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COMMON FIXED POINT THEOREMS IN G-FUZZY METRIC SPACES WITH APPLICATIONS

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Abstract. In this paper, we prove common fixed point theorems for six weakly compatible mappings in G-fuzzy metric spaces introduced by Sun and Yang [16] which is actually generalization of G-metric spaces. G-metric spaces coined by Mustafa and Sims [13]. The paper concerns our sustained efforts for the materialization of G-fuzzy metric spaces and their properties. We also exercise the concept of symmetric G-fuzzy metric space, ϕ-function and weakly compatible mappings. The results present in this paper generalize the well-known comparable results in the literature. We justify our results by suitable examples. Some applications are also given in support of our results.

1. Introduction

The concept of fuzzy sets was introduced by Zadeh [18] in 1965. Kramosil and Michalek [11] introduced the concept of fuzzy metric space in 1975, which can be regarded as a generalization of the statistical metric space. Clearly this work plays an essential role in the construction of fixed point theory in fuzzy metric spaces. Mustafa and Sims [13] introduced a new notion of a generalized metric space called G-metric space. Rao et al. [14] proved two unique common

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coupled fixed point theorems for three mappings in symmetric G- fuzzy metric spaces. Sun and Yang [16] introduced the concept of G-fuzzy metric spaces and proved two common fixed point theorems for four mappings. Subsequently, in 1988, Grabiec [2] defined a G-complete fuzzy metric space and extended the complete fuzzy metric spaces. Following Grabiec's work, many authors [9, 12] etc. introduced and generalized the different types of fuzzy contractive mappings and investigate some fixed point theorems in fuzzy metric spaces. In 1994, George and Veeramani [1] modified the notion of M-complete fuzzy metric space with the help of continuous t-norms.

A number of fixed point theorem has been obtained by various authors [4, 6, 12, 17] in fuzzy metric spaces by using the concept of implicit relations, compatible maps, weakly compatible maps, R-weakly compatible maps, E.A. property. In 2019, Gupta et al. [3] proved fixed point theorem in V-fuzzy metric space employing the effectiveness of E.A. property and CLRg property.

Imdad et al. [5] proved common fixed point theorems in modified intuitionistic fuzzy metric spaces in 2012. Jeyaraman et al. [8] validated unique common fixed point theorems for six weakly compatible mappings in intuitionistic generalized fuzzy metric spaces in 2020. Before giving our main result, we recall some of the basic concepts and results in G-metric spaces and G-fuzzy metric spaces.

2. Preliminaries

Now, we begin with some basic concepts.

Definition 2.1. ([15]) A binary operation $*: [0,1] \times [0,1] \to [0,1]$ is called a continuous triangular norm (in short, continuous t-norm) if it satisfies the following conditions:

(TN-1) $*$ is commutative and associative.
(TN-2) $*$ is continuous.
(TN-3) $a * 1 = a$ for every $a \in [0,1]$.
(TN-4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ for all $a, b, c, d \in [0,1]$.

Definition 2.2. ([1]) An ordered triple $(X, M, *)$ is called a fuzzy metric space such that X is a nonempty set, $*$ defined a continuous t-norm and M is a fuzzy set on $X \times X \times (0, \infty)$, satisfying the following conditions: for all $x, y, z \in X$, and $s, t > 0$:

(FM-1) $M(x, y, t) > 0$.
(FM-2) $M(x, y, t) = 1$ iff $x = y$.
(FM-3) $M(x, y, t) = M(y, x, t)$.
(FM-4) $(M(x, y, t) * M(y, z, s)) \leq M(x, z, t + s)$.
(FM-5) $M(x, y, *) : (0, \infty) \to (0, 1]$ is left continuous.

Definition 2.3. ([13]) Let X be a nonempty set. Then $G : X\times X\times X \to [0,\infty)$ is said to be a generalized metric(or a G-metric) on X, if it satisfies the following conditions:

(G-1) $G(x, y, z) = 0$ if $x = y = z$,
(G-2) $0 < G(x, x, y)$ for all $x, y \in X$ with $x \neq y$,
(G-3) $G(x, x, y) \leq G(x, y, z)$ for all $x, y, z \in X$ with $y \neq z$,
(G-4) $G(x, y, z) = G(x, z, y) = G(y, z, x) = \cdots$, symmetry in all three variables,
(G-5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all $x, y, z, a \in X$.

In this case, the pair (X, G) is called a G-metric space.

Definition 2.4. ([13]) The G-metric space (X, G) is called symmetric if $G(x, x, y) = G(x, y, y)$ for all $x, y \in X$.

Definition 2.5. ([16]) A 3-tuple $(X, G, *)$ is said to be a G-fuzzy metric space (denoted by GF space) if X is an arbitrary nonempty set, $*$ is a continuous t-norm and G is a fuzzy set on $G : X \times X \times X \to (0, +\infty)$ satisfying the following conditions: for each $t, s > 0$:

(GF-1) $G(x, x, y, t) > 0$ for all $x, y \in X$ with $x \neq y$,
(GF-2) $G(x, x, y, t) \geq G(x, y, z, t)$ for all $x, y, z \in X$ with $y \neq z$,
(GF-3) $G(x, y, z, t) = 1$ if and only if $x = y = z$,
(GF-4) $G(x, y, z, t) = G(p(x, y, z), t)$, where p is a permutation function,
(GF-5) $G(x, a, a, t) * G(a, y, z, s) \leq G(x, y, z, t + s)$ (the triangle inequality),
(GF-6) $G(x, y, z, \cdot) : (0,\infty) \to [0, 1]$ is continuous.

Remark 2.6. ([16]) Let $x = w, y = u, z = u, a = v$ in $(GF - 5)$, we have

$$G(w, u, u, t + s) \geq G(w, v, v, t) * G(v, u, u, s),$$

which implies that

$$G(u, u, w, s + t) \geq G(u, u, v, s) * G(v, v, w, t),$$

for all $u, v, w \in X$ and $s, t > 0$.

A GF space is said to be symmetric if $G(x, x, y, t) = G(x, y, y, t)$ for all $x, y \in X$ and for each $t > 0$.

Definition 2.7. ([16]) Let $(X, G, *)$ be a GF space. Then

(1) a sequence $\{x_n\}$ in X is said to be convergent to x (denoted by $\lim_{n\to\infty} x_n = x$) if $\lim_{n\to\infty} G(x_n, x_n, x, t) = 1$ for all $t > 0$.
(2) a sequence $\{x_n\}$ in X is said to be a Cauchy sequence if $\lim_{m\to\infty} G(x_n, x_n, x_m, t) = 1$, as $n, m \to \infty$ that is, for any $\epsilon > 0$ and for each $t > 0$, there exists $n_0 \in N$ such that $G(x_n, x_n, x_m, t) > 1 - \epsilon$, for all $n, m \geq n_0$.

(3) A GF space $(X, G, *)$ is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.8. ([10]) A pair (F, G) of self-mappings F and G is weakly compatible, if there exists a point $x \in X$ such that $Fx = Gx$ implies $FGx = GFx$, that is, they commute at their coincidence points.

Lemma 2.9. ([16]) *Let $(X, G, *)$ be a GF space. Then $G(x, y, z, t)$ is non-decreasing with respect to t for all $x, y, z \in X$.*

Lemma 2.10. ([7]) *Φ denote the set of all continuous non decreasing function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi^n(t) \to 0$ as $n \to \infty$ for all $t > 0$. It is clear that $\phi(t) > t$ for all $t > 0$ and $\phi(0) = 0$.*

The objective of this work is to proved an unique common fixed point theorems in G-fuzzy metric spaces. Our results generalize or improve many recent fixed point theorems in the literature. We furnish two examples to validate our results.

3. Main results

In this section, we establish fixed point theorem in G-fuzzy metric space.

Theorem 3.1. *Let $(X, G, *)$ be a G-fuzzy metric space. Let $f, g, h : X \to X$, and $A, B, C : X \to X$ be satisfying.*

(i) *$f(X) \subseteq B(X), g(X) \subseteq C(X)$ and $h(X) \subseteq A(X)$,*
(ii) *One of the $f(X), g(X)$ and $h(X)$ is a closed subspace of X,*
(iii) *The pairs $(f, A), (g, B)$ and (h, C) are weakly compatible and*
(iv)

$$G(fx, gy, hz, t) \geq \phi\left(\min \left\{ \begin{array}{r} G(Ax, By, Cz, t), \dfrac{1}{2}(G(Ax, fy, gy, t) \\ +G(By, gz, hz, t)), \dfrac{1}{5}(G(Ax, gy, hz, t) \\ +G(Cz, fy, hy, t) + G(Bx, fz, gz, t)) \end{array} \right\} \right), \tag{3.1}$$

for all $x, y, z \in X$, where $\phi \in \Phi$.

Then either one of the pairs $(f, A), (g, B), (h, C)$ has a coincide point or the maps f, g, h, A, B and C have a unique common fixed point in X.

Proof. Let x_0 be an arbitrary point in X. By (i), there exist $x_1, x_2, x_3 \in X$ such that $fx_0 = Bx_1 = y_0, gx_1 = Cx_2 = y_1$ and $hx_2 = Ax_3 = y_2$. Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{3n} = fx_{3n} = Bx_{3n+1}, y_{3n+1} =$

$gx_{3n+1} = Cx_{3n+2}$ and $y_{3n+2} = hx_{3n+2} = Ax_{3n+3}$, where $n = 0, 1, 2....$ If $y_{3n} = y_{3n+1}$ then x_{3n+1} is a coincidence point of B and g. If $y_{3n+1} = y_{3n+2}$ then x_{3n+2} is a coincidence point of C and h. If $y_{3n+2} = y_{3n+3}$ then x_{3n+2} is a coincidence point of A and f. Now assume that $y_n \neq y_{n+1}$ for all n. Denote $d_n = G(y_n, y_{n+1}, y_{n+2}, t)$. Putting $x = x_{3n}, y = x_{3n+1}, z = x_{3n+2}$ in (3.1), then we get

$$
\begin{aligned}
d_{3n} &= G(y_{3n}, y_{3n+1}, y_{3n+2}, t) = G(fx_{3n}, gx_{3n+1}, h_{3n+2}, t)\\
&\geq \phi\left(\min\left\{\begin{array}{r} G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}, t), \frac{1}{2}(G(Ax_{3n}, fx_{3n+1}, gx_{3n+1}, t) \\ +G(Bx_{3n+1}, gx_{3n+2}, hx_{3n+2}, t), \frac{1}{5}(G(Ax_{3n}, gx_{3n+1}, hx_{3n+2}, t) \\ +G(Cx_{3n+2}, hx_{3n}, fx_{3n}, t) + G(Bx_{3n}, fx_{3n+2}, gx_{3n+2}, t)) \end{array}\right\}\right)\\
&\geq \phi\left(\min\left\{\begin{array}{r} G(y_{3n-1}, y_{3n}, y_{3n+1}, t), \frac{1}{2}(G(y_{3n-1}, y_{3n+1}, y_{3n+1}, t) \\ +G(y_{3n}, y_{3n+2}, y_{3n+2}, t)), \frac{1}{5}(G(y_{3n-1}, y_{3n+1}, y_{3n+2}, t) \\ +G(y_{3n+1}, y_{3n}, y_{3n}, t) + G(y_{3n-1}, y_{3n+2}, y_{3n+2}, t)) \end{array}\right\}\right)\\
&\geq \phi\left(\min\left\{d_{3n-1}, \frac{1}{2}(d_{3n-1} + d_{3n}), \frac{1}{5}((d_{3n-1} + d_{3n}) + d_{3n} + (d_{3n} + d_{3n-1}))\right\}\right).
\end{aligned} \tag{3.2}
$$

If $d_{3n} \leq d_{3n-1}$ then from Lemma 2.9, we have $d_{3n} \geq (\phi)d_{3n} \geq d_{3n}$. It is a contradiction. Hence $d_{3n} \geq d_{3n-1}$. Now from Lemma 2.9,

$$d_{3n} \geq \phi\ (d_{3n-1}).$$

Similarly, by putting $x = x_{3n+3}, y = x_{3n+1}, z = x_{3n+2}$ and $x = x_{3n+3}, y = x_{3n+4}, z = x_{3n+2}$ in (3.1), we get

$$d_{3n+1} \geq \phi(d_{3n}) \tag{3.3}$$

and

$$d_{3n+2} \geq \phi(d_{3n+1}). \tag{3.4}$$

Thus, from Lemma 2.9, equations (3.1), (3.2) and (3.3), we have

$$
\begin{aligned}
G(y_n, y_{n+1}, y_{n+2}, t) &\geq \phi(G(y_{n-1}, y_n, y_{n+1}, t))\\
&\geq \phi^2(G(y_{n-2}, y_{n-1}, y_n, t))\\
&\vdots\\
&\geq \phi^n(G(y_0, y_1, y_2, t))
\end{aligned} \tag{3.5}
$$

and $G(y_n, y_n, y_{n+1}, t) \geq G(y_n, y_{n+1}, y_{n+2}, t) \geq \phi^n(G(y_0, y_1, y_2, t))$, and now for $m > n$, we have

$$\begin{aligned} G(y_n, y_n, y_m, t) &\geq G(y_n, y_n, y_{n+1}, t) + G(y_{n+1}, y_{n+1}, y_{n+2}, t) \\ &\quad + \cdots + G(y_{m-1}, y_{m-1}, y_m, t) \\ &\geq \phi^n(G(y_0, y_1, y_2, t)) + \phi^{n+1}(G(y_0, y_1, y_2, t)) \\ &\quad + \cdots + \phi^{m-1}(G(y_0, y_1, y_2, t)) \\ &\to 1 \quad \text{as} \ \ n \to \infty. \end{aligned}$$

Since $\phi^n(t) \to 1$ as $n \to \infty$ for all $t > 0$, $\{y_n\}$ is G-Cauchy. Suppose $f(x)$ is G-complete. Then there exist $p, t \in X$ such that $y_{3n+2} \to p = At$. Since $\{y_n\}$ is G-Cauchy, it follows that $y_{3n} \to p$ and $y_{3n+1} \to p$ as $n \to \infty$, and

$$G(ft, gx_{3n+1}, hx_{3n+2}, t)$$

$$\geq \phi\left(\min\left\{\begin{array}{r} G(At, Bx_{3n+1}, Cx_{3n+2}, t), \dfrac{1}{2}(G(At, fx_{3n+1}, gx_{3n+1}, t) \\ +G(Bx_{3n+1}, gx_{3n+2}, hx_{3n+2}, t)), \dfrac{1}{5}(G(At, gx_{3n+1}, hx_{3n+2}, t) \\ +G(Cx_{3n+2}, ht, ft, t) + G(Bt, fx_{3n+2}, gx_{3n+2}, t)) \end{array}\right\}\right).$$

Letting $n \to \infty$, we get

$$G(fp, p, p, t)$$

$$\geq \phi\left(\min\left\{1, \frac{1}{2}(G(p, ft, p, t) + 1)\frac{1}{5}(1 + G(p, p, ft, t) + G(p, ft, p, t))\right\}\right),$$

that is, $G(fp, p, p, t) \geq 2G(fp, p, p, t)$, we have $G(fp, p, p, t) \geq \phi(G(fp, p, p, t))$. ϕ is nondecreasing, thus $fp = p$. Therefore $fp = Ap = p$. Since the pair (f, A) is weakly compatible, we have $Ap = fp$. Since $p = fp \in A(X)$, putting $x = p, y = x_{3n+1}, z = x_{3n+2}$ in (3.1), we get

$$G(fp, gx_{3n+1}, hx_{3n+2}, t)$$

$$\geq \phi\left(\min\left\{\begin{array}{r} G(Ap, Bx_{3n+1}, Cx_{3n+2}, t), \dfrac{1}{2}(G(Ap, fx_{3n+1}, gx_{3n+1}, t) \\ +G(Bx_{3n+1}, gx_{3n+2}, hx_{3n+2}, t), \dfrac{1}{5}(G(Ap, gx_{3n+1}, hx_{3n+2}, t) \\ +G(Cx_{3n+2}, hp, fp, t) + G(Bp, fx_{3n+2}, gx_{3n+2}, t)) \end{array}\right\}\right).$$

Letting $n \to \infty$, we have

$$G(fp, p, p, t)$$

$$\geq \phi\left(\min\left\{\begin{array}{c} G(p,p,p,t), \frac{1}{2}(G(fp,p,p,t)+G(p,p,p,t)), \\ \frac{1}{5}(G(fp,p,p,t)+G(p,p,fp,t)+G(p,fp,p,t)) \end{array}\right\}\right).$$

Since $G(fp,p,p,t) \geq \phi\ G(fp,p,p,t)$. $Ap = fp = p$. Since $p = fp \in B(X)$, there exist $u \in X$ such that $p = gu$. Putting $x = p, y = u, z = x_{3n+2}$ in (3.1), we get

$G(fp, gu, hx_{3n+2}, t)$

$$\geq \phi\left(min\left\{\begin{array}{r} G(Ap,Bu,Cu,t), \frac{1}{2}(G(Ap,fu,gu,t) \\ +G(Bu,gx_{3n+2},hx_{3n+2},t)), \frac{1}{5}(G(Ap,gu,hx_{3n+2},t) \\ +G(Cx_{3n+2},hp,fp,t)+G(Bp,fx_{3n+2},gx_{3n+2},t)) \end{array}\right\}\right).$$

Letting $n \to \infty$, we deduce that

$G(p, gu, p, t)$

$$\geq \phi\left(min\left\{ G(p,gu,p,t), \frac{1}{2}(G(p,p,gu,t)+1)\frac{1}{5}(G(p,gu,p,t)+1+1)\right\}\right)$$

$$So\ G(p,gu,p,t) \geq \phi(G(p,gu,p,t)).$$

Since ϕ is non decreasing. $gu = p$, so that $p = gu = Bu$. Since the pair (g, B) is weakly compatible, we have $gp = Bp$. Putting $x = p, y = p, z = x_{3n+2}$ in (3.1), we get

$G(fp, gp, hx_{3n+2}, t)$

$$\geq \phi\left(\min\left\{\begin{array}{r} G(Ap,Bp,Cx_{3n+2},t), \frac{1}{2}(G(Ap,fp,gp,t) \\ +G(Bp,hx_{3n+2},gx_{3n+2},t))\frac{1}{5}(G(Ap,gp,hx_{3n+2},t) \\ +G(Cx_{3n+2},hp,fp,t)+G(Bp,fx_{3n+2},gx_{3n+2},t)) \end{array}\right\}\right).$$

Letting $n \to \infty$, we have

$G(p, gp, p, t)$

$$\geq \phi\left(\min\left\{ G(p,p,p,t), \frac{1}{2}(G(p,p,gp,t)+1), \frac{1}{5}(G(p,gp,p,t)+1+1)\right\}\right).$$

Since $G(p, gp, p, t) \geq \phi(G(p, gp, p, t))$, $gp = p$. Hence $Bp = gp = p$. Since $p = gp \in C(X)$, there exists $v \in X$ such that $p = Cv$. Putting $x = p, y = p, z = v$ in (3.1) we get

$$G(fp,gp,hv,t) \geq \phi\left(\min\left\{\begin{array}{c}(G(Ap,Bp,Cv,t)),\frac{1}{2}(G(Ap,fp,gp,t)+G(Bp,gv,hv,t)),\\ \frac{1}{5}(G(Ap,gp,hv,t)+G(Cp,hp,fp,t)+G(Bp,fv,gv,t))\end{array}\right\}\right).$$

Since

$$G(p,p,hv,t) \geq \phi\left(\min\left\{\begin{array}{r}1,\frac{1}{2}(1+G(p,p,hv,t)+1),\frac{1}{5}(G(p,p,hv,t)\\ +G(p,p,p,t)+G(p,p,p,t)\end{array}\right\}\right),$$

$$G(p,p,hv,t) \geq \phi(G(p,p,hv,t)),$$

and ϕ is non decreasing, $hv = p$, so that $p = Cv = hv$. Since the pair (h,C) is weakly compatible, we have $hp = Cp$. Putting $x = p, y = p, z = p$ in (3.1), we get

$$\begin{aligned}&G(p,p,hp,t)\\ &\geq \phi\left(\min\left\{\begin{array}{c}G(Ap,Bp,Cp,t),\frac{1}{2}(G(Ap,fp,gp,t)+G(Bp,gp,hp,t)),\\ \frac{1}{5}(G(Ap,gp,hp,t)+G(Cp,hp,fp,t)+G(Bp,fp,gp,t))\end{array}\right\}\right)\\ &\geq \phi\left(\min\left\{\begin{array}{c}G(p,p,p,t),\frac{1}{2}(G(p,p,p,t)+G(p,p,hp,t)),\\ \frac{1}{5}(G(p,p,hp,t)+G(p,hp,p,t)+G(p,p,p,t))\end{array}\right\}\right).\end{aligned}$$

Hence we have

$$G(p,p,hp,t) \geq \phi\ (G(p,p,hp,t)). \tag{3.6}$$

Thus $hp = Cp = p$. It follows that p is a common point of f,g,h,A,B and C. Uniqueness of common fixed point follows easily from (iv). Similarly, we can prove the theorem when $B(X)$ or $C(X)$ is a complete subspace of X. This completes the proof. □

Now, we give an example to validate our result.

Example 3.2. Let $X = [0,1]$ be endowed with the G-fuzzy metric space and $G(x,y,z,t) = 1 + |x-y| + |y-z| + |z-x|$ for all $x,y,z \in X$. Then $(X,G,*)$ is a complete G-fuzzy metric space. Define the mapping $f,g,h : X \to X$ and $A,B,C : X \to X$ be satisfying

$$fx = \frac{x^2}{3}, gx = \frac{x^2}{2}, hx = x^2,$$
$$Ax = x, Bx = \frac{x}{3}, Cx = \frac{x}{2},$$

Then the mappings f, g, h, A, B, C have a unique common fixed point in X.

Solution: Clearly, we get the pairs $f(X) \subseteq B(X), g(X) \subseteq C(X), h(X) \subseteq A(X)$. Since $fX = \{\frac{x^2}{3}\}, BX = \{\frac{x}{3}\}, gX = \{\frac{x^2}{2}\}, CX = \{\frac{x}{2}\}, hX = \{x^2\}$, and $AX = \{x\}$, so that $f(X) \subseteq B(X), g(X) \subseteq C(X), h(X) \subseteq A(X)$.

By the definition of weakly compatible mappings of f and A, only for $x \in [0, 1]$, at this time $f(Ax) = f(x) = \frac{x^2}{3} = A(x) = Afx$, so (f, A) is weakly compatible. Similarly, we can show that, the pair (g, B) and the pair (h, C) are weakly compatible for all $x \in [0, 1]$.

Now we prove that the mappings f, g, h, A, B, C are satisfying condition (3.1) of Theorem 3.1 with $\phi \in \Phi$. Let

$$G(fx, gy, hz, t) = G(\frac{x^2}{3}, \frac{x^2}{2}, x^2) = 1 + \frac{4x^2}{3},$$

$$G(fx, gy, hz, t) \geq \phi\left(\min\left\{\begin{array}{r} G(Ax, By, Cz, t), \frac{1}{2}(G(Ax, fy, gy, t) \\ +G(By, gz, hz, t)), \frac{1}{5}(G(Ax, gy, hz, t) \\ +G(Cz, fy, hy, t) + G(Bx, fz, gz, t)) \end{array}\right\}\right),$$

$$1 + \frac{4x^2}{3} \geq \phi\left(\min\left\{\begin{array}{r} G(x, \frac{x}{3}, \frac{x}{2}, t), \frac{1}{2}(G(x, \frac{x^2}{3}, \frac{x^2}{2}, t) \\ +G(\frac{x}{3}, \frac{x^2}{2}, x^2, t)), \frac{1}{5}(G(x, \frac{x^2}{2}, x^2, t) \\ +G(\frac{x}{2}, \frac{x^2}{3}, x^2, t) + G(\frac{x}{3}, \frac{x^2}{3}, \frac{x^2}{2}), t)) \end{array}\right\}\right),$$

$$1 + \frac{4x^2}{3} \geq \phi\left(\min\left\{1 + \frac{4x}{3}, 1 + \frac{4x}{3} - \frac{3x^2}{4}, \frac{3}{5} + \frac{11x}{15} - \frac{7x^2}{15}\right\}\right),$$

$$1 + \frac{4x^2}{3} \geq \phi\ (\frac{3}{5} + \frac{11x}{15} - \frac{7x^2}{15}) > \frac{3}{5} + \frac{11x}{15} - \frac{7x^2}{15}. \tag{3.6}$$

Thus in all the above cases, the mappings f, g, h, A, B, C are satisfying condition (3.1) of Theorem 3.1 with ϕ, so all the conditions of Theorem 3.1 are satisfied and '0' is the common fixed point of mappings f, g, h, A, B and C.

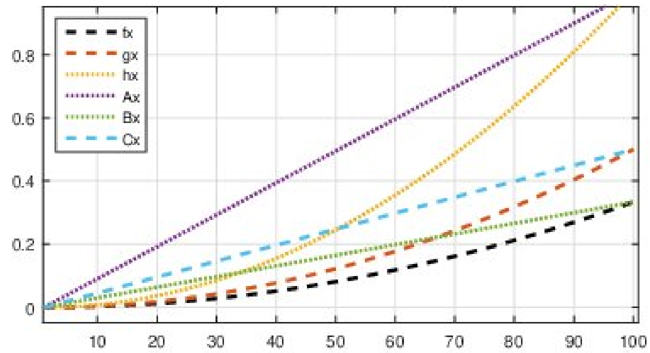

FIGURE 1. Graph of mappings f, g, h, A, B, and C of Example 3.2.

4. APPLICATION

Theorem 4.1. *Let $(X, G, *)$ be a G-complete metric space and let f, g, h, A, B and C be mappings from X into itself such that the following conditions are satisfied:*

(i) $f(X) \subseteq B(X), g(X) \subseteq C(X)$ *and* $h(X) \subseteq A(X)$,
(ii) *One of the* $f(X), g(X)$ *and* $h(X)$ *is a closed subspace of* X,
(iii) *The pairs* $(f, A), (g, B)$ *and* (h, C) *are weakly compatible and*
(iv) *Let*

$$\int_0^{G(fx,gy,hz,t)} \varphi(t)dt \geq \int_0^{\lambda(x,y,z)} \varphi(t)dt, \tag{4.1}$$

where

$$\lambda(x,y,z) = \phi\left(min \left\{ \begin{array}{r} G(Ax, By, Cz, t), \frac{1}{2}(G(Ax, fy, gy, t) \\ +G(By, gz, hz, t)), \frac{1}{5}(G(Ax, gy, hz, t) \\ +G(Cz, fy, hy, t) + G(Bx, fz, gz, t)), \end{array} \right\} \right),$$

for all $x, y, z \in X$, $\varphi : [0, \infty) \to [0, \infty)$ *is a Lebesgue integrable mapping which is summable on each compact subset of* $[0, \infty)$ *non-negative and such that for every* $\epsilon > 0, \int_0^\epsilon \varphi(t)dt > 0$ *and* $\phi : [0, \infty) \to [0, \infty)$.

Then the maps f, g, h, A, B *and* C *have a unique common fixed point in* X.

Proof. Let x_0 be any arbitrary point in X. By (i), there exist $x_1, x_2, x_3 \in X$ such that $fx_0 = Bx_1 = y_0, gx_1 = Cx_2 = y_1$ and $hx_2 = Ax_3 = y_2$. Inductively, there exist sequences $\{x_n\}$ and $\{y_n\}$ in X such that $y_{3n} = fx_{3n} = Bx_{3n+1}, y_{3n+1} = gx_{3n+1} = Cx_{3n+2}$ and $y_{3n+2} = hx_{3n+2} = Ax_{3n+3}$, where

$n = 0, 1, 2...$ If $y_{3n} = y_{3n+1}$ then x_{3n+1} is a coincidence point of B and g. If $y_{3n+1} = y_{3n+2}$ then x_{3n+2} is a coincidence point of C and h. If $y_{3n+2} = y_{3n+3}$ then x_{3n+2} is a coincidence point of A and f. Now assume that $y_n \neq y_{n+1}$ for all n. Denote $d_n = G(y_n, y_{n+1}, y_{n+2}, t)$. Putting $x = x_{3n}, y = x_{3n+1}, z = x_{3n+2}$ in (4.1), we get $d_{3n} = G(y_{3n}, y_{3n+1}, y_{3n+2}, t) = G(fx_{3n}, gx_{3n+1}, h_{3n+2}, t)$.

For

$$\lambda(x_{3n}, x_{3n+1}, x_{3n+2})$$

$$= \phi\left(\min\left\{\begin{array}{c} G(Ax_{3n}, Bx_{3n+1}, Cx_{3n+2}, t), \frac{1}{2}(G(Ax_{3n}, fx_{3n+1}, gx_{3n+1}, t) \\ +G(Bx_{3n+1}, gx_{3n+2}, hx_{3n+2}, t), \frac{1}{5}(G(Ax_{3n}, gx_{3n+1}, hx_{3n+2}, t) \\ +G(Cx_{3n+2}, hx_{3n}, fx_{3n}, t) + G(Bx_{3n}, fx_{3n+2}, gx_{3n+2}, t)) \end{array}\right\}\right),$$

$$\int_0^{G(fx_{3n}, gx_{3n+1}, hx_{3n+2}, t)} \varphi(t)dt \geq \int_0^{\lambda(x_{3n}, x_{3n+1}, x_{3n+2})} \varphi(t)dt.$$

From Theorem 3.1, we have

$$\begin{aligned} \int_0^{G(y_n, y_{n+1}, y_{n+2}, t)} \varphi(t)dt &\geq \int_0^{\phi(G(y_{n-1}, y_n, y_{n+1}, t))} \varphi(t)dt \\ &\geq \int_0^{\phi^2(G(y_{n-2}, y_{n-1}, y_n, t)) \geq \cdots \geq \phi^n(G(y_0, y_1, y_2, t))} \varphi(t)dt. \end{aligned}$$

Since $\varphi(t)$ is a Lebesgue integrable function we have

$$G(y_n, y_n, y_{n+1}, t) \geq G(y_n, y_{n+1}, y_{n+2}, t) \geq \phi^n(G(y_0, y_1, y_2, t)),$$

and now for $m > n$, we have

$$\begin{aligned} G(y_n, y_n, y_m, t) &\geq G(y_n, y_n, y_{n+1}, t) + G(y_{n+1}, y_{n+1}, y_{n+2}, t) \\ &\quad + \cdots + G(y_{m-1}, y_{m-1}, y_m, t) \\ &\geq \phi^n(G(y_0, y_1, y_2, t)) + \phi^{n+1}(G(y_0, y_1, y_2, t)) \\ &\quad + \cdots + \phi^{m-1}(G(y_0, y_1, y_2, t)) \\ &\to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Since $\phi^n(t) \to 1$ as $n \to \infty$ for all $t > 0$. Hence $\{y_n\}$ is G-Cauchy. Suppose $f(x)$ is G-complete. Then there exist $p_0, t_0 \in X$ such that $y_{3n+2} \to p_0 = At_0$. Since $\{y_n\}$ is G-Cauchy, it follows that $y_{3n} \to p_0$ and $y_{3n+1} \to p_0$ as $n \to \infty$, and $G(ft, gx_{3n+1}, hx_{3n+2}, t)$

$$\int_0^{G(fp_0, p_0, p_0, t)} \varphi(t)dt \geq \int_0^{\phi(G(fp_0, p_0, p_0, t))} \varphi(t)dt.$$

Since $\varphi(t)$ is a Lebesgue integrable function we have

$$G(fp_0, p_0, p_0, t) \geq \phi(G(fp_0, p_0, p_0, t)).$$

Since, ϕ is non-decreasing, $fp_0 = p_0$. Therefore $fp_0 = Ap_0 = p_0$. Since the pair (f, A) is weakly compatible, we have $Ap_0 = fp_0$. Since $p_0 = fp_0 \in A(X)$. Similarly, we can show that (g, B) and (h, C) are weakly compatible. It follows from Theorem 3.1 that the mappings f, g, h, A, B and C have a unique common fixed point in X. □

Remark 4.2. If we take $\varphi(t) = 1$ in equation (4.1), we obtain the result of Theorem 3.1.

Example 4.3. Consider the set $X = [0, 1]$ with the G-fuzzy metric space $G(x, y, z, t) = 1 + |x - y| + |y - z| + |z - x|$ for all $x, y, z \in X$. Then $(X, G, *)$ is a complete G-fuzzy metric space. Define the mapping $f, g, h, A, B, C : X \to X$ by

$$fx = \frac{x}{2}, gx = \frac{x}{3}, hx = \frac{2x}{3},$$
$$Ax = x, Bx = x, Cx = x.$$

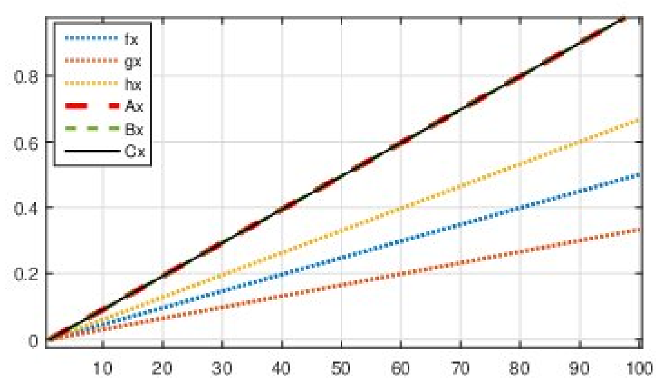

FIGURE 2. Graph of mappings f, g, h, A, B, and C of Example 4.3.

Let $\varphi(t) = t$ and $\phi(t) = \frac{t}{2}$. Then all hypothesis of Theorem 4.1 are satisfied and '0' is the unique common fixed point of mappings f, g, h, A, B and C.

Conclusions: From our investigations, we conclude that, six weakly compatible mappings have common fixed point in G-fuzzy metric spaces. The paper concerns our sustained efforts for the materialization of G-fuzzy metric spaces and their properties. Our investigations and results obtained were supported by the suitable examples and an application which provides a new path for researchers in the concerned field.

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