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EXISTENCE AND UNIQUENESS OF A SOLUTION FOR FIRST ORDER NONLINEAR LIOUVILLE-CAPUTO FRACTIONAL DIFFERENTIAL EQUATIONS

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Abstract. In this paper, first order nonlinear Liouville-Caputo fractional differential equations is studied. The existence and uniqueness of a solution are investigated by using Krasnoselskii and Banach fixed point theorems and the method of lower and upper solutions. Finally, an example is given to illustrate our results.

1. Introduction

In recent years, theory of fractional differential equations has become an important investigation area(Kilbas et al. [6], Podlubny [17], Miller and Ross [12] and Samko et al. [18]). The basic theory for initial value problems for fractional differential equations involving the Riemann-Liouville and Liouville-Caputo differential operator was discussed by Diethelm [5]. Many interesting results of the existence of solutions of various classes of fractional differential equations involving Riemann-Liouville and Caputo type fractional derivatives with the initial condition, the integral boundary conditions have been studied

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extensively by several researchers (see [1, 7, 8, 9, 10, 13, 14, 15, 16, 20, 21] and the references therein).

Recently, Matar [11] used the method of upper and lower solutions and Schauder and Banach fixed point theorems to obtain the existence and uniqueness of positive solution for the nonlinear fractional differential equations:

$$\begin{cases} {}^cD^{\mu}w(r) = f(r, w(r)), & 0 < r \leq 1, \\ w(0) = 0, \quad w^{'}(0) = \zeta > 0, \end{cases}$$

where ${}^cD^{\mu}$ is the standard Caputo fractional derivative of order $1 < \mu \leq 2$ and $f : [0, 1] \times [0, \infty) \to [0, \infty)$ is continuous functions.

By employing the upper and lower solutions and Schauder and Banach fixed point theorems, Boulares et al. [4] investigated existence and uniqueness of positive solutions for the nonlinear fractional differential equations

$$\begin{cases} {}^cD^{\mu}w(r) = f(r, w(r)) + {}^cD^{\mu-1}h(r, w(r)), & 0 < r \leq T, \\ w(0) = \zeta_1, \quad w^{'}(0) = \zeta_2 > 0, \end{cases}$$

where ${}^cD^{\mu}$ is the standard Liouville-Caputo's fractional derivative of order $1 < \mu \leq 2$, $h, f : [0, T] \times [0, \infty) \to [0, \infty)$ are given continuous functions, h is nondecreasing on w and $\zeta_2 \geq h(0, \zeta_1)$. Ardjouni et al. [2] studied the existence and uniqueness of positive solutions for the first-order nonlinear Caputo-Hadamard fractional differential equations. Also Ardjouni et al. [3] studied the existence and uniqueness of positive solutions for the first-order nonlinear Liouville-Caputo fractional differential equations by using the upper and lower solutions and use Krasnoselskii and Banach fixed point theorems.

Inspired by the aforementioned works, in this paper, using the method of upper and lower solutions and the Krasnoselskii and Banach fixed point theorems, we study the existence and uniqueness of solutions of nonlinear fractional differential equation

$$\begin{cases} {}^cD^{\mu}(w(r) - h(r, w(r))) = f(r, w(r)), & 0 < r \leq T, \\ w(0) = w_0 > h(0, w_0) > 0, \end{cases} \tag{1.1}$$

where $0 < \mu \leq 1$ and $h, f : [0, T] \times [0, \infty) \to [0, \infty)$ are continuous functions.

The layout of paper is as follows: In Section 2, we introduce some basic definitions and lemmas that will be used to prove main results. Section 3 is devoted to existence and uniqueness of solution for the problem (1.1) and we provide an example to illustrate results.

2. Preliminaries

Let $B = C([0,T])$ be the Banach space of all real-valued continuous functions defined on the compact interval $[0,T]$, endowed with the norm $\|w\| = \max\limits_{0\leq r\leq T} |w(r)|$.

Let K be a nonempty closed subset of B defined as

$$K = \{w \in B : \|w\| \leq l,\, l > 0\}.$$

Let $c, d \in \mathbb{R}^+$ with $c < d$ and for any $w \in [c,d] \subset \mathbb{R}^+$, we define the upper and lower control functions respectively as follows:

$$M(r,w) = \sup_{c\leq \eta\leq w} f(r,\eta), \quad m(r,w) = \inf_{w\leq \eta\leq d} f(r,\eta).$$

It is obvious that $m(r,w)$ and $M(r,w)$ are monotonic non-decreasing on [c, d] and $m(r,w) \leq f(r,w) \leq M(r,w)$.

We give some definitions and their properties for our main results.

Definition 2.1. ([6, 17]) The fractional integral of order $\mu > 0$ of a function $w : \mathbb{R}^+ \to \mathbb{R}$ is given by

$$I^\mu w(r) = \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} w(s)\, ds,$$

provided the right side is pointwise defined on $\mathbb{R}^+$.

Definition 2.2. ([5, 16]) The Liouville-Caputo fractional derivative of order $\mu > 0$ of a function $w : \mathbb{R}^+ \to \mathbb{R}$ is given by

$${}^cD^\mu w(r) = I^{n-\mu} w^n(r) = \frac{1}{\Gamma(n-\mu)} \int_0^r (r-s)^{n-\mu-1} w^{(n)}(s)\, ds,$$

where $n = [\mu] + 1$, provided the right side is pointwise defined on $\mathbb{R}^+$.

Lemma 2.3. ([6, 17]) *Let $Re(\mu) > 0$, $w \in C^{n-1}([0,+\infty))$ and $w^{(n)}$ exists almost everywhere on any bounded interval of $\mathbb{R}^+$. Then*

$$(I^\mu\, {}^cD_0^\mu w)(r) = w(r) - \sum_{k=0}^{n-1} \frac{w^{(k)}(0)}{k!} r^k.$$

In particular, when $0 < Re(\mu) < 1$, $(I^\mu\, {}^cD_0^\mu w)(r) = w(r) - w(0)$.

Lemma 2.4. *Let $w \in C([0,T])$, w' and $\frac{\partial h}{\partial r}$ exist. Then $w(r)$ is a solution of* (1.1) *if and only if*

$$w(r) = w_0 - h(0,w_0) + h(r,w(r)) + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s,w(s))\, ds. \quad (2.1)$$

Proof. Suppose $w(r)$ satisfies (1.1). Then applying I_0^μ to both sides of (1.1), we have

$$I^\mu[{}^cD^\mu(w(r) - h(r, w(r)))] = I^\mu f(r, w(r)),\ 0 < r \leq T.$$

In view of Lemma 2.3 and the initial condition of problem (1.1), we get

$$w(r) = w_0 - h(0, w_0) + h(r, w(r)) + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, w(s))\, ds.$$

Conversely, suppose that $w(r)$ satisfies equation (2.1). Then applying ${}^cD^\mu$ to both sides of equation (2.1), we obtain

$$\begin{aligned}
{}^cD^\mu w(r) &= {}^cD^\mu[w_0 - h(0, w_0) + h(r, w(r)) \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, w(s))\, ds] \\
&= {}^cD^\mu h(r, w(r)) + {}^cD^\mu I^\mu f(r, w(r)) \\
&= {}^cD^\mu h(r, w(r)) + f(r, w(r)).
\end{aligned}$$

Then ${}^cD^\mu[w(r) - h(r, w(r))] = f(r, w(r))$ and the initial condition $w(0) = w_0$ holds. □

Lastly, we state the fixed point theorems which is useful to prove the existence and uniqueness of a solution of (1.1).

Definition 2.5. Let $(B, \|.\|)$ be a Banach space and $\phi : B \to B$. The operator ϕ is a contraction operator if there is an $\gamma \in (0, 1)$ such that $u, v \in B$ implying

$$\|\phi u - \phi v\| \leq \gamma \|u - v\|.$$

Theorem 2.6. ([19]) *Let E be a nonempty closed convex subset of a Banach space B and $\Phi : E \to E$ be a contraction operator. Then there is a unique $w \in E$ with $\Phi w = w$.*

Theorem 2.7. (Krasnoselskii fixed point theorem, [19]) *Let E be a nonempty closed convex subset of a Banach space B and let P and Q two operators defined on E with values in B such that $Pu + Qv \in E$, for every pair $u, v \in E$, the operator P is completely continuous and the operator Q is a contraction.Then there exist $w \in E$ such that $w = Pw + Qw$.*

3. Existence and uniqueness of solution

In this section, first we need to construct two mappings, one is contraction and other is completely continuous. Now we define the operator $\Phi : K \to B$

by

$$\begin{aligned}(\Phi w)(r) &= w_0 - h(0, w_0) + h(r, w(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, w(s))\, ds \\ &= (Pw)(r) + (Qw)(r), \end{aligned} \tag{3.1}$$

where the operator $P : K \to B$ is defined as

$$(Pw)(r) = \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, w(s))\, ds$$

and the operator $Q : K \to X$ is defined as

$$(Qw)(r) = w(0) - h(0, w_0) + h(r, w(r)).$$

Throughout this paper, we assume that the following conditions hold.

(C1) $h, f \in C([0,T] \times [0,\infty), [0,\infty))$ and h is non-decreasing on w.

(C2) Let $w^*, w_* \in K$ such that $c \leq w_* \leq w^* \leq d$ and satisfying

$$^cD^\mu(w^*(r) - h(r, w^*(r))) \geq M(r, w^*(r))$$

and

$$^cD^\mu(w_*(r) - h(r, w_*(r))) \leq m(r, w_*(r)),$$

for any $r \in [0,T]$. The function w^* and w_* are respectively called a pair of upper and lower solutions for the equation (1.1).

(C3) For $u, v \in B$ and $r \in [0,T]$, there exist $\alpha \in (0,1)$ and $\beta < 1$ such that

$$|h(r,u) - h(r,v)| \leq \alpha \|u - v\|$$

and

$$|f(r,u) - f(r,v)| \leq \beta \|u - v\|.$$

We need the following lemmas to establish our results.

Lemma 3.1. *Assume that* $(C1)$ *holds. Then, the operator* $P : K \to B$ *is completely continuous.*

Proof. By $(C1)$, f is a continuous and nonnegative function, we get that $P : K \to B$ is continuous. If the function $f : [0,T] \times K \to [0,\infty)$ is bounded, then there exists $\lambda > 0$ such that $0 \leq f(r, w(r)) \leq \lambda$. Therefore, we obtain

$$\begin{aligned}|(Pw)(r)| &\leq \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} |f(s, w(s))|\, ds \\ &\leq \frac{\lambda}{\Gamma(\mu)} \left[\frac{(r-s)^\mu}{-\mu}\right]_0^r = \frac{\lambda r^\mu}{\Gamma(\mu+1)} \\ &\leq \frac{\lambda T^\mu}{\Gamma(\mu+1)}. \end{aligned}$$

Hence $P(K)$ is uniformly bounded.

Now we will prove equicontinuity of P. Let $w \in K$, $\epsilon > 0$, $\delta > 0$ and for any $r_1, r_2 \in [0,T]$ with $r_1 < r_2$ such that $|r_2 - r_1| < \delta$. If $\delta = \left[\frac{\epsilon\Gamma(\mu+1)}{2\lambda}\right]^{\frac{1}{\mu}}$, then we have

$$\begin{aligned}
|(Pw)(r_1) - (Pw)(r_2)| &\leq \frac{1}{\Gamma(\mu)} \int_0^{r_1} \left|(r_1 - s)^{\mu-1} - (r_2 - s)^{\mu-1}\right| |f(s, w(s))| \, ds \\
&\quad + \frac{1}{\Gamma(\mu)} \int_{r_1}^{r_2} \left|(r_2 - s)^{\mu-1}\right| |f(s, w(s)| \, ds \\
&\leq \frac{\lambda}{\Gamma(\mu)} \int_0^{r_1} \left[(r_1 - s)^{\mu-1} - (r_2 - s)^{\mu-1}\right] ds \\
&\quad + \frac{\lambda}{\Gamma(\mu)} \int_{r_1}^{r_2} (r_2 - s)^{\mu-1} \, ds \\
&= \frac{\lambda}{\Gamma(\mu+1)} [r_1^\mu - r_2^\mu + 2(r_2 - r_1)^\mu] \\
&\leq \frac{2\lambda}{\Gamma(\mu+1)} (r_2 - r_1)^\mu \\
&< \epsilon.
\end{aligned}$$

Therefore $P(K)$ is equicontinuous. Then by Arzela-Ascoli theorem, $P : K \to B$ is completely continuous. □

Lemma 3.2. *Assume that* $(C1)$ *and* $(C3)$ *hold. Then the operator* $Q : K \to B$ *is contraction.*

Proof. By $(C1)$ and initial conditions of problem (1.1), the operator $Q : K \to B$ is continuous. For $u, v \in K$ and $\alpha \in (0,1)$, we have

$$|(Qu)(r) - (Qv)(r)| = |h(r, u(r)) - h(r, v(r))| \leq \alpha \|u - v\|.$$

Hence Q is contraction. □

Theorem 3.3. *Assume that* $(C1)$ *and* $(C2)$ *hold. Then there exists at least one solution* $w(r) \in B$ *of the problem* (1.1) *satisfying* $w_*(r) \leq w(r) \leq w^*(r)$, *for* $r \in [0,T]$.

Proof. Let $U = \{w \in K : w_*(r) \leq w(r) \leq w^*(r), r \in [0,T]\}$, endowed with the norm $\|w\| = \max_{0 \leq r \leq T} |w(r)|$. Then we have $\|w\| \leq l$. Hence U is a bounded, closed and convex subset of a Banach space B. Moreover, by $(C1)$, the continuity of h, f implies that the operator Φ defined by (3.1) is continuous on U. By Lemma 3.1, $P : U \to K$ is completely continuous. Also by Lemma 3.2, $Q : U \to K$ is contraction.

Now, we show that if $u(r), v(r) \in U$, then $(Pu)(r) + (Qv)(r) \in U$. For any $u(r), v(r) \in U$, we have $w_*(r) \leq u(r), v(r) \leq w^*(r)$, then

$$\begin{aligned}(Pu)(r) + (Qv)(r) &= u_0 - h(0, u_0) + h(r, v(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, u(s))\, ds \\ &\leq u_0 - h(0, u_0) + h(r, w^*(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} M(s, w^*(s))\, ds \\ &\leq w^*(r) \end{aligned} \tag{3.2}$$

and

$$\begin{aligned}(Pu)(r) + (Qv)(r) &= u_0 - h(0, u_0) + h(r, v(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} f(s, u(s))\, ds \\ &\geq u_0 - h(0, u_0) + h(r, w_*(r)) \\ &\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} m(s, w_*(s))\, ds \\ &\geq w_*(r). \end{aligned} \tag{3.3}$$

Thus from (3.2) and (3.3), $w_*(r) \leq (Pu)(r) + (Qv)(r) \leq w^*(r)$ implying that $(Pu)(r) + (Qv)(r) \in U$. Hence by Krasnoselskii fixed point theorem, there exists fixed point $w(r) \in U$ such that $w(r) = (Pw)(r) + (Qw)(r)$, $r \in [0, T]$ in U. Therefore the problem (1.1) has at least one solution $w(r) \in U$ and $w_*(r) \leq w(r) \leq w^*(r)$, $r \in [0, T]$. □

Corollary 3.4. *Assume that $(C1) - (C3)$ hold and there exists ζ_1, ζ_2, ζ_3, ζ_4 such that*

$$0 < \zeta_1 \leq h(r, w) \leq \zeta_2 < \infty, \quad (r, w(r)) \in [0, T] \times [0, +\infty) \tag{3.4}$$

and

$$0 < \zeta_3 \leq f(r, w) \leq \zeta_4 < \infty, \quad (r, w(r)) \in [0, T] \times [0, \infty). \tag{3.5}$$

Then the problem (1.1) *has at least one solution $w \in B$. Moreover,*

$$w_0 - h(0, w_0) + \zeta_1 + \zeta_3 \frac{r^\mu}{\Gamma(\mu+1)} \leq w(r) \leq w_0 - h(0, w_0) + \zeta_2 + \zeta_4 \frac{r^\mu}{\Gamma(\mu+1)}. \tag{3.6}$$

Proof. By (3.5) and definition of control functions, we have

$$\zeta_3 \leq m[r, w] \leq M[r, w] \leq \zeta_4, \ (r, w(r)) \in [0, T] \times [0, +\infty). \tag{3.7}$$

Now, we consider the equations

$$
\begin{aligned}
{}^cD_0^\mu[w(r)-\zeta_1] &= \zeta_3, \quad w(0)=w_0,\\
{}^cD_0^\mu[w(r)-\zeta_2] &= \zeta_4, \quad w(0)=w_0.
\end{aligned}
\tag{3.8}
$$

Then, equation (3.8) are equivalent to

$$
\begin{aligned}
w(r) &= w_0 - h(0,w_0) + \zeta_1 + \frac{\zeta_3}{\Gamma(\mu)}\int_0^r (r-s)^{\mu-1}\,ds\\
&= w_0 - h(0,w_0) + \zeta_1 + \zeta_3\frac{r^\mu}{\Gamma(\mu+1)}
\end{aligned}
$$

and

$$
\begin{aligned}
w(r) &= w_0 - h(0,w_0) + \zeta_2 + \frac{\zeta_4}{\Gamma(\mu)}\int_0^r (r-s)^{\mu-1}\,ds\\
&= w_0 - h(0,w_0) + \zeta_2 + \zeta_4\frac{r^\mu}{\Gamma(\mu+1)}.
\end{aligned}
$$

Now taking into account (3.4), (3.7) we have

$$
\begin{aligned}
w_*(r) &= w_0 - h(0,w_0) + \zeta_1 + \frac{\zeta_3}{\Gamma(\mu)}\int_0^r (r-s)^{\mu-1}\,ds\\
&\le w_0 - h(0,w_0) + h(r,w_*(r)) + \frac{1}{\Gamma(\mu)}\int_0^r (t-s)^{\mu-1}m(r,w_*(r))\,ds
\end{aligned}
$$

and

$$
\begin{aligned}
w^*(r) &= w_0 - h(0,w_0) + \zeta_2 + \frac{\zeta_4}{\Gamma(\mu)}\int_0^r (r-s)^{\mu-1}\,ds\\
&\ge w_0 - h(0,w_0) + h(r,w^*(r)) + \frac{1}{\Gamma(\mu)}\int_0^r (r-s)^{\mu-1}M(r,w^*(r))\,ds.
\end{aligned}
$$

Then it is clear that, $w_*(r)$ and $w^*(r)$ are respectively the lower and upper solutions of equation (3.8). Therefore, an application of Theorem 3.3, yields that the problem (1.1) has at least one solution $w \in U \subset B$ and satisfies equation (3.6). □

Theorem 3.5. *Assume that* $(C1)$ *and* $(C3)$ *hold and*

$$
\alpha + \frac{\beta T^\mu}{\Gamma(\mu+1)} < 1. \tag{3.9}
$$

Then the problem (1.1) *has a unique solution* $w \in U$.

Proof. It follows from Theorem 3.3 that the problem (1.1) has at least one solution in U. For uniqueness of solution, we need only to prove that the operator $\Phi : K \to B$ defined in equation (3.1) is a contraction on B. For all $\xi_1,\, \xi_2 \in U$, we have

$$\begin{aligned}
|\Phi(\xi_1) - \Phi(\xi_2)| &\leq |h(r, \xi_1(r)) - h(r, \xi_2(r))| \\
&\quad + \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1} \left| f(s, \xi_1(s)) - f(s, \xi_2(s)) \right| ds \\
&\leq \alpha \left| \xi_1 - \xi_2 \right| + \beta \left| \xi_1 - \xi_2 \right| \frac{1}{\Gamma(\mu)} \int_0^r (r-s)^{\mu-1}\, ds \\
&= \alpha \left| \xi_1 - \xi_2 \right| + \beta \left| \xi_1 - \xi_2 \right| \frac{r^\mu}{\Gamma(\mu+1)} \\
&\leq \left(\alpha + \frac{\beta T^\mu}{\Gamma(\mu+1)} \right) \left| \xi_1 - \xi_2 \right| .
\end{aligned}$$

Thus,

$$\left\| \Phi(\xi_1) - \Phi(\xi_2) \right\| \leq \left(\alpha + \frac{\beta T^\mu}{\Gamma(\mu+1)} \right) \left\| \xi_1 - \xi_2 \right\| .$$

Hence by equation (3.9), the operator Φ is a contraction mapping. Then by contraction mapping principle, we conclude that the problem (1.1) has a unique solution $w \in U$. $\square$

Example 3.6. We consider the following nonlinear fractional differential equation:

$$\begin{cases} {}^cD^{\frac{1}{4}} \left[w(r) - \frac{1+w(r)}{3+w(r)} \right] = \frac{1}{2+r} \left[2 + \frac{rw(r)}{2+w(r)} \right], \quad 0 < r \leq 1, \\ w(0) = 1, \end{cases} \tag{3.10}$$

where $w_0 = 1$, $T = 1$, $h(r,w) = \frac{1+w(r)}{3+w(r)}$, $f(r,w) = \frac{1}{2+r} \left[2 + \frac{rw(r)}{2+w(r)} \right]$ and $h(0, w_0) = \frac{1}{2}$. Since h is non-decreasing on w, $\lim\limits_{w \to \infty} \frac{1+w(r)}{3+w(r)} = 1$, $\lim\limits_{w \to \infty} \frac{1}{2+r} \left[2 + \frac{rw(r)}{1+w(r)} \right] = 1$ and $\frac{1}{3} \leq h(r,w) \leq 1$, $\frac{2}{3} \leq f(r,w) \leq 1$, for $(r,w) \in [0,1] \times [0, \infty)$. Hence from Corollary 3.4, equation (3.10) has a solution which satisfies $w_*(r) \leq w(r) \leq w^*(r)$, where $w^*(r) = \frac{3}{2} + \frac{4r^{\frac{1}{4}}}{\Gamma(\frac{1}{4})}$, $w_*(r) = \frac{5}{6} + \frac{8r^{\frac{1}{4}}}{3\Gamma(\frac{1}{4})}$ are respectively the upper and lower solutions of (3.10). Also $\alpha + \frac{\beta T^{\frac{1}{4}}}{\Gamma(q)} \approx 0.3106 < 1$, then by Theorem 3.5 and (3.10) has a unique solution which is bounded by $w_*(r)$ and $w^*(r)$.

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