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ANALYTICAL AND APPROXIMATE SOLUTIONS FOR GENERALIZED FRACTIONAL QUADRATIC INTEGRAL EQUATION

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Abstract. In this paper, we study the analytical and approximate solutions for a fractional quadratic integral equation involving Katugampola fractional integral operator. The existence and uniqueness results obtained in the given arrangement are not only new but also yield some new particular results corresponding to special values of the parameters ρ and ϑ. The main results are obtained by using Banach fixed point theorem, Picard Method, and Adomian decomposition method. An illustrative example is given to justify the main results.

1. Introduction

The subject of fractional order of differential equations has newly developed as an interesting field of research. In fact, fractional derivatives types supply an excellent tool for the description of memory and hereditary properties of different materials and processes. More authors have found that fractional

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order differential equations play important roles in many research fields, such as chemical technology, physics, biotechnology, population dynamics, and economics see [7, 19, 22].

On the other hand, fractional calculus and its applications have also a fundamental role in the theory of differential equations and applied mathematics. We refer the readers to [12, 31, 32, 33, 34, 35].

Picard Method (PM) [10] generates a sequence of increasingly precise algebraic approximations of the curtained exact solution of the first order differential equation with an initial value. The PM of successive approximations is applied to the proof of the existence of a solution of such equations.

The Adomian decomposition method (ADM) is an analytical method for solving broad types of functional equations. In ongoing decades, there has been a lot of enthusiasm for the ADM. The technique was effectively applied to a lot of utilizations in applied sciences. For additional insights concerning the technique and its application, see [1, 2, 3, 8, 9, 29].

The PM was first contrasted with the ADM by [29] and [5] on a variety of examples. In [18] the author indicated that the ADM for a linear differential equation was equivalent to the PM. Nonetheless, this equivalence doesn't hold for nonlinear differential equations (DEs). The authors in [16] contrasted the two techniques for a quadratic integral equation (QIE).

The QIEs can be very applicable in many applications such as the theory of radiative exchange, the traffic theory, the dynamic theory of gases, etc. The QIEs have been concentrated in sundry papers and monographs, see [4, 13, 14, 15, 16, 17, 23, 24, 25, 26, 27]. For instance, in [16] the authors discussed the Picard method and the Adomian method with proving the existence and uniqueness of solution for

$$x(t) = a(t) + \mathfrak{g}(t, x(t)) \int_0^t \mathfrak{F}(\tau, x(\tau)) d\tau.$$

In [17] the authors concerned with Picard and Adomian methods and the existence of the solution to the fractional order QIE

$$x(t) = a(t) + \mathfrak{g}(t, x(t)) \int_0^t \frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \mathfrak{F}(\tau, x(\tau)) d\tau, \ \vartheta > 0.$$

In this work, we give the analytical and approximate solutions for the fractional quadratic integral equation (FQIE)

$$x(t) = a(t) + \mathfrak{g}(t, x(t))^{\rho}\mathfrak{I}_{0^+}^{\vartheta} \ \mathfrak{F}(t, x(t)), \ t \in J = [0, 1], \ \vartheta > 0, \tag{1.1}$$

where $\mathfrak{I}_{0^+}^{\vartheta;\rho}$ is the Katugampola fractional integral (KFI) defined in the following form:

$$\mathfrak{I}_{0^+}^{\vartheta;\rho}\ \mathfrak{F}(t,x(t))=\frac{1}{\Gamma(\vartheta)}\int_0^t \tau^{\rho-1}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x(\tau))d\tau.$$

Moreover, we obtain the existence and uniqueness theorem for equation (1.1).

The paper is composed as follows. In Section 2, we give notations and definitions utilized all through the paper. In Section 3, we prove the existence and uniqueness results for FQIE (1.1) involving Katugampola fractional integral. Moreover, we discuss the analytical and approximate solution of the proposed equation by using Picard and Adomian methods.

2. Preliminaries

Let $J=[0,1]\subset\mathbb{R}^+$ and $C(J)$ be the Banach space of all continuous functions on J. For $z\in C(J)$, we have

$$\|z\|_C=\sup_{t\in J}\{|z(t)|:t\in J\}.$$

For $a<b$, $c\in\mathbb{R}^+$ and $1\le p<\infty$, define the function space

$$X_c^p(a,b)=\left\{z:J\to\mathbb{R}:\|z\|_{X_c^p}=\left(\int_a^b |t^c z(t)|^p\,\frac{dt}{t}\right)^{\frac{1}{p}}<\infty\right\},$$

for $p=\infty$,

$$\|z\|_{X_c^p}=ess\sup_{a\le t\le T}\left[|t^c z(t)|\right].$$

Definition 2.1. ([20]) Let $\vartheta>0$, $\rho>0$, $c\in\mathbb{R}^+$ and $z\in X_c^p(a,b)$. Then the Katugampola fractional integral of order ϑ with a parameter ρ is defined by

$$\mathfrak{I}_{a^+}^{\vartheta;\rho}z(t)=\int_a^t\frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}z(\tau)d\tau. \tag{2.1}$$

Definition 2.2. ([21]) Let $n-1<\vartheta<n$, $(n=[\vartheta]+1)$, $\rho>0$, $c\in\mathbb{R}^+$ and $z\in X_c^p(a,b)$. Then the Katugampola and Caputo-Katugampola fractional derivative of order ϑ with a parameter ρ are defined by

$$D_{a^+}^{\vartheta;\rho}z(t)=\left(t^{1-\rho}\frac{d}{dt}\right)^n\mathfrak{I}_{a^+}^{n-\vartheta;\rho}z(t) \tag{2.2}$$

and

$$D_{a^+}^{\vartheta;\rho}z(t)=\mathfrak{I}_{a^+}^{n-\vartheta;\rho}\ z_\rho^{(n)}(t), \tag{2.3}$$

respectively, where $z_{\rho}^{(n)}(t) = \left(t^{1-\rho}\frac{d}{dt}\right)^{n} z(t)$.

Lemma 2.3. ([20]) *Let $\vartheta, \delta, \beta > 0$ and $z \in X_c^p(a,b)$. Then*

(i) $\mathfrak{I}_{a^+}^{\vartheta;\rho}$ *is bounded on the function space $X_c^p(a,b)$.*
(ii) $\mathfrak{I}_{a^+}^{\vartheta;\rho}\mathfrak{I}_{a^+}^{\beta;\rho}z(t) = \mathfrak{I}_{a^+}^{\vartheta+\beta;\rho}z(t)$.
(iii) $\mathfrak{I}_{a^+}^{\vartheta;\rho}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\delta-1} = \frac{\Gamma(\delta)}{\Gamma(\delta+\vartheta)}\left(\frac{t^{\rho}-a^{\rho}}{\rho}\right)^{\vartheta+\delta-1}$.

Theorem 2.4. ([6]) *Let (X,d) be a nonempty complete metric space and $Q : X \to X$ be a contraction mapping. Then the mapping Q has a fixed point in X.*

For further properties of generalized fractional integral operator (see [19, 20, 21]).

3. Main results

The FQDE (1.1) will be investigated under the following hypotheses:

(i) $a : J \to \mathbb{R}$ is continuous on J.
(ii) $\mathfrak{F}, \mathfrak{g} : J \times \mathbb{R} \to \mathbb{R}$ are continuous and bounded with $k_1 = \sup_{(t,x)\in J\times\mathbb{R}} |\mathfrak{g}(t,x)|$ and $k_2 = \sup_{(t,x)\in J\times\mathbb{R}} |\mathfrak{F}(t,x)|$.
(iii) There exist two constants $\ell_1,\ell_2 > 0$ such that

$$|\mathfrak{g}(t,x) - \mathfrak{g}(t,y)| \leq \ell_1 |x-y|$$

and

$$|\mathfrak{F}(t,x) - \mathfrak{F}(t,y)| \leq \ell_2 |x-y|,$$

for all $t \in J$ and $x, y \in \mathbb{R}$.

Define the operator $\mathfrak{N}$ as

$$(\mathfrak{N}x)(t) = a(t) + \mathfrak{g}(t,x(t)) \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^{\rho}-\tau^{\rho}}{\rho}\right)^{\vartheta-1} \mathfrak{F}(\tau,x(\tau))d\tau, \ t \in J, \quad \vartheta > 0.$$

Theorem 3.1. (**Uniqueness Theorem**) *Assume (i), (ii) and (iii) hold. If $\Lambda := \left(\frac{\ell_1 k_2+\ell_2 k_1}{\Gamma(\vartheta+1)}\right)\rho^{-\vartheta} < 1$, then the nonlinear FQIE* (1.1) *has a unique solution $x \in C(J)$.*

Proof. It is obvious that $\mathfrak{N} : C(J) \to C(J)$. Now, let $\mathbb{B}_\lambda \subset C(J)$ such that

$$\mathbb{B}_\lambda = \{x(t) \in C(J) : |x(t) - a(t)| \leq \lambda, \text{ for } t \in J\}.$$

Then $\mathbb{B}_\lambda$ is a closed subset of $C(J)$ and for $\lambda = \frac{k_1 k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}$, the operator $\mathfrak{N} : \mathbb{B}_\lambda \to \mathbb{B}_\lambda$. Indeed, for $x \in \mathbb{B}_\lambda$, we have

$$
\begin{aligned}
|x(t)-a(t)| &\leq |\mathfrak{g}(t,x(t))| \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} |\mathfrak{F}(\tau,x(\tau))|\, d\tau \\
&\leq k_1 k_2 \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} d\tau \\
&\leq \frac{k_1 k_2}{\Gamma(\vartheta+1)} \left(\frac{t^\rho}{\rho}\right)^{\vartheta} \\
&\leq \frac{k_1 k_2}{\Gamma(\vartheta+1)} \rho^{-\vartheta} \\
&= \lambda.
\end{aligned}
$$

Now we prove that $\mathfrak{N}$ is a contraction. Since

$$
\begin{aligned}
&(\mathfrak{N}x)(t)-(\mathfrak{N}y)(t) \\
&= \mathfrak{g}(t,x(t)) \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} \mathfrak{F}(\tau,x(\tau)) d\tau \\
&\quad - \mathfrak{g}(t,y(t)) \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} \mathfrak{F}(\tau,y(\tau)) d\tau \\
&\quad + \mathfrak{g}(t,x(t)) \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} \mathfrak{F}(\tau,y(\tau)) d\tau \\
&\quad - \mathfrak{g}(t,x(t)) \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} \mathfrak{F}(\tau,y(\tau)) d\tau \\
&= [\mathfrak{g}(t,x(t)-\mathfrak{g}(t,y(t)] \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} \mathfrak{F}(\tau,y(\tau)) d\tau \\
&\quad + \mathfrak{g}(t,x(t)) \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} \left[\mathfrak{F}(\tau,x(\tau))-\mathfrak{F}(\tau,x(\tau)\right] d\tau,
\end{aligned}
$$

we have

$$
\begin{aligned}
&|(\mathfrak{N}x)(t)-(\mathfrak{N}y)(t)| \\
&\leq |\mathfrak{g}(t,x(t)-\mathfrak{g}(t,y(t)| \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} |\mathfrak{F}(\tau,y(\tau))|\, d\tau \\
&\quad + |\mathfrak{g}(t,x(t))| \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} |\mathfrak{F}(\tau,x(\tau))-\mathfrak{F}(\tau,x(\tau)|\, d\tau
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{\ell_1 k_2}{\Gamma(\vartheta+1)}\left(\frac{t^\rho}{\rho}\right)^\vartheta |x(t)-y(t)| \\
&\quad + \ell_2 k_1 \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} |x(\tau)-y(\tau)|\, d\tau \\
&\leq \frac{\ell_1 k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta} |x(t)-y(t)| \\
&\quad + \ell_2 k_1 \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} |x(\tau)-y(\tau)|\, d\tau,
\end{aligned}
$$

which implies that

$$
\begin{aligned}
\|(\mathfrak{N}x)(t)-(\mathfrak{N}y)(t)\| &= \sup_{t\in J} |(\mathfrak{N}x)(t)-(\mathfrak{N}y)(t)| \\
&\leq \frac{\ell_1 k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}\|x-y\| \\
&\quad + \ell_2 k_1 \|x-y\| \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} d\tau \\
&\leq \frac{\ell_1 k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}\|x-y\| + \frac{\ell_2 k_1}{\Gamma(\vartheta+1)}\rho^{-\vartheta}\|x-y\| \\
&= \Lambda \|x-y\|.
\end{aligned}
$$

Since $\Lambda < 1$, $\mathfrak{N}$ is a contraction. Hence, Theorem 2.4 shows that FQIE (1.1) has a unique solution $x \in C(J)$. □

3.1. **Picard method:** Applying the Picard method to the FQEI (1.1), the solution is structured by the sequence

$$
\begin{cases}
x_n(t) = a(t) + \mathfrak{g}(t, x_{n-1}(t)) \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1} \mathfrak{F}(\tau, x_{n-1}(\tau))d\tau, \ n = 1, 2, ..., \\
x_0(t) = a(t).
\end{cases}
\tag{3.1}
$$

Then the functions $\{x_n(t)\}_{n\geq 1}$ are continuous and x_n can be written as

$$
x_n = x_0 + \sum_{j=1}^{n} [x_j - x_{j-1}].
$$

If the infinite series $\sum [x_j - x_{j-1}]$ converges, then the sequence $\{x_n(t)\}$ will converge to $x(t)$. Thus, the solution will be

$$
x(t) = \lim_{n\to\infty} x_n(t).
$$

Now, we show that $\{x_n(t)\}_{n\geq 1}$ is uniform convergence. Consider the infinite series

$$\sum_{n=1}^{\infty}\left[x_n(t)-x_{n-1}(t)\right].$$

By (3.1) for $n=1$, we obtain

$$x_1(t)-x_0(t)=\mathfrak{g}(t,x_0(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_0(\tau))d\tau.$$

Thus

$$|x_1(t)-x_0(t)|\leq k_1k_2\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}d\tau\leq \frac{k_1k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}t^{\rho\vartheta}. \tag{3.2}$$

Now, we estimate the express $x_n(t)$ $-x_{n-1}(t)$, for $n\geq 2$ as follows:

$$\begin{aligned}
&x_n(t)-x_{n-1}(t)\\
&=\mathfrak{g}(t,x_{n-1}(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-1}(\tau))d\tau\\
&\quad-\mathfrak{g}(t,x_{n-2}(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-2}(\tau))d\tau\\
&\quad+\mathfrak{g}(t,x_{n-1}(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-2}(\tau))d\tau\\
&\quad-\mathfrak{g}(t,x_{n-1}(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-2}(\tau))d\tau\\
&=\mathfrak{g}(t,x_{n-1}(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\left[\mathfrak{F}(\tau,x_{n-1}(\tau))-\mathfrak{F}(\tau,x_{n-2}(\tau)\right]d\tau\\
&\quad+\left[\mathfrak{g}(t,x_{n-1}(t)-\mathfrak{g}(t,x_{n-2}(t)\right]\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-2}(\tau))d\tau.
\end{aligned}$$

Using hypotheses (ii) and (iii), we obtain

$$\begin{aligned}
&|x_n(t)-x_{n-1}(t)|\\
&\leq|\mathfrak{g}(t,x_{n-1}(t))|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\left|\mathfrak{F}(\tau,x_{n-1}(\tau))-\mathfrak{F}(\tau,x_{n-2}(\tau)\right|d\tau\\
&\quad+\left|\mathfrak{g}(t,x_{n-1}(t)-\mathfrak{g}(t,x_{n-2}(t)\right|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\left|\mathfrak{F}(\tau,x_{n-2}(\tau))\right|d\tau
\end{aligned}$$

$$\leq \ell_2 k_1 \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\vartheta-1} |x_{n-1}(\tau) - x_{n-2}(\tau)|\, d\tau$$
$$+ \ell_1 k_2 |x_{n-1}(t) - x_{n-2}(t)| \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\vartheta-1} d\tau.$$

Imposing $n = 2$, then utilizing (3.2), we obtain

$$\begin{aligned}
|x_2(t) - x_1(t)| &\leq \ell_2 k_1 \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\vartheta-1} |x_1(\tau) - x_0(\tau)|\, d\tau \\
&\quad + \ell_1 k_2 |x_1(t) - x_0(t)| \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\vartheta-1} d\tau \\
&\leq \frac{\ell_2 k_1^2 k_2}{\Gamma(\vartheta+1)} \rho^{-\vartheta} \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\vartheta-1} \tau^{\rho\vartheta} d\tau \\
&\quad + \frac{\ell_1 k_2^2 k_1}{\Gamma(\vartheta+1)} \rho^{-\vartheta} t^{\rho\vartheta} \frac{\rho^{-\vartheta} t^{\rho\vartheta}}{\Gamma(\vartheta+1)} \\
&\leq \frac{\ell_2 k_1^2 k_2}{\Gamma(\vartheta+1)} \rho^{-\vartheta} \frac{\Gamma(\vartheta+1)}{\Gamma(2\vartheta+1)} \rho^{-\vartheta} t^{2\rho\vartheta} \\
&\quad + \frac{\ell_1 k_2^2 k_1}{\Gamma(\vartheta+1)\Gamma(\vartheta+1)} \rho^{-2\vartheta} t^{2\rho\vartheta} \\
&\leq \frac{k_1 k_2}{\Gamma(\vartheta+1)} \left[\ell_2 k_1 \frac{\Gamma(\vartheta+1)}{\Gamma(2\vartheta+1)} \rho^{-2\vartheta} + \frac{\ell_1 k_2}{\Gamma(\vartheta+1)} \rho^{-2\vartheta}\right] t^{2\rho\vartheta}.
\end{aligned}$$

Similarly, for $n = 3$

$$\begin{aligned}
|x_3(t) - x_2(t)| &\leq \ell_2 k_1 \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\vartheta-1} |x_2(t) - x_1(t)|\, d\tau \\
&\quad + \ell_1 k_2 |x_2(t) - x_1(t)| \int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)} \left(\frac{t^\rho - \tau^\rho}{\rho}\right)^{\vartheta-1} d\tau \\
&\leq \frac{k_1 k_2}{\Gamma(\vartheta+1)} \left(\ell_2 k_1 \frac{\Gamma(\vartheta+1)}{\Gamma(2\vartheta+1)} \rho^{-2\vartheta} + \frac{\ell_1 k_2}{\Gamma(\vartheta+1)} \rho^{-2\vartheta}\right) \\
&\quad \times \left(\ell_2 k_1 \frac{\Gamma(2\vartheta+1)}{\Gamma(3\vartheta+1)} \rho^{-3\vartheta} + \frac{\ell_1 k_2}{\Gamma(\vartheta+1)} \rho^{-3\vartheta}\right) t^{3\rho\vartheta}.
\end{aligned}$$

Repeating this process, we get

$$\begin{aligned}|x_n(t)-x_{n-1}(t)| &\leq \frac{k_1k_2}{\Gamma(\vartheta+1)}\left(\ell_2k_1\frac{\Gamma(\vartheta+1)}{\Gamma(2\vartheta+1)}\rho^{-2\vartheta}+\frac{\ell_1k_2}{\Gamma(\vartheta+1)}\rho^{-2\vartheta}\right)\\ &\quad\times\left(\ell_2k_1\frac{\Gamma(2\vartheta+1)}{\Gamma(3\vartheta+1)}\rho^{-3\vartheta}+\frac{\ell_1k_2}{\Gamma(\vartheta+1)}\rho^{-3\vartheta}\right)\\ &\quad\vdots\\ &\quad\times\left(\ell_2k_1\frac{\Gamma((n-1)\vartheta+1)}{\Gamma(n\vartheta+1)}\rho^{-n\vartheta}+\frac{\ell_1k_2}{\Gamma(\vartheta+1)}\rho^{-n\vartheta}\right)t^{n\rho\vartheta}\\ &\leq \frac{k_1k_2}{\Gamma(\vartheta+1)}\left(\ell_2k_1\frac{\Gamma(\vartheta+1)}{\Gamma(\vartheta+1)}\rho^{-\vartheta}+\frac{\ell_1k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}\right)\\ &\quad\times\left(\ell_2k_1\frac{\Gamma(2\vartheta+1)}{\Gamma(\vartheta+1)}\rho^{-\vartheta}+\frac{\ell_1k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}\right)\\ &\quad\vdots\\ &\quad\times\left(\ell_2k_1\frac{\Gamma((n-1)\vartheta+1)}{\Gamma((n-1)\,\vartheta+1)}\rho^{-\vartheta}+\frac{\ell_1k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}\right)\\ &\leq \frac{k_1k_2}{\Gamma(\vartheta+1)}\left(\left(\ell_2k_1+\ell_1k_2\right)\rho^{-\vartheta}\right)\times\left(\left(\ell_2k_1+\ell_1k_2\right)\rho^{-\vartheta}\right)\\ &\quad\vdots\\ &\quad\times\left(\left(\ell_2k_1+\ell_1k_2\right)\rho^{-\vartheta}\right)\\ &\leq \frac{k_1k_2}{\Gamma(\vartheta+1)}\left(\left(\ell_2k_1+\ell_1k_2\right)\rho^{-\vartheta}\right)^n.\end{aligned}$$

Since $\left(\frac{\ell_1k_2+\ell_2k_1}{\Gamma(\vartheta+1)}\right)\rho^{-\vartheta}<1$, the series $\sum_{n=1}^{\infty}\left[x_n(t)-x_{n-1}(t)\right]$ and the sequence $\{x_n(t)\}$ are uniformly convergent.

Due to $\mathfrak{F}(t,x)$ and $\mathfrak{g}(t,x)$ are continuous in x, it follows that

$$\begin{aligned}x(t) &= \lim_{n\to\infty}\mathfrak{g}(t,x_n(t))\int_0^t\frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_n(\tau))d\tau\\ &= \mathfrak{g}(t,x(t))\int_0^t\frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x(\tau))d\tau.\end{aligned}$$

This proves the existence of a solution.

Now we need to prove that the solution is unique. In order to get this, let $y(t)$ be a continuous solution of (1.1), that is,

$$y(t)=a(t)+\mathfrak{g}(t,y(t))\int_0^t\frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,y(\tau))d\tau,\quad t\in[0,1],\ \vartheta>0.$$

Then

$$
\begin{aligned}
&y(t)-x_n(t)\\
&=\mathfrak{g}(t,y(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,y(\tau))d\tau\\
&\quad-\mathfrak{g}(t,x_{n-1}(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-1}(\tau))d\tau\\
&\quad+\mathfrak{g}(t,y(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-1}(\tau))d\tau\\
&\quad-\mathfrak{g}(t,y(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-1}(\tau))d\tau\\
&=\mathfrak{g}(t,y(t))\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\left[\mathfrak{F}(\tau,y(\tau))-\mathfrak{F}(\tau,x_{n-1}(\tau)\right]d\tau\\
&\quad+\left[\mathfrak{g}(t,y(t)-\mathfrak{g}(t,x_{n-1}(t)\right]\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\mathfrak{F}(\tau,x_{n-1}(\tau))d\tau.
\end{aligned}
$$

By utilizing suppositions (ii) and (iiii), we obtain

$$
\begin{aligned}
&|y(t)-x_n(t)|\\
&\leq|\mathfrak{g}(t,y(t))|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\left|\mathfrak{F}(\tau,y(\tau))-\mathfrak{F}(\tau,x_{n-1}(\tau)\right|d\tau\\
&\quad+\left|\mathfrak{g}(t,y(t)-\mathfrak{g}(t,x_{n-1}(t)\right|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\left|\mathfrak{F}(\tau,x_{n-1}(\tau))\right|d\tau\\
&\leq\ell_2k_1\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}\left|y(\tau)-x_{n-1}(\tau)\right|d\tau\\
&\quad+\ell_1k_2\left|y(t)-x_{n-1}(t)\right|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left(\frac{t^\rho-\tau^\rho}{\rho}\right)^{\vartheta-1}d\tau. \qquad (3.3)
\end{aligned}
$$

But we have

$$|y(t)-a(t)|\leq\frac{k_1k_2}{\Gamma(\vartheta+1)}\rho^{-\vartheta}t^{\rho\vartheta}.$$

Hence with using (3.3), we get

$$|y(t)-x_n(t)|\leq\frac{k_1k_2}{\Gamma(\vartheta+1)}\left[\left(\ell_2k_1+\ell_1k_2\right)\rho^{-\vartheta}\right]^n.$$

Consequently

$$\lim_{n\to\infty}x_n(t)=y(t)=x(t).$$

Hence, we have the desired result.

Corollary 3.2. *Under the assumptions of Theorem* 3.1*. If $\rho \to 1$, then the FQEI* (1.1) *reduces to*

$$x(t) = a(t) + \mathfrak{g}(t, x(t)) \int_0^t \frac{(t-\tau)^{\vartheta-1}}{\Gamma(\vartheta)} \mathfrak{F}(\tau, x(\tau)) d\tau,$$

which has a unique solution ([17]).

Corollary 3.3. *Under the assumptions of Theorem* 3.1*. If $\vartheta, \rho \to 1$, then the FQEI* (1.1) *reduces to*

$$x(t) = a(t) + \mathfrak{g}(t, x(t)) \int_0^t \mathfrak{F}(\tau, x(\tau)) d\tau,$$

which has a unique solution ([16]).

In particular, if $\mathfrak{g}(t, x(t)) \equiv 1$, we get the Picard Theorem ([10, 11]).

Corollary 3.4. *Under the assumptions of Theorem* 3.1 *with $\mathfrak{g}(t, x(t)) \equiv 1$, $a(t) = x_0(t)$ and $\vartheta, \rho \to 1$. If $\ell_2 < 1$, then the FQEI* (1.1) *reduces to*

$$x(t) = x_0(t) + \int_0^t \mathfrak{F}(\tau, x(\tau)) d\tau.$$

which has a unique solution ([11]).

3.2. **AD method (ADM).** In this part, we will study ADM for the FQEI (1.1). The solution algorithm of the FQEI (1.1) using ADM is

$$x_0(t) = a(t), \tag{3.4}$$

$$x_k(t) = A_{(k-1)}(t)\ \mathfrak{I}_{0+}^{\vartheta;\rho} B_{(k-1)}(t), \tag{3.5}$$

where A_k and B_k are Adomian polynomials of the nonlinear terms $\mathfrak{g}(t, x)$ and $\mathfrak{F}(\tau, x)$, respectively, which takes the following form

$$A_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\mathfrak{g} \left(t, \sum_{k=0}^{\infty} \lambda^k x_k \right) \right) \right]_{\lambda=0}, \tag{3.6}$$

$$B_n = \frac{1}{n!} \left[\frac{d^n}{d\lambda^n} \left(\mathfrak{F} \left(t, \sum_{k=0}^{\infty} \lambda^k x_k \right) \right) \right]_{\lambda=0}. \tag{3.7}$$

Here we will express the solution as

$$x(t) = \sum_{k=0}^{\infty} x_k. \tag{3.8}$$

3.3. Convergence analysis:

Theorem 3.5. *Let $x(t)$ is a solution of FQIE* (1.1) *and there exists a positive constant M such that $|x_1(t)| < M$. Then the series solution* (3.8) *of FQIE* (1.1) *using ADM converges.*

Proof. Set $\{S_\gamma\}$ be a sequence such that $S_\gamma = \sum_{k=0}^{\gamma} x_k$ is a sequence of partial sums from the series (3.8) and we have

$$\mathfrak{g}(t,x) = \sum_{k=0}^{\infty} A_k \quad \text{and} \quad \mathfrak{F}(t,x) = \sum_{k=0}^{\infty} B_k.$$

Let S_γ and S_ε be two arbitrary partial sums with $\gamma > \varepsilon$. Now, we go ahead to demonstrate that $\{S_\gamma\}$ is a Cauchy sequence in $C(J)$.

$$\begin{aligned}
S_\gamma - S_\varepsilon &= \sum_{k=0}^{\gamma} x_k - \sum_{k=0}^{\varepsilon} x_k \\
&= \sum_{k=0}^{\gamma} A_{(k-1)}(t) \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \sum_{k=0}^{\gamma} B_{(k-1)}(t) \right) \\
&\quad - \sum_{k=0}^{\varepsilon} A_{(k-1)}(t) \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \sum_{k=0}^{\varepsilon} B_{(k-1)}(t) \right) \\
&= \sum_{k=0}^{\gamma} A_{(k-1)}(t) \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \sum_{k=0}^{\gamma} B_{(k-1)}(t) \right) \\
&\quad - \sum_{k=0}^{\varepsilon} A_{(k-1)}(t) \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \sum_{k=0}^{\gamma} B_{(k-1)}(t) \right) \\
&\quad + \sum_{k=0}^{\varepsilon} A_{(k-1)}(t) \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \sum_{k=0}^{\gamma} B_{(k-1)}(t) \right) \\
&\quad - \sum_{k=0}^{\varepsilon} A_{(k-1)}(t) \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \sum_{k=0}^{\varepsilon} B_{(k-1)}(t) \right) \\
&= \left[\sum_{k=0}^{\gamma} A_{(k-1)}(t) - \sum_{k=0}^{\varepsilon} A_{(k-1)}(t) \right] \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \sum_{k=0}^{\gamma} B_{(k-1)}(t) \right) \\
&\quad + \sum_{k=0}^{\varepsilon} A_{(k-1)}(t) \left(\mathfrak{I}_{0^+}^{\vartheta;\rho} \left[\sum_{k=0}^{\gamma} B_{(k-1)}(t) - \sum_{k=0}^{\varepsilon} B_{(k-1)}(t) \right] \right).
\end{aligned}$$

Hence we have

$$
\begin{aligned}
&\|S_\gamma - S_\varepsilon\| \\
&\le \max_{t\in J}\left|\sum_{k=\varepsilon+1}^{\gamma} A_{(k-1)}(t)\left(\mathfrak{I}_{0^+}^{\vartheta;\rho}\sum_{k=0}^{\gamma} B_{(k-1)}(t)\right)\right| \\
&\quad + \max_{t\in J}\left|\sum_{k=0}^{\varepsilon} A_{(k-1)}(t)\left(\mathfrak{I}_{0^+}^{\vartheta;\rho}\sum_{k=\varepsilon+1}^{\gamma} B_{(k-1)}(t)\right)\right| \\
&\le \max_{t\in J}\left|\sum_{k=\varepsilon}^{\gamma-1} A_k(t)\right|\left|\mathfrak{I}_{0^+}^{\vartheta;\rho}\sum_{k=0}^{\gamma} B_{(k-1)}(t)\right| \\
&\quad + \max_{t\in J}\left|\sum_{k=0}^{\varepsilon} A_{(k-1)}(t)\right|\left|\sum_{k=\varepsilon}^{\gamma-1} B_k(t)\right| \\
&\le \max_{t\in J}\left|\mathfrak{g}(t,S_{\gamma-1})-\mathfrak{g}(t,S_{\varepsilon-1})\right|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left[\frac{t^\rho-\tau^\rho}{\rho}\right]^{\vartheta-1}\left|\mathfrak{F}(\tau,S_p)\right|d\tau \\
&\quad + \max_{t\in J}\left|\mathfrak{g}(t,S_\varepsilon)\right|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left[\frac{t^\rho-\tau^\rho}{\rho}\right]^{\vartheta-1}\left|\mathfrak{F}(\tau,S_{\gamma-1})-\mathfrak{F}(\tau,S_{\varepsilon-1})\right|d\tau \\
&\le \ell_1 k_2 \max_{t\in J}\left|S_{\gamma-1}-S_{\varepsilon-1}\right|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left[\frac{t^\rho-\tau^\rho}{\rho}\right]^{\vartheta-1}d\tau \\
&\quad + \ell_2 k_1 \max_{t\in J}\left|S_{\gamma-1}-S_{\varepsilon-1}\right|\int_0^t \frac{\tau^{\rho-1}}{\Gamma(\vartheta)}\left[\frac{t^\rho-\tau^\rho}{\rho}\right]^{\vartheta-1}d\tau \\
&\le \frac{1}{\Gamma(\vartheta+1)}\left[\left(\ell_2 k_1+\ell_1 k_2\right)\rho^{-\vartheta}\right]\max_{t\in J}\left|S_{\gamma-1}-S_{\varepsilon-1}\right| \\
&\le \Lambda\left\|S_{\gamma-1}-S_{\varepsilon-1}\right\|.
\end{aligned}
$$

Let $\gamma = \varepsilon + 1$. Then

$$
\|S_{\varepsilon+1}-S_\varepsilon\| \le \Lambda\|S_\varepsilon - S_{\varepsilon-1}\| \le \Lambda^2\|S_{\varepsilon-1}-S_{\varepsilon-2}\| \le \cdots \le \Lambda^\varepsilon\|S_1 - S_0\|.
$$

Therefore, we have

$$
\begin{aligned}
\|S_\gamma - S_\varepsilon\| &\le \|S_{\varepsilon+1}-S_\varepsilon\| + \|S_{\varepsilon+2}-S_{\varepsilon+1}\| + \cdots + \|S_\gamma - S_{\gamma-1}\| \\
&\le \left[\Lambda^\varepsilon + \Lambda^{\varepsilon+1} + \cdots + \Lambda^{\gamma-1}\right]\|S_1 - S_0\| \\
&\le \Lambda^\varepsilon\left[1+\Lambda+\cdots+\Lambda^{\gamma-\varepsilon-1}\right]\|S_1 - S_0\| \\
&\le \Lambda^\varepsilon\left[\frac{1-\Lambda^{\gamma-\varepsilon}}{1-\Lambda}\right]\|x_1\|.
\end{aligned}
$$

The assumptions $0 < \Lambda < 1$, and $\gamma > \varepsilon$ lead to $(1 - \Lambda^{\gamma-\varepsilon}) \leq 1$. Hence,

$$\begin{aligned} \|S_\gamma - S_\varepsilon\| &\leq \frac{\Lambda^\varepsilon}{1-\Lambda}\|x_1\| \\ &\leq \frac{\Lambda^\varepsilon}{1-\Lambda}\max_{t\in J}|x_1(t)|. \end{aligned}$$

Since $|x_1(t)| < M$ and as $\varepsilon \to \infty$, $\|S_\gamma - S_\varepsilon\| \to 0$ and hence, $\{S_\gamma\}$ is a Cauchy sequence in $C(J)$ and the series $\sum_{k=0}^{\infty} x_k(t)$ converges. □

4. Numerical example

In this section, we apply the Picard method and the ADM method through a numerical example.

Example 4.1. Consider the following nonlinear FQIE,

$$x(t) = \left(t^2 - \frac{204t^{\frac{19}{2}}}{1501}\right) + \frac{1}{4}x(t)\mathfrak{I}_{0^+}^{\frac{1}{2};\frac{1}{2}}x^3(t), \tag{4.1}$$

which has the exact solution $x(t) = t^2$.

Applying Picard method to Eq. (4.1), we get

$$x_n(t) = \left(t^2 - \frac{204t^{\frac{19}{2}}}{1501}\right) + \frac{1}{4}x_{n-1}(t)\mathfrak{I}_{0^+}^{\frac{1}{2};\frac{1}{2}}x_{n-1}^3(t), \quad n = 1, 2, \cdots,$$

$$x_0(t) = \left(t^2 - \frac{204t^{\frac{19}{2}}}{1501}\right),$$

and the solution will be

$$x(t) = x_n(t).$$

Applying ADM to Eq. (4.1), we get

$$x_0(t) = \left(t^2 - \frac{204t^{\frac{19}{2}}}{1501}\right),$$

$$x_i(t) = \frac{1}{4}x_{i-1}(t)\mathfrak{I}_{0^+}^{\frac{1}{2};\frac{1}{2}}A_{i-1}(t), \quad i = 1, 2, \cdots,$$

where A_i are Adomian polynomials of the nonlinear term x^3, and the solution will be

$$x(t) = \sum_{i=0}^{q} x_i(t).$$

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