Banach spaces*, *J. of Comput. Anal. Appl.*, **22**(2) (2017), 327–335,
- [17] J.K. Kim, Salahuddin and W.H. Lim, *Solutions of general variational inequality problems in Banach spaces*, *Linear and Nonlinear Anal.*, **6**(3) (2020), 333–345.
- [18] J.K. Kim, P.N. Anh and T.T.H. Anh and N.D. Hien, *Projection methods for solving the variational inequalities involving unrelated nonexpansive mappings*, *J. Nonlinear and Convex Anal.*, **21**(11) (2020), 2517–2537.
- [19] J.K. Kim and Salahuddin, *Local sharp vector variational type inequality and optimization problems*, *Mathematics*, **8**(10):1844, (2020), 1–10. <https://doi.org/10.3390/math8101844>
- [20] J.K. Kim and Salahuddin, *System of hierarchical nonlinear mixed variational inequalities*, *Nonlinear Funct. Anal. and Appl.*, **24**(2) (2019), 207–220.
- [21] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and Their Applications*, Academic Press, New York, 1980.
- [22] R. Larsen, *Functional Analysis*, Marcel Dekker, Inc. New York, (1973).
- [23] E.S. Levitin and B.T. Polyak, *Convergence of minimizing sequence in conditional extremum problems*, *Sov. Math. Dokl.* **7** (1966), 764–767.

- [24] R. Lucchetti and F. Patrone, *A characterization of Tykhonov well-posedness for minimum problems with applications to variational inequalities*, Numer. Funct. Anal. Optim., **3** (1981), 461–476.
- [25] R. Lucchetti and F. Patrone, *Some properties of well posed variational equalities governed by linear operators*, Numer. Funct. Anal. Optim., **5** (1982-1983), 349–361.
- [26] S.N. Mishra, P.K. Das and S.K. Mishra, *On generalized harmonic vector variational inequalities using HC_* -condition*, Nonlinear Funct. Anal. Appl, **24**(3) (2019), 639-649, <https://doi.org/10.22771/nfaa.2019.24.03.14>
- [27] M.A. Noor, *On a class of variational inequalities*, J. Math Anal. Appl., **128** (1987), 138–155.
- [28] M.A. Noor, *Well-posed variational inequalities*, J. Appl. Math and Computing Vol., **11** (2003), 165–172.
- [29] A.H. Siddiqi, Q.H. Ansari and M.F. Khan *Variational-like inequalities for multivalued maps*, Indian Journal of Pure and Applied Mathematics, **30** (1999), 161–166.
- [30] M. Sofonea and Y.B. Xiao, *Tykonov Well-posedness of elliptic variational-hemivariational inequalities*, Electron. J. Differ. Equ. **64** (2018), 1–19.
- [31] M. Sofonea and Y.B. Xiao, *On the well-posedness in the sense of Tykhonov*, Journal of Optimization Theory and Applications **183** (2019), 139–157.
- [32] A.N. Tykhonov, *On the stability of functional optimization problems*, USSR, Comput. Math. Math. Phys. **6** (1966), 28–33.
- [33] Y.M. Wang, *Equivalence of well-posedness between systems of hemivariational inequalities and inclusion problems*, J. Nonlinear Sci. App. **9** (2016), 1178–1192.
- [34] G. Wang, S.S. Chang, Salahuddin and J.A Liu, *Generalized vector variational inequalities and applications*, PanAmerican Mathematical Journal, **26** (2016), 77–88.