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EXISTENCE AND STABILITY RESULTS OF GENERALIZED FRACTIONAL INTEGRODIFFERENTIAL EQUATIONS

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Abstract. This paper gives sufficient conditions to ensure the existence and stability of solutions for generalized nonlinear fractional integrodifferential equations of order α $(1 < \alpha < 2)$. The main theorem asserts the stability results in a weighted Banach space, employing the Krasnoselskii's fixed point technique and the existence of at least one mild solution satisfying the asymptotic stability condition. Two examples are provided to illustrate the theory.

1. Introduction

During the last decade, with the advent of fractional differential equations as a frontier area of research, different concepts of fractional derivatives and

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integrals are being introduced and used by various researchers, for example, Riemann-Lioville, Caputo, Grunwald-Letnikov, Riesz and Hadamard derivatives and integrals [18, 21]. It is known that fractional derivatives essentially capture the non-local nature of the dynamics. However, while dealing with increasingly complex systems, different types of non-locality arise and researchers try to fit in by generalizing the existing fractional derivatives. Almeida et al. [3] introduced a generalized fractional integral and derivative that interpolates the Caputo and Caputo-Hadamard fractional derivatives. Here the kernel is generalized using a parameter ρ that helps in capturing a variety of non-local phenomena and some direct applications found in the literature are image encryption and quantum mechanics related to chaos problems arising in fractional dynamical systems. For more properties of this derivative, we refer the readers to [2, 3, 13].

A variety of literature can be found in discussing the existence, uniqueness and qualitative properties of the fractional integrodifferential equations, for example, see [1, 5, 9, 14, 15]. Stability of these systems plays a crucial part in stabilizability and controllability problems. There are many notions of stability and various methods to study stability problems. We choose the fixed point method to establish stability results since it only requires conditions of averaging nature which is advantageous over other methods using pointwise conditions. Only fewer works are reported using this method, for example, see [4, 16] and references therein. Burton and Zhang [10] generalized Schaefer's and Kranoselskii's fixed point theorems suitable to study fractional differential equations (FDEs). Ge and Kou [12] adopted these fixed point theorems to study the stability of FDEs and Makhlouf et al. [7] followed a similar method to study the existence and stability of generalized FDEs. On the other hand, very recently several researchers investigated the existence of solutions of generalized FDEs and some of the works discussed various notions of stability on generalized fractional differential equations [7, 8, 17, 22, 23, 25].

Motivated by the above works, we study the existence and stability of the following generalized nonlinear fractional integrodifferential equations

$$
\begin{aligned}
{}^{C}D^{\alpha,\rho}_{t_0^+}x(t) &= f(t,x(t),\int_{t_0}^{t} h(t,s,x(s))ds), \quad t \geq t_0, && (1.1)\\
x(t_0) &= x_0, \quad x'(t_0) = x_1, && (1.2)
\end{aligned}
$$

where $1 < \alpha < 2$, $0 < \rho \neq 1$, $x_0, x_1 \in \mathbb{R}$, $I := [t_0, \infty)$ and the functions $f : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and $h : I \times I \times \mathbb{R} \to \mathbb{R}$ are continuous with $f(t, 0, 0) = 0$. The nonlinear function f of this type with integral kernel h occur in mathematical problems concerning electromagnetic waves in dielectric media [24], ruin probabilities in financial risk theory [11], among many more applications found in applied physics.

First the initial value problem (1.1)-(1.2) is converted into an equivalent integral equation which is a sum of two operators. One is a contraction and the other one is compact and hence Krasnoselskii's fixed point theorem is applied. Existence and stability results are established in a weighted Banach space.

The rest of the paper is organized as follows. In Section 2, basic notations, results, lemmas and theorems are stated. The weighted Banach space and modified compactness criterion are given in this section. In Section 3, the main results are presented. Examples are provided in section 4 to validate the theory.

2. Preliminaries

Let us recall some basic definitions from fractional calculus needed for our work. We adopt the definitions and other properties of generalized fractional integrals and derivatives from [3] and [7].

Definition 2.1. (Generalized fractional integral)
Let $f \in L^1[a,b]$. Then the generalized fractional integral of the function f is defined as

$$I_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1} f(s)}{(t^\rho - s^\rho)^{1-\alpha}} \mathrm{d}s,$$

where $a > 0$ and $\alpha, \rho > 0$.

Definition 2.2. (Generalized fractional derivative)
Let $f \in AC^n[a,b]$. Then the generalized fractional derivative of the function f is defined as

$${}^C D_{a^+}^{\alpha,\rho} f(t) = \frac{\rho^{\alpha-n+1}}{\Gamma(n-\alpha)} \int_a^t \frac{s^{(\rho-1)(1-n)}}{(t^\rho - s^\rho)^{\alpha-n+1}} f^{(n)}(s) \mathrm{d}s,$$

where $a > 0$, $\alpha, \rho > 0$ and n is the smallest integer greater than α.

The relationship between these two fractional operators is the following:

$${}^C D_{a+}^{\alpha,\rho} f(t) = I_{a+}^{1-\alpha,\rho} \left(t^{1-\rho} \frac{\mathrm{d}}{\mathrm{d}t} f \right)(t).$$

Remark 2.3. ([6])

(i) (Semigroup property) For any continuous function f, the semigroup property for generalized fractional integral operators holds.

$$I_{a^+}^{\alpha,\rho} I_{a^+}^{\beta,\rho} f(t) = I_{a^+}^{\alpha+\beta,\rho} f(t)$$

for $\alpha, \beta > 0$ and $\rho > 0$.

(ii) If f is a constant, then the generalized fractional derivative
$$ {}^C D^{\alpha,\rho}_{a^+} f(t) = 0. $$

Lemma 2.4. ([7]) *For $1 < \alpha < 2$, the following relationship holds*
$$ {}^C D^{\alpha,\rho}_{a^+} f(t) = {}^C D^{\alpha-1,\rho}_{a^+} \left(\int_a^t s^{1-\rho} f^{(2)}(s) \mathrm{d}s \right). $$

Lemma 2.5. ([7]) *Let $u \in C(I, \mathbb{R})$, $1 < \alpha < 2$ and $0 < \rho \neq 1$. Then x is a solution of the initial value problem:*
$$ {}^C D^{\alpha,\rho}_{t_0^+} x(t) = u(t), \quad t \geq t_0, \tag{2.1} $$
$$ x(t_0) = x_0, \quad x'(t_0) = x_1, \tag{2.2} $$
if and only if
$$ \begin{aligned} x(t) = {} & x_0 \left[\left(\frac{t}{t_0} \right)^{1-\rho} - \frac{(1-\rho) t^{1-\rho}}{t_0^\rho} \int_{t_0}^t \frac{1}{s^{2-2\rho}} \mathrm{d}s \right] \\ & + x_1 (t t_0)^{1-\rho} \int_{t_0}^t \frac{1}{s^{2-2\rho}} \mathrm{d}s \\ & + \rho(1-\rho) t^{1-\rho} \int_{t_0}^t \int_\tau^t \frac{1}{s^{2-2\rho}} \frac{x(\tau)}{\tau^{1+\rho}} \mathrm{d}s \, \mathrm{d}\tau \\ & + \frac{\rho^{2-\alpha} t^{1-\rho}}{\Gamma(\alpha-1)} \int_{t_0}^t \int_\tau^t \frac{1}{s^{2-2\rho}} \frac{\tau^{\rho-1}}{(s^\rho - \tau^\rho)^{2-\alpha}} u(\tau) \mathrm{d}s \, \mathrm{d}\tau. \end{aligned} \tag{2.3} $$

Using the above Lemma 2.5, we define the solution to our initial value problem (1.1)-(1.2) as follows. For simplicity we take $z_x(t) := \int_{t_0}^t h(t, s, x(s)) \mathrm{d}s$.

Definition 2.6. The function x is defined as a mild solution of the initial value problem (1.1)-(1.2) if it satisfies
$$ \begin{aligned} x(t) = {} & x_0 \left[\left(\frac{t}{t_0} \right)^{1-\rho} - \frac{(1-\rho) t^{1-\rho}}{t_0^\rho} \int_{t_0}^t \frac{1}{s^{2-2\rho}} \mathrm{d}s \right] \\ & + x_1 (t t_0)^{1-\rho} \int_{t_0}^t \frac{1}{s^{2-2\rho}} \mathrm{d}s \\ & + \rho(1-\rho) t^{1-\rho} \int_{t_0}^t \int_\tau^t \frac{1}{s^{2-2\rho}} \frac{x(\tau)}{\tau^{1+\rho}} \mathrm{d}s \, \mathrm{d}\tau \\ & + \frac{\rho^{2-\alpha} t^{1-\rho}}{\Gamma(\alpha-1)} \int_{t_0}^t \int_\tau^t \frac{1}{s^{2-2\rho}} \frac{\tau^{\rho-1}}{(s^\rho - \tau^\rho)^{2-\alpha}} f(\tau, x(\tau), z_x(\tau)) \mathrm{d}s \mathrm{d}\tau. \end{aligned} \tag{2.4} $$

Definition 2.7. The generalized FDE (1.1) subject to the initial conditions (1.2) is said to be stable in a Banach space $\mathbb{E}$, if for every $\epsilon > 0$, there exists a $\delta > 0$ (δ depends only on ϵ) such that $|x_0| + |x_1| < \delta$ implies that there exists a mild solution $x(t)$ defined on I which satisfies $\|x\| \leq \epsilon$.

Let us introduce the following weighted Banach space and establish our stability results in it. Let $0 < \rho \neq 1$, $1 < \alpha < 2$ and $g : I \to \mathbb{R}^+$ be the weight function such that $g(t) \geq t^{\alpha\rho^2+3}$ and $g(\frac{t}{s})g(s) \leq g(t)$ for all $t, s \geq t_0$. Let us define the weighted space as

$$\mathbb{E} := \left\{ x \in C(I, \mathbb{R}) : \sup_{t \geq t_0} \frac{|x(t)|}{g(t)} < \infty \right\}.$$

Observe that $\mathbb{E}$ is a Banach space equipped with the norm $\|x\| = \sup\limits_{t \geq t_0} \dfrac{|x(t)|}{g(t)}$. For further details on this Banach space, refer [19]. For any $\epsilon > 0$, let

$$\mathcal{F}(\epsilon) = \{x \in \mathbb{E} : \|x\| \leq \epsilon\}.$$

Lemma 2.8. ([7]) *Let $0 < \rho \neq 1$ and $1 < \alpha < 2$. Then for all $t \geq t_0$, there exist $M_1 := M_1(t_0, \alpha, \rho) > 0$, $0 < M_2 := M_2(\rho) < 1$, $M_3 := M_3(\alpha, \rho) > 0$ such that*

$$\left| \frac{(\rho - 1)t^{1-\rho}}{g(t)} \int_{t_0}^{t} \frac{1}{s^{2-2\rho}} \mathrm{d}s \right| \leq M_1, \tag{2.5}$$

$$\left| \frac{(1 - \rho)\rho t^{1-\rho}}{g(t)} \int_{t_0}^{t} \int_{\tau}^{t} \frac{1}{s^{2-2\rho}} \frac{x(\tau)}{\tau^{1+\rho}} \mathrm{d}s \ \mathrm{d}\tau \right| \leq M_2\|x\|, \tag{2.6}$$

$$\frac{k(t, \tau)}{g(\frac{t}{\tau})} \leq M_3 \frac{\tau^{\alpha\rho^2}}{t^{\alpha\rho^2-\alpha\rho+1}}, \tag{2.7}$$

where

$$k(t, \tau) = \begin{cases} \dfrac{\rho^{2-\alpha}}{\Gamma(\alpha - 1)} \left(\dfrac{t}{\tau}\right)^{1-\rho} \displaystyle\int_{\tau}^{t} \frac{1}{s^{2-2\rho}} \frac{1}{(s^\rho - \tau^\rho)^{2-\alpha}} \mathrm{d}s, & t - \tau > 0, \\ 0, & t - \tau \leq 0. \end{cases}$$

Lemma 2.9. ([20]) *Let X be a nonempty closed convex subset of a Banach space $(Y, \|.\|)$. Suppose that A and B map X into Y such that*

(1) *$Ax + By \in X$ for all $x, y \in X$,*
(2) *A is continuous and AX is contained in a compact set of Y,*
(3) *B is a contraction with contraction constant $c < 1$.*

Then there is an $x \in X$ with $Ax + Bx = x$.

Lemma 2.10. ([19]) *Let $\mathcal{F}$ be a subset of the Banach space $\mathbb{E}$. Then $\mathcal{F}$ is relatively compact in $\mathbb{E}$ if the following conditions are satisfied:*

(1) $\{x(t)/g(t) : x(t) \in \mathcal{F}\}$ *is uniformly bounded,*
(2) $\{x(t)/g(t) : x(t) \in \mathcal{F}\}$ *is equicontinuous on any compact interval of* $\mathbb{R}^+$,
(3) $\{x(t)/g(t) : x(t) \in \mathcal{F}\}$ *is equiconvergent at infinity, that is, for any given* $\epsilon > 0$, *there exists a* $T > t_0$ *such that for all* $x \in \mathcal{F}(\epsilon)$ *and* $t_1, t_2 > T$, $|x(t_2)/g(t_2) - x(t_1)/g(t_1)| < \epsilon$ *holds.*

3. Main results

Before proving the main results, let us introduce the following hypotheses:

(H1) There exist constants $\eta_1, \eta_2 > 0$ and a continuous function $\psi : \mathbb{R}^+ \times (0, \eta_1] \times (0, \eta_2] \to \mathbb{R}^+$ such that

$$\frac{|f(t, ug(t), vg(t))|}{g(t)} \le \psi(t, |u|, |v|) \tag{3.1}$$

for all $t \ge t_0$, $0 < |u| \le \eta_1$, $0 < |v| \le \eta_2$.

(H2) There exists an $L^1([t_0, \infty))$ integrable function $m : [t_0, \infty) \times [t_0, \infty) \to \mathbb{R}$ such that

$$|h(t, s, x(s))| \le m(t, s)|x(s)| \tag{3.2}$$

and

$$\sup_{t \ge t_0} \int_{t_0}^{t} m(t, s) \mathrm{d}s < \infty. \tag{3.3}$$

(H3) There exists a constant $\beta_1 > 0$ such that

$$\sup_{t \ge t_0} \int_{t_0}^{t} \frac{k(t, \tau)}{g(t/\tau)} \frac{\psi(\tau, r_1, r_2)}{r} \mathrm{d}\tau \le \beta_1 < 1 - M_2 \tag{3.4}$$

holds for every $0 < r_1 \le \eta_1$ and $0 < r_2 \le \eta_2$, where $r = \min(r_1, r_2)$, M_2 is as defined in (2.6), $\psi(t, r_1, r_2)$ is nondecreasing in r_1, r_2 for fixed t and $t^{\alpha\rho^2}\psi(t, r_1, r_2) \in L^1([t_0, \infty))$ in t for fixed r_1, r_2.

Theorem 3.1. *Let* $1 < \alpha < 2$ *and* $0 < \rho \ne 1$. *Suppose that hypotheses* $(H1) - (H3)$ *hold. Then*

(i) *the generalized nonlinear FDEs* (1.1)-(1.2) *is stable in the Banach space* $\mathbb{E}$;

(ii) *there exists at least one mild solution to the generalized nonlinear FDEs* (1.1)-(1.2) *such that*

$$\lim_{t\to\infty}\frac{x(t)}{g(t)}=0.$$

Proof. (*i*) Let $0<\epsilon<\eta_1,\eta_2$ and take $\delta<\dfrac{(1-M_2-\beta_1)\epsilon}{M_4+M_1(t_0^{-\rho}+t_0^{1-\rho})}$. Let us prove that if $|x_0|+|x_1|<\delta$ then there exists a mild solution $x(t)$ defined on $[t_0,+\infty)$ which satisfies $\|x\|\le\epsilon$.

Consider the nonempty closed convex subset

$$\mathcal{F}(\epsilon)=\{x:x\in\mathbb{E},\|x\|\le\epsilon\}\subset\mathbb{E},$$

which is also a Banach space.

We define two mappings A,B on $\mathcal{F}(\epsilon)$ as follows:

$$\begin{aligned}
Ax(t) &= \frac{\rho^{2-\alpha}t^{1-\rho}}{\Gamma(\alpha-1)}\int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{1}{s^{2-2\rho}}\frac{\tau^{\rho-1}}{(s^\rho-\tau^\rho)^{2-\alpha}}f(\tau,x(\tau),z_x(\tau))\mathrm{d}s\right]\mathrm{d}\tau\\
&= \int_{t_0}^{t}k(t,\tau)f(\tau,x(\tau),z_x(\tau))\mathrm{d}\tau, \qquad (3.5)\\
Bx(t) &= x_0\left(\left(\frac{t}{t_0}\right)^{1-\rho}+\frac{(\rho-1)t^{1-\rho}}{t_0^{\rho}}\int_{t_0}^{t}\frac{1}{s^{2-2\rho}}\mathrm{d}s\right)+x_1(tt_0)^{1-\rho}\\
&\quad\times\int_{t_0}^{t}\frac{1}{s^{2-2\rho}}\mathrm{d}s+(1-\rho)\rho t^{1-\rho}\int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{1}{s^{2-2\rho}}\frac{x(\tau)}{\tau^{1+\rho}}\mathrm{d}s\right]\mathrm{d}\tau. \qquad (3.6)
\end{aligned}$$

Obviously, for $x\in\mathcal{F}(\epsilon)$, both Ax and Bx are continuous functions on $[t_0,+\infty)$.

By Hypothesis (H2), we see that

$$|z_x(t)|\le\int_{t_0}^{t}|h(t,s,x(s))|\mathrm{d}s\le\int_{t_0}^{t}m(t,s)|x(s)|\mathrm{d}s\le c|x(t)|,$$

where $c:=\sup\limits_{t\ge t_0}\displaystyle\int_{t_0}^{t}m(t,s)\mathrm{d}s<\infty$. Thus we have

$$\|z_x\|\le c\|x\|. \qquad (3.7)$$

Since $\psi(t, r_1, r_2)$ is nondecreasing in r_1, r_2 for fixed t and by (3.1)-(3.4) and (3.7), for any $t \geq t_0$ and $x \in \mathcal{F}(\epsilon)$, we have

$$\begin{aligned}
\frac{|Ax(t)|}{g(t)} &= \frac{1}{g(t)}\left|\int_{t_0}^{t} k(t,\tau) f(\tau, x(\tau), z_x(\tau))\mathrm{d}\tau\right| \\
&\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g(t/\tau)} \frac{|f(\tau, x(\tau), z_x(\tau))|}{g(\tau)}\mathrm{d}\tau \\
&\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g(t/\tau)} \psi\left(\tau, \frac{|x(\tau)|}{g(\tau)}, \frac{|z_x(\tau)|}{g(\tau)}\right)\mathrm{d}\tau \\
&\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g(t/\tau)} \psi\left(\tau, \epsilon, c\epsilon\right)\mathrm{d}\tau \\
&\leq \beta_1 \min(\epsilon, c\epsilon) \leq \beta_1 \epsilon.
\end{aligned}$$

Now,

$$\frac{|Ax(t)|}{g(t)} \leq \beta_1 \epsilon < \infty. \tag{3.8}$$

On the other hand, there exists $M_4 = M_4(t_0, \alpha, \rho)$ such that

$$\frac{(t/t_0)^{1-\rho}}{g(t)} \leq \frac{t^{1-\rho}}{t_0^{1-\rho} t^{\alpha\rho^2+3}} \leq \frac{1}{t_0^{1-\rho} t^{\alpha\rho^2+\rho+2}} \leq \frac{1}{t_0^{\alpha\rho^2+3}} := M_4. \tag{3.9}$$

By (2.5), (2.6) and (3.9), we see that

$$\begin{aligned}
\frac{|Bx(t)|}{g(t)} &\leq \frac{|x_0|}{g(t)}\left|\left(\frac{t}{t_0}\right)^{1-\rho} + \frac{(\rho-1)t^{1-\rho}}{t_0^{\rho}}\int_{t_0}^{t}\frac{1}{s^{2-2\rho}}\mathrm{d}s\right| + \frac{|x_1|}{g(t)}(tt_0)^{1-\rho} \\
&\quad \times \int_{t_0}^{t}\frac{1}{s^{2-2\rho}}\mathrm{d}s + \frac{(1-\rho)\rho t^{1-\rho}}{g(t)}\int_{t_0}^{t}\left[\int_{\tau}^{t}\frac{1}{s^{2-2\rho}}\frac{|x(\tau)|}{\tau^{1+\rho}}\mathrm{d}s\right]\mathrm{d}\tau \\
&\leq |x_0|\left(M_4 + \frac{1}{t_0^{\rho}}M_1\right) + |x_1| t_0^{1-\rho} M_1 + M_2\epsilon < \infty.
\end{aligned} \tag{3.10}$$

Then $A\mathcal{F}(\epsilon) \subseteq \mathbb{E}$ and $B\mathcal{F}(\epsilon) \subseteq \mathbb{E}$.

Next, we shall use Lemma 2.9 to prove that there exists at least one fixed point to the operator $A + B$ in $\mathcal{F}(\epsilon)$. For clarity, let us divide the proof into three steps.

Step 1: We will prove that $Ax+By \in \mathcal{F}(\epsilon)$ for all $x, y \in \mathcal{F}(\epsilon)$. Let $x, y \in \mathcal{F}(\epsilon)$, from (3.8) and (3.10), we obtain,

$$\begin{aligned}
\left|\frac{Ax(t) + By(t)}{g(t)}\right| &\leq |x_0|(M_4 + t_0^{-\rho}M_1) + |x_1|t_0^{1-\rho}M_1 + M_2\epsilon + \beta_1\epsilon \\
&\leq (M_4 + M_1(t_0^{-\rho} + t_0^{1-\rho}))\delta + (M_2 + \beta_1)\epsilon \leq \epsilon,
\end{aligned}$$

which implies that $Ax + By \in \mathcal{F}(\epsilon)$ for all $x, y \in \mathcal{F}(\epsilon)$.

Step 2: We will prove that A is continuous and $A\mathcal{F}(\epsilon)$ is relatively compact in $\mathbb{E}$. First, we will show that A is continuous. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence such that $x_n \to x$ in $\mathcal{F}(\epsilon)$.

Using (3.1), we get

$$\begin{aligned}
&\tau^{\alpha\rho^2}\frac{|f(\tau,x_n(\tau),z_{x_n}(\tau))-f(\tau,x(\tau),z(\tau))|}{g(\tau)}\\
&\quad\leq \tau^{\alpha\rho^2}\frac{|f(\tau,x_n(\tau),z_{x_n}(\tau))|+|f(\tau,x(\tau),z(\tau))|}{g(\tau)}\\
&\quad\leq \tau^{\alpha\rho^2}\left(\psi\left(\tau,\frac{|x_n(\tau)|}{g(\tau)},\frac{|z_{x_n}(\tau)|}{g(\tau)}\right)+\psi\left(\tau,\frac{|x(\tau)|}{g(\tau)},\frac{|z(\tau)|}{g(\tau)}\right)\right)\\
&\quad\leq 2\tau^{\alpha\rho^2}\psi(\tau,\epsilon,c\epsilon)\in L^1([t_0,+\infty)).
\end{aligned}$$

It follows from (2.7) that for any $t\geq t_0$,

$$\begin{aligned}
&\frac{|Ax_n(t)-Ax(t)|}{g(t)}\\
&\quad= \frac{1}{g(t)}\left|\int_{t_0}^{t}k(t,\tau)[f(\tau,x_n(\tau),z_{x_n}(\tau))-f(\tau,x(\tau),z_x(\tau))]\mathrm{d}\tau\right|\\
&\quad\leq \int_{t_0}^{t}\frac{k(t,\tau)}{g(t/\tau)}\frac{|f(\tau,x_n(\tau),z_{x_n}(\tau))-f(\tau,x(\tau),z_x(\tau))|}{g(\tau)}\mathrm{d}\tau\\
&\quad\leq \frac{M_3}{t_0^{\alpha\rho^2-\alpha\rho+1}}\int_{t_0}^{t}\tau^{\alpha\rho^2}\frac{|f(\tau,x_n(\tau),z_{x_n}(\tau))-f(\tau,x(\tau),z_x(\tau))|}{g(\tau)}\mathrm{d}\tau\\
&\quad\leq \frac{M_3}{t_0^{\alpha\rho^2-\alpha\rho+1}}\int_{t_0}^{\infty}\tau^{\alpha\rho^2}\frac{|f(\tau,x_n(\tau),z_{x_n}(\tau))-f(\tau,x(\tau),z_x(\tau))|}{g(\tau)}\mathrm{d}\tau
\end{aligned}$$

and hence

$$\|Ax_n-Ax\|\leq \frac{M_3}{t_0^{\alpha\rho^2-\alpha\rho+1}}\int_{t_0}^{\infty}\tau^{\alpha\rho^2}\frac{|f(\tau,x_n(\tau),z_{x_n}(\tau))-f(\tau,x(\tau),z_x(\tau))|}{g(\tau)}\mathrm{d}\tau.$$

Note that

$$\begin{aligned}
|z_{x_n}(t)-z_x(t)| &= \left|\int_{t_0}^{t}h(t,s,x_n(s))\mathrm{d}s-\int_{t_0}^{t}h(t,s,x(s))\mathrm{d}s\right|\\
&\leq \int_{t_0}^{t}\left(|h(t,s,x_n(s))|-|h(t,s,x(s))|\right)\mathrm{d}s\\
&\leq \int_{t_0}^{t}m(t,s)\left(|x_n(s)|-|x(s)|\right)\mathrm{d}s\\
&\leq c\left(|x_n(t)|-|x(t)|\right)\leq c|x_n(t)-x(t)|.
\end{aligned}$$

Hence we see that $|z_{x_n}(\tau)-z_x(\tau)|\leq c|x_n(\tau)-x(\tau)|$.

Now we have for any $\tau \geq t_0$,

$$\left|\frac{x_n(\tau)-x(\tau)}{g(\tau)}\right| \leq \|x_n - x\|,$$

so $\lim_{n\to\infty}|x_n(\tau)-x(\tau)| = 0$ and similarly $\lim_{n\to\infty}|z_{x_n}(\tau)-z_x(\tau)| = 0$ for all $\tau \geq t_0$.

Since we know f is continuous in $[t_0,\infty)\times\mathbb{R}\times\mathbb{R}$, we have

$$\lim_{n\to\infty}\frac{|f(\tau,x_n(\tau),z_{x_n}(\tau)) - f(\tau,x(\tau),z_x(\tau))|}{g(\tau)} = 0 \text{ for all } \tau \geq t_0.$$

Then by dominated convergence theorem, it follows that $\|Ax_n - Ax\| \to 0$ as $n \to \infty$. Thus, A is continuous in $\mathcal{F}(\epsilon)$.

Secondly let us prove that $A\mathcal{F}(\epsilon)$ is relatively compact in $\mathbb{E}$. For all $t \geq t_0$, it follows from (2.7) that there exists a constant $M_3 = M_3(\alpha,\rho)$ such that

$$\frac{k(t,\tau)}{g(t/\tau)} \leq M_3\frac{\tau^{\alpha\rho^2}}{t^{\alpha\rho^2-\alpha\rho+1}}.$$

Moreover, for any $T \geq t_0$, the function $\dfrac{k(t,\tau)}{g(t/\tau)}$ is uniformly continuous on $\{(t,\tau): t_0 \leq \tau \leq t \leq T\}$.

We have for any $x \in \mathcal{F}(\epsilon)$, $t_1, t_2 \in [t_0,T]$ and $t_1 < t_2$,

$$\begin{aligned}
&\left|\frac{Ax(t_2)}{g(t_2)} - \frac{Ax(t_1)}{g(t_1)}\right| \\
&\qquad \leq \left|\int_{t_0}^{t_2}\frac{k(t_2,\tau)}{g(t_2)}f(\tau,x(\tau),z_x(\tau))\mathrm{d}\tau - \int_{t_0}^{t_1}\frac{k(t_1,\tau)}{g(t_1)}f(\tau,x(\tau),z_x(\tau))\mathrm{d}\tau\right| \\
&\qquad \leq \int_{t_0}^{t_1}\left|\frac{k(t_2,\tau)}{g(t_2)} - \frac{k(t_1,\tau)}{g(t_1)}\right||f(\tau,x(\tau),z_x(\tau))|\mathrm{d}\tau \\
&\qquad\quad + \int_{t_1}^{t_2}\left|\frac{k(t_2,\tau)}{g(t_2)}\right||f(\tau,x(\tau),z_x(\tau))|\mathrm{d}\tau \\
&\qquad \leq \int_{t_0}^{t_1}\left|\frac{k(t_2,\tau)g(\tau)}{g(t_2)} - \frac{k(t_1,\tau)g(\tau)}{g(t_1)}\right|\psi(\tau,\epsilon,c\epsilon)\mathrm{d}\tau \\
&\qquad\quad + \int_{t_1}^{t_2}\left|\frac{k(t_2,\tau)g(\tau)}{g(t_2)}\right|\psi(\tau,\epsilon,c\epsilon)\mathrm{d}\tau \\
&\qquad \leq \int_{t_0}^{t_1}\left|\frac{k(t_2,\tau)g(\tau)}{g(t_2)} - \frac{k(t_1,\tau)g(\tau)}{g(t_1)}\right|\psi(\tau,\epsilon,c\epsilon)\mathrm{d}\tau \\
&\qquad\quad + \frac{M_3}{t_0^{\alpha\rho^2-\alpha\rho+1}}\int_{t_1}^{t_2}\tau^{\alpha\rho^2}\psi(\tau,\epsilon,c\epsilon)\mathrm{d}\tau,
\end{aligned}$$

as $t_2 \to t_1$ which tells us that $\left\{\frac{Ax(t)}{g(t)}, x(t) \in \mathcal{F}(\epsilon)\right\}$ is equicontinuous on any compact interval of $[t_0, \infty)$. To prove that $A\mathcal{F}(\epsilon)$ is relatively compact in $\mathbb{E}$, by Lemma 2.9, it still remains to show that $\left\{\frac{Ax(t)}{g(t)}, x(t) \in \mathcal{F}(\epsilon)\right\}$ is equiconvergent at infinity. By (2.7), we have

$$\begin{aligned}\frac{|Ax(t)|}{g(t)} &\leq \int_{t_0}^{t} \frac{k(t,\tau)}{g(t)} |f(\tau, x(\tau), z_x(\tau)| \mathrm{d}\tau \leq \int_{t_0}^{t} \frac{k(t,\tau)}{g(t/\tau)} \psi(\tau, \epsilon, c\epsilon) \mathrm{d}\tau \\ &\leq \frac{M_3}{t^{\alpha\rho^2 - \alpha\rho + 1}} \int_{t_0}^{t} \tau^{\alpha\rho^2} \psi(\tau, \epsilon, c\epsilon) \mathrm{d}\tau \\ &\to 0 \text{ as } t \to \infty. \end{aligned} \tag{3.11}$$

Thus, we can assert that $\left\{\frac{Ax(t)}{g(t)}, x(t) \in \mathcal{F}(\epsilon)\right\}$ is equiconvergent at infinity.

Step 3: Let us show that $B : \mathcal{F}(\epsilon) \to \mathbb{E}$ is a contraction mapping. By (2.6), for any $x_1, x_2 \in \mathcal{F}(\epsilon)$, we get that

$$\begin{aligned}\sup_{t \geq t_0} \left| \frac{Bx_1(t)}{g(t)} - \frac{Bx_2(t)}{g(t)} \right| &\leq \frac{(1-\rho)\rho t^{1-\rho}}{g(t)} \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{1}{s^{2-2\rho}} \frac{|x_1(\tau) - x_2(\tau)|}{\tau^{1+\rho}} \mathrm{d}s \right] \mathrm{d}\tau \\ &\leq M_2 \|x_1 - x_2\|.\end{aligned}$$

Through Lemma 2.8, we can assert that there exists atleast one fixed point of the operator $A + B$ in $\mathcal{F}(\epsilon)$, which is a mild solution of (1.1)-(1.2). Hence the generalized nonlinear FDEs (1.1)-(1.2) is stable in the Banach space $\mathbb{E}$.

(ii) For any $0 < \epsilon < \eta_1, \eta_2$, let us define

$$\mathcal{F}^*(\epsilon) = \left\{ x \in \mathcal{F}(\epsilon), \lim_{t \to \infty} \frac{x(t)}{g(t)} = 0 \right\}.$$

We show that $Ax + By \in \mathcal{F}^*(\epsilon)$ for any $x, y \in \mathcal{F}^*(\epsilon)$, that is, we need to show that $\dfrac{Ax(t) + By(t)}{g(t)} \to 0$ as $t \to \infty$. In fact,

$$\begin{aligned}\frac{Ax(t) + Bx(t)}{g(t)} = \frac{1}{g(t)} \Bigg[&\int_{t_0}^{t} k(t,\tau) f(\tau, x(\tau), z_x(\tau)) \mathrm{d}\tau \\ &+ x_0 \left(\left(\frac{t}{t_0}\right)^{1-\rho} + \frac{(\rho - 1) t^{1-\rho}}{t_0^{\rho}} \int_{t_0}^{t} \frac{1}{s^{2-2\rho}} \mathrm{d}s \right) \\ &+ x_1 \left((t t_0)^{1-\rho} \int_{t_0}^{t} \frac{1}{s^{2-2\rho}} \mathrm{d}s \right) \\ &+ (1-\rho)\rho t^{1-\rho} \int_{t_0}^{t} \left[\int_{\tau}^{t} \frac{1}{s^{2-2\rho}} \frac{x(\tau)}{\tau^{1+\rho}} \mathrm{d}s \right] \mathrm{d}\tau \Bigg].\end{aligned}$$

Since we know $g(t) \geq t^{\alpha\rho^2+3}$, we obtain

$$\frac{t^{1-\rho}}{g(t)} \leq \frac{t^{1-\rho}}{t^{\alpha\rho^2+3}} = \frac{1}{t^{\alpha\rho^2+\rho+2}} \to 0 \text{ as } t \to \infty \tag{3.12}$$

and

$$\frac{t^{1-\rho}}{g(t)} \int_{t_0}^{t} \frac{1}{s^{2-2\rho}} \mathrm{d}s \to 0 \text{ as } t \to \infty. \tag{3.13}$$

Moreover, we see that

$$\frac{t^{1-\rho}}{g(t)} \int_{t_0}^{t} \left(\int_{\tau}^{t} \frac{1}{s^{2-2\rho}} \frac{x(\tau)}{\tau^{1+\rho}} \mathrm{d}s \right) \mathrm{d}\tau \leq t^{1-\rho} \int_{t_0}^{t} \left(\int_{\tau}^{t} \frac{1}{g(t/\tau)} \frac{1}{s^{2-2\rho}} \frac{1}{\tau^{1+\rho}} \frac{x(\tau)}{g(\tau)} \mathrm{d}s \right) \mathrm{d}\tau,$$

and let us consider the following two cases.

Case 1: For $\rho \in (0,1)$, we have

$$t^{1-\rho} \int_{t_0}^{t} \left(\int_{\tau}^{t} \frac{1}{g(t/\tau)} \frac{1}{s^{2-2\rho}} \frac{1}{\tau^{1+\rho}} \frac{x(\tau)}{g(\tau)} \mathrm{d}s \right) \mathrm{d}\tau \ \leq \ \frac{1}{t^{\alpha\rho^2+\rho+1}} \int_{t_0}^{t} \tau^{\alpha\rho^2+\rho} \frac{x(\tau)}{g(\tau)} \mathrm{d}\tau.$$

Since $\lim_{\tau\to\infty} \frac{x(\tau)}{g(\tau)} = 0$, there exists $T_1 > t_0$ such that for all $t \geq T_1$,

$$\frac{|x(t)|}{g(t)} < (\alpha\rho^2 + \rho + 1)\frac{\epsilon}{2}.$$

Moreover there exists a $T_2 > T_1$, such that for all $t \geq T_2$,

$$\frac{1}{t} \int_{t_0}^{T_1} \frac{|x(\tau)|}{g(\tau)} \mathrm{d}\tau < \frac{\epsilon}{2}.$$

Then we have for $t \geq T_2$,

$$\begin{aligned}
&\frac{1}{t^{\alpha\rho^2+\rho+1}} \int_{t_0}^{t} \tau^{\alpha\rho^2+\rho} \frac{|x(\tau)|}{g(\tau)} \mathrm{d}\tau \\
&\quad = \frac{1}{t^{\alpha\rho^2+\rho+1}} \int_{t_0}^{T_1} \tau^{\alpha\rho^2+\rho} \frac{|x(\tau)|}{g(\tau)} \mathrm{d}\tau + \frac{1}{t^{\alpha\rho^2+\rho+1}} \int_{T_1}^{t} \tau^{\alpha\rho^2+\rho} \frac{|x(\tau)|}{g(\tau)} \mathrm{d}\tau \\
&\quad \leq \frac{1}{t} \int_{t_0}^{T_1} \frac{|x(\tau)|}{g(\tau)} \mathrm{d}\tau + \frac{1}{t^{\alpha\rho^2+\rho+1}} \frac{|x(t)|}{g(t)} \int_{T_1}^{t} \tau^{\alpha\rho^2+\rho} \mathrm{d}\tau \\
&\quad < \frac{\epsilon}{2} + (\alpha\rho^2 + \rho + 1)\frac{\epsilon}{2} \left(\frac{1}{\alpha\rho^2 + \rho + 1} \right) \\
&\quad < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Case 2: For $\rho \in (1,\infty)$, we have

$$t^{1-\rho} \int_{t_0}^{t} \left(\int_{\tau}^{t} \frac{1}{g(t/\tau)} \frac{1}{s^{2-2\rho}} \frac{1}{\tau^{1+\rho}} \frac{x(\tau)}{g(\tau)} \mathrm{d}s \right) \mathrm{d}\tau \ \leq \ \frac{1}{t^{\alpha\rho^2-\rho+3}} \int_{t_0}^{t} \tau^{\alpha\rho^2-\rho+2} \frac{x(\tau)}{g(\tau)} \mathrm{d}\tau.$$

Since $\lim_{\tau\to\infty}\dfrac{x(\tau)}{g(\tau)}=0$, there exists $T_1>t_0$ such that for all $t\geq T_1$,

$$\frac{|x(t)|}{g(t)}<(\alpha\rho^2-\rho+3)\frac{\epsilon}{2}.$$

Moreover there exists a $T_2>T_1$, such that for all $t\geq T_2$,

$$\frac{1}{t}\int_{t_0}^{T_1}\frac{|x(\tau)|}{g(\tau)}\mathrm{d}\tau<\frac{\epsilon}{2}.$$

Then we have for $t\geq T_2$,

$$\begin{aligned}
&\frac{1}{t^{\alpha\rho^2-\rho+3}}\int_{t_0}^{t}\tau^{\alpha\rho^2-\rho+2}\frac{|x(\tau)|}{g(\tau)}\mathrm{d}\tau\\
&\quad= \frac{1}{t^{\alpha\rho^2-\rho+3}}\int_{t_0}^{T_1}\tau^{\alpha\rho^2-\rho+2}\frac{|x(\tau)|}{g(\tau)}\mathrm{d}\tau+\frac{1}{t^{\alpha\rho^2-\rho+3}}\int_{T_1}^{t}\tau^{\alpha\rho^2-\rho+3}\frac{|x(\tau)|}{g(\tau)}\mathrm{d}\tau\\
&\quad\leq \frac{1}{t}\int_{t_0}^{T_1}\frac{|x(\tau)|}{g(\tau)}\mathrm{d}\tau+\frac{1}{t^{\alpha\rho^2-\rho+3}}\frac{|x(t)|}{g(t)}\int_{T_1}^{t}\tau^{\alpha\rho^2-\rho+2}\mathrm{d}\tau\\
&\quad< \frac{\epsilon}{2}+(\alpha\rho^2-\rho+3)\frac{\epsilon}{2}\left(\frac{1}{\alpha\rho^2-\rho+3}\right)\\
&\quad< \frac{\epsilon}{2}+\frac{\epsilon}{2}=\epsilon.
\end{aligned}$$

Based on the previous calculations, we get

$$\frac{t^{1-\rho}}{g(t)}\int_{t_0}^{t}\left(\int_{\tau}^{t}\frac{1}{s^{2-2\rho}}\frac{x(\tau)}{\tau^{1+\rho}}\mathrm{d}s\right)\mathrm{d}\tau\to 0 \text{ as } t\to\infty. \tag{3.14}$$

Also by (3.11) we obtain

$$\frac{Ax(t)}{g(t)}=\int_{t_0}^{t}k(t,\tau)f(\tau,x(\tau),z_x(\tau))\mathrm{d}\tau\to 0 \text{ as } t\to\infty. \tag{3.15}$$

Then by using (3.12)-(3.15), we assert that

$$\frac{Ax(t)+By(t)}{g(t)}\to 0 \text{ as } t\to\infty. \tag{3.16}$$

Hence we proved that there exists at least one mild solution to (1.1)-(1.2) such that

$$\lim_{t\to\infty}\frac{x(t)}{g(t)}=0.$$

□

4. Examples

Example 4.1. Consider the nonlinear fractional initial value problem:

$$ {}^{C}D^{\frac{3}{2},\frac{1}{2}}x(t) = \frac{1}{10}\left[\frac{x}{t^6}+\frac{x^3}{1+t^{12}}+\frac{1}{1+t^8}\left(\int_1^t \frac{x(s)}{e^{t+s}}\mathrm{d}s\right)^2\right], \tag{4.1}$$

$$ x(1) = x_0,\ x'(1)=x_1. \tag{4.2}$$

Let us choose $g(t)=t^4$ and let

$$\mathbb{E}=\left\{x\in C([1,\infty),\mathbb{R}) : \sup_{t\geq 1}\frac{|x(t)|}{t^4}<\infty\right\}.$$

Set

$$f(t,x,z_x)=\frac{1}{10}\left(\frac{x}{t^6}+\frac{x^3}{1+t^{12}}+\frac{z_x^2}{1+t^8}\right)$$

where $z_x := \int_1^t \frac{x(s)}{e^{t+s}}\mathrm{d}s$. Then we have $|z_x(t)|\leq \frac{1}{e^2}|x(t)|$ and observe that

$$\frac{|f(t,ug(t),z_u g(t))|}{g(t)}=\frac{1}{10}\frac{\left|\frac{ut^4}{t^6}+\frac{(ut^4)^3}{1+t^{12}}+\frac{(z_u t^4)^2}{1+t^8}\right|}{t^4}\leq\frac{1}{10}\left(\frac{|u|}{t^6}+\frac{|u|^3}{t^4}+\frac{|z_u|^2}{t^4}\right).$$

So let us take

$$\psi(t,r_1,r_2)=\frac{1}{10}\left(\frac{r_1}{t^6}+\frac{r_1^3}{t^4}+\frac{r_2^2}{t^4}\right)$$

and further

$$\int_1^\infty t^{3/8}\psi(t,r_1,r_2)\mathrm{d}t\leq\frac{1}{10}\left(\frac{r_1}{4}+\frac{r_1^3}{2}+\frac{r_2^2}{2}\right)$$

for fixed r_1, r_2 and hence $t^{3/8}\psi(t,r_1,r_2)\in L^1([1,\infty))$.

We see that

$$\sup_{t\geq 1}\int_1^t \frac{k(t,\tau)}{g(t/\tau)}\frac{\psi(t,r_1,r_2)}{r}\mathrm{d}\tau\leq\frac{2\sqrt{2}}{\sqrt{\pi}}\left(\frac{r_1}{5r}+\frac{r_1^3}{3r}+\frac{r_2^2}{3r}\right),$$

where $r:=\min(r_1,r_2)$.

Then there exist $\eta_1,\eta_2>0$ such that

$$\sup_{t\geq 1}\int_1^t \frac{k(t,\tau)}{g(t/\tau)}\frac{\psi(t,r_1,r_2)}{r}\mathrm{d}\tau\leq\frac{4}{5}<1-\frac{2}{15}=\frac{13}{15},$$

for all $0<r_1\leq\eta_1$ and $0<r_2\leq\eta_2$. Hence by Theorem 3.1, we conclude that the nonlinear fractional integrodifferential equation (4.1)-(4.2) is stable in the Banach space $\mathbb{E}$ and there exists at least one mild solution which satisfies $\lim_{t\to\infty}\frac{x(t)}{g(t)}=0$.

Example 4.2. Consider the nonlinear fractional initial value problem:

$$ {}^{C}D^{\frac{4}{3},3}x(t) = \frac{1}{t^5}\arctan\left(t^3 + x^{1/3} + \left(\int_1^t \frac{\sin(x(s))}{(t-s)^2}\mathrm{d}s\right)^{1/3}\right), \quad (4.3) $$

$$ x(1) = x_0,\ x'(1) = x_1. \quad (4.4) $$

Let us choose $g(t) = t^{15}$ and let

$$ \mathbb{E} = \left\{x \in C([1,\infty),\mathbb{R}) : \sup_{t\geq 1} \frac{|x(t)|}{t^{15}} < \infty\right\}. $$

Take

$$ f(t,x,z_x) = \frac{1}{t^5}\arctan(t^3 + x^{1/3} + z_x^{1/3}) $$

where $z_x := \displaystyle\int_1^t \frac{\sin(x(s))}{(t-s)^2}\mathrm{d}s$. Note that $|z_x(t)| \leq \dfrac{1}{4}|x(t)|$ and

$$ \begin{aligned} \frac{|f(t,ug(t),z_u g(t))|}{g(t)} &= \frac{1}{t^5}\frac{\left|\arctan\left(t^3 + (ut^{15})^{1/3} + (z_u t^{15})^{1/3}\right)\right|}{t^{15}} \\ &\leq \frac{\left(\frac{t^3}{t^5} + \frac{|u^{1/3}|t^5}{t^5} + \frac{|z_u|^{1/3}t^5}{t^5}\right)}{t^{15}} \leq \frac{1}{t^{17}} + \frac{|u|^{1/3}}{t^{15}} + \frac{|z_u|^{1/3}}{t^{15}}. \end{aligned} $$

So let us take

$$ \psi(t,r_1,r_2) = \left(\frac{1}{t^{17}} + \frac{r_1^{1/3}}{t^{15}} + \frac{r_2 1/3}{t^{15}}\right) $$

and further

$$ \int_1^\infty t^{12}\psi(t,r_1,r_2)\mathrm{d}t \leq \frac{1}{4} + \frac{r_1^{1/3}}{2} + \frac{r_2^{1/3}}{2} $$

for fixed r_1, r_2 and hence $t^{12}\psi(t,r_1,r_2) \in L^1([1,\infty))$.

We see that

$$ \sup_{t\geq 1}\int_1^t \frac{k(t,\tau)}{g(t/\tau)}\frac{\psi(t,r_1,r_2)}{r}\mathrm{d}\tau \leq \frac{1}{3^{1/3}\Gamma(4/3)}\left(\frac{1}{16r} + \frac{r_1^{1/3}}{14r} + \frac{r_2^{1/3}}{14r}\right), $$

where $r := \min(r_1, r_2)$. Then there exist $\eta_1, \eta_2 > 0$ such that

$$ \sup_{t\geq 1}\int_1^t \frac{k(t,\tau)}{g(t/\tau)}\frac{\psi(t,r_1,r_2)}{r}\mathrm{d}\tau \leq \frac{1}{5} < 1 - \frac{1}{2} = \frac{1}{2}, $$

for all $0 < r_1 \leq \eta_1$ and $0 < r_2 \leq \eta_2$. Hence by Theorem 3.1, we conclude that the nonlinear fractional integrodifferential equation (4.3)-(4.4) is stable in the Banach space $\mathbb{E}$ and there exists at least one mild solution which satisfies $\displaystyle\lim_{t\to\infty}\frac{x(t)}{g(t)} = 0$.

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