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CONVERGENCE THEOREMS FOR SP-ITERATION SCHEME IN A ORDERED HYPERBOLIC METRIC SPACE

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Abstract. In this paper, we study the $\Delta-$convergence and strong convergence of SP-iteration scheme involving a nonexpansive mapping in partially ordered hyperbolic metric spaces. Also, we give an example to support our main result and compare SP-iteration scheme with the Mann iteration and Ishikawa iteration scheme. Thus, we generalize many previous results.

1. Introduction

Fixed point theory is an active and vital branch of mathematics. It has many applications in mathematics and outside mathematics. Banach contraction principle was the first fundamental fixed point theorem given by Banach in 1922. He proved that every contraction mapping on a complete metric space has a unique fixed point. Moreover, the Picard iteration generated by a contraction mapping converges to that unique fixed point. After Banach, many researchers generalized Banach contraction principle in several ways. Some researchers weaken the contraction while some researchers extended the metric structure. In course of study of periodic differential equations, Nieto

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and Rodriguez-Löpez [17] provided order theoretic version of Banach contraction principle. Almost similar results were also given by Ran and Reuring [20] while solving matrix equation.

In 1965, Browder [5, 6], Göhde [8] and Kirk [12] generalized the underlying mapping from contraction mapping to nonexpansive mapping and gave existence fixed point theorem in Banach spaces.

Existence theorem guarantee the existence of fixed point but not gives method how to calculate the fixed point. Convergence theorems use an iteration process and give method to calculate the fixed point. Many iteration processes like Mann [16], Ishikawa [11], Agarwal [1], Noor [18], SP-iterations [19], Garodia and Uddin [9, 10] play central role of attraction in approximating fixed point. Recently, Ali and Uddin [4] studied convergence behaviour of SP-iteration for generalized nonexpansive mapping.

In this paper, we prove some strong and $\Delta-$convergence theorems to approximate the fixed point of nonexpansive mapping using SP-iteration scheme in partially ordered hyperbolic metric spaces.

2. Preliminaries

Let's start this section with some basic definitions and lemmas which will be useful in our main section.

Let $(X, d, \preceq)$ be a metric space with partial order. Let $T : K \to K$ be a map preserve monotone order if $x \preceq y$ then $T(x) \preceq T(y)$, for any $x, y \in X$. A point $x \in X$ is said to be a fixed point of T whenever $Tx = x$. The set of fixed point of T is denoted by $F(T)$. T is said to be monotone nonexpansive if T is monotone and $d(Tx, Ty) \leq d(x, y)$ for any $x, y \in X$ such that x and y are comparable.

Due to natural linearity and convex structure in Banach spaces. Fixed point theory is richly developed in Banach spaces. It is natural to ask, can we generalize the results of linear spaces to nonlinear spaces? To keep in mind this situation, in 2005, Kohlenbach [13] introduced the following definition of a hyperbolic metric space.

Definition 2.1. If (X, d) is a metric space, then (X, d, W) will be the hyperbolic metric space if the function $W : X \times X \times [0, 1] \to X$ satisfies

(i) $d(z, W(x, y, \alpha)) \leq (1 - \alpha)d(z, x) + \alpha d(z, y)$,
(ii) $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|d(x, y)$,
(iii) $W(x, y, \alpha) = W(x, y, 1 - \alpha)$,
(iv) $d(W(x, y, \alpha), W(z, w, \alpha)) \leq (1 - \alpha)d(x, z) + \alpha d(y, w)$,
for all $x, y, z, w \in X$ and $\alpha, \beta \in [0, 1]$.

Linear example of hyperbolic metric space is normed linear space. Hadamard manifolds, the Hilbert open unit ball equipped with the hyperbolic metric and the CAT(0) spaces are nonlinear example of hyperbolic metric space. For some recent finding in hyperbolic metric space one can see [2], [3]. Hyperbolic metric space endowed with partial order is called partially ordered hyperbolic metric space.

In this paper, we will assume that order interval are convex and closed. Recall that an order interval is any of the subset $[a, \rightarrow) = \{x \in X; a \preceq x\}$ and $(\leftarrow, b] = \{x \in X; x \preceq b\}$, for any $a, b \in X$.

The following notion of uniformly convex hyperbolic metric space is given by Shimizu and Takahashi [21].

Definition 2.2. Let (X, d) be a hyperbolic metric space. We say that X is uniformly convex if for any $a \in X$, for every $r > 0$, and for each $\varepsilon > 0$

$$\delta(r,\varepsilon) = \inf\left\{1-\frac{1}{r}d\left(W(x,y,\frac{1}{2}),a\right); d(x,a) \le r, d(y,a) \le r, d(x,y) \ge r\varepsilon\right\} > 0.$$

In 1976, Lim [14] introduced the concept of $\Delta-$convergence in metric spaces.

Definition 2.3. Let X be a complete hyperbolic metric space and $\{x_n\}$ be a bounded sequence in X. Then the type function $r(., \{x_n\}) : X \to [0, \infty)$ is defined by

$$r(x, \{x_n\}) = \limsup_{n\to\infty} d(x, x_n).$$

The asymptotic radius $r(\{x_n\})$ is given by

$$r(\{x_n\}) = \inf\{r(x, \{x_n\}) : x \in X\}$$

and the asymptotic center $A(\{x_n\})$ of $\{x_n\}$ is defined as

$$A(\{x_n\}) = \{x \in X : r(x, \{x_n\}) = r(\{x_n\})\}.$$

Definition 2.4. A bounded sequence $\{x_n\}$ in X is said to be $\Delta-$convergent to $x \in X$, if x is the unique asymptotic center of every subsequence $\{u_n\}$ of $\{x_n\}$. We write $x_n \rightharpoonup x$ ($\{x_n\}$ $\Delta-$converges to x).

Lemma 2.5. ([15]) *Let K be a nonempty, closed and convex subset of a complete uniformly convex hyperbolic metric space X. Then, every bounded sequence $\{x_n\}$ in X has a unique asymptotic center with respect to K.*

Lemma 2.6. ([7]) *Let X be a uniformly convex hyperbolic space. Let $R \in [0, \infty)$ be such that*

$$\limsup_{n\to\infty} d(x_n, a) \le R, \ \limsup_{n\to\infty} d(y_n, a) \le R$$

and

$$\lim_{n\to\infty} d(a, \alpha_n x_n \oplus (1 - \alpha_n) y_n) = R,$$

where $\alpha_n \in [a, b]$, with $0 < a \le b < 1$. Then, we have $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Now, we present SP-iteration [19] in the setting of hyperbolic space as follows:

Let $T : K \to K$ be any mapping defined on convex subset

$$\begin{cases} x_1 \in K, \\ z_n = W(x_n, Tx_n, \gamma_n), \\ y_n = W(z_n, Tz_n, \beta_n), \\ x_{n+1} = W(y_n, Ty_n, \alpha_n), \end{cases} \tag{2.1}$$

for each $n \geq 1$, where α_n, β_n, $\gamma_n \in (0, 1)$.

Our aim is to prove some strong and $\Delta-$convergence for iteration scheme (2.1) in the hyperbolic metric spaces.

3. Δ-convergence and strong convergence theorem

Let us start with following useful lemma:

Lemma 3.1. *Let K be a nonempty, closed and convex subset of partially ordered hyperbolic metric space X and $T : K \to K$ be a monotone nonexpansive mapping. If $\{x_n\}$ is a sequence defined by* (2.1) *such that $x_1 \preceq T(x_1)$ (or $T(x_1) \preceq x_1$). Then,*

(i) *$x_n \preceq T(x_n)$ (or $T(x_n) \preceq x_n$);*

(ii) *$x_n \preceq p$ (or $p \preceq x_n$), provided $\{x_n\}$ $\Delta-$converge to a point $p \in K$ for all $n \in \boldsymbol{N}$.*

Proof. We shall use induction to prove this result. Since by assumption, we have $x_1 \preceq T(x_1)$. Thus, the result is true for $n = 1$. Now suppose that the result is true for n, that is,

$$x_n \preceq Tx_n.$$

By the convexity of order interval $[x_n, Tx_n]$ and by (2.1) we have

$$x_n \preceq z_n \preceq Tx_n.$$

By the monotonicity of T, we get

$$Tx_n \preceq Tz_n,$$

which implies

$$z_n \preceq Tz_n.$$

Now, by convexity of ordered interval $[z_n, Tz_n]$, we get

$$z_n \preceq y_n \preceq Tz_n.$$

Again using monotonicity of T, we get

$$Tz_n \preceq Ty_n,$$

this gives

$$y_n \preceq Ty_n.$$

Owing to convexity of ordered interval $[y_n, Ty_n]$, we have,

$$y_n \preceq x_{n+1} \preceq Ty_n.$$

As T is monotone, hence relation

$$Ty_n \preceq Tx_{n+1}$$

which yields

$$x_{n+1} \preceq Tx_{n+1}.$$

This proves (i). Next, suppose p is a $\Delta-$limit of $\{x_n\}$. As the sequence $\{x_n\}$ is monotone increasing and the order interval $[x_m, \rightarrow)$ is closed and convex. We claim that $p \in [x_m, \rightarrow)$ for a fixed $m \in N$. If $p \notin [x_m \rightarrow)$, then the asymptotic center of subsequence $\{x_r\}$ of $\{x_n\}$ defined by leaving first $m-1$ terms of sequence $\{x_n\}$ will not equal to p which is a contradiction as p is a $\Delta-$limit of $\{x_n\}$. This completes the proof. □

Lemma 3.2. *Let K be a nonempty, closed and convex subset of a complete uniformly convex partially ordered hyperbolic metric space X and let $T : K \rightarrow K$ be a monotone nonexpansive mapping with $F(T) \neq \emptyset$. Assume that the sequence $\{x_n\}$ is defined by* (2.1) *and $x_1 \preceq Tx_1$ (or $Tx_1 \preceq x_1$). If $p \preceq x_1$ (or $x_1 \preceq p$) for some $p \in F(T)$, then*

$$\lim_{n\to\infty} d(x_n, Tx_n) = 0.$$

Proof. It follows from Lemma 3.1, $p \preceq x_n$ and hence by monotonicity of T

$$Tp \preceq Tx_n.$$

For any $p \in F(T)$

$$\begin{aligned} d(x_{n+1}, p) &= d(W(y_n, Ty_n, \alpha_n), p) \\ &\leq \alpha_n d(y_n, p) + (1-\alpha_n) d(Ty_n, p) \\ &\leq d(y_n, p) \\ &\leq d(W(z_n, Tz_n, \beta_n), p) \\ &\leq \beta_n d(z_n, p) + (1-\beta_n) d(Tz_n, p) \\ &\leq d(z_n, p) \\ &\leq d(W(x_n, Tx_n, \gamma_n), p) \\ &\leq \gamma_n d(x_n, p) + (1-\gamma_n) d(Tx_n, p) \\ &\leq d(x_n, p), \end{aligned}$$

which implies that $\{d(x_n, p)\}$ is a decreasing and bounded below sequence of real numbers. Hence, $\lim_{n\to\infty} d(x_n, p)$ exists. Let $\lim_{n\to\infty} d(x_n, p) = c$. As

$d(Tx_n, p) \leq d(x_n, p)$, therefore $\limsup_{n\to\infty} d(Tx_n, p) \leq c$. Again as from above $d(x_{n+1}, p) \leq d(z_n, p) \leq d(x_n, p)$. This gives $\lim_{n\to\infty} d(z_n, p) = c$. Also,

$$\lim_{n\to\infty} d(z_n, p) = \lim_{n\to\infty} d(\gamma_n x_n \oplus (1-\gamma_n)Tx_n, p) = c.$$

In view of Lemma 2.6, $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. □

Now, we will prove our first main convergence theorem.

Theorem 3.3. *Let K be a nonempty, closed and convex subset of a uniformly convex partially ordered hyperbolic metric space X and let $T : K \to K$ be a monotone nonexpansive mapping with $F(T) \neq \emptyset$. If sequence $\{x_n\}$ is defined by (2.1) with $x_1 \preceq Tx_1$ (or $Tx_1 \preceq x_1$). If $p \preceq x_1$ (or $x_1 \preceq p$) for some $p \in F(T)$, then $\{x_n\}$ $\Delta-$ converges to a fixed point x^* of T.*

Proof. From Lemma 3.2, $\lim_{n\to\infty} d(x_n, p)$ exists for each $p \in F(T)$, so that the sequence is bounded and $\lim_{n\to\infty} d(x_n, Tx_n) = 0$. By Lemma 2.4, $\{x_n\}$ has unique asymptotic center. Let $A(x_n) = x^*$ and $\{u_n\}$ is any subsequence of $\{x_n\}$ such that $A(u_n) = u$. Now claim $x^* = u$.
On contrary, suppose that $x^* \neq u$. Now,

$$\begin{aligned}\limsup_{n\to\infty} d(u_n, u) &< \limsup_{n\to\infty} d(u_n, x^*)\\ &\leq \limsup_{n\to\infty} d(x_n, x^*)\\ &< \limsup_{n\to\infty} d(x_n, u)\\ &= \limsup_{n\to\infty} d(u_n, u),\end{aligned}$$

which is a contradiction and hence $\Delta - \lim_{n\to\infty} x_n = x^*$.
Now, we claim that $x^* \in F(T)$. As $x^* \preceq x_n \Rightarrow Tx^* \preceq Tx_n$

$$\begin{aligned}\limsup_{n\to\infty} d(Tx^*, x_n) &\leq \limsup_{n\to\infty} d(Tx^*, Tx_n) + \limsup_{n\to\infty} d(Tx_n, x_n)\\ &\leq \limsup_{n\to\infty} d(x^*, x_n).\end{aligned}$$

Since $\Delta - \lim_{n\to\infty} x_n = x^*$,

$$\limsup_{n\to\infty} d(x^*, x_n) < \limsup_{n\to\infty} d(Tx^*, x_n).$$

Thus, we have $Tx^* = x^*$. □

Next, we prove strong convergence with some strong conditions.

Theorem 3.4. *Let K be a nonempty, closed and convex subset of a complete uniformly convex partially ordered hyperbolic metric space X and $T : K \to K$ be a monotone nonexpansive mapping with $F(T) \neq \emptyset$. If sequence $\{x_n\}$ is defined by (2.1) with $x_1 \preceq Tx_1$ (or $Tx_1 \preceq x_1$). If $p \preceq x_1$ (or $x_1 \preceq p$) for some $p \in F(T)$, then $\{x_n\}$ converges to a fixed point of T if and only if*

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

Proof. It is easy to see that if $\{x_n\}$ converges to a point $x \in F(T)$, then

$$\liminf_{n\to\infty} d(x_n, F(T)) = 0.$$

For converse part, suppose that $\liminf_{n\to\infty} d(x_n, F(T)) = 0$. From Lemma 3.2, for any $p \in F(T)$, we have

$$\inf_{p\in F(T)} d(x_{n+1}, p) \leq \inf_{p\in F(T)} d(x_n, p)$$

so that

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)).$$

Hence, $\lim_{n\to\infty} d(x_n, F(T))$ exists. Since $\liminf_{n\to\infty} d(x_n, F(T)) = 0$, we get

$$\lim_{n\to\infty} d(x_n, F(T)) = 0.$$

Let $\varepsilon > 0$ be arbitrary chosen that there exist $n_0 \in \mathbf{N}$ such that

$$d(x_n, F(T)) < \frac{\varepsilon}{4}$$

for all $n \geq n_0$. In particular

$$\inf\{d(x_{n_0}, p) : p \in F(T)\} < \frac{\varepsilon}{4}$$

or

$$d(x_{n_0}, p) < \frac{\varepsilon}{2}.$$

Now, for $m, n \geq n_0$ we have

$$\begin{aligned} d(x_{m+n,x_n}) &\leq d(x_{n+m}, p) + d(p, x_n) \\ &< 2\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\{x_n\}$ is a Cauchy sequence in K hence converges in K. Set $\lim_{n\to\infty} x_n = q$, then

$$\begin{aligned} d(q, Tq) &\leq d(q, x_n) + d(x_n, Tx_n) + d(Tx_n, Tq) \\ &\leq d(q, x_n) + d(x_n, Tx_n) + d(x_n, q) \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Thus, $Tq = q$. □

4. Example

In this section, we present an example of nonexpansive mapping and approximate the fixed point by using SP-iteration, Mann-iteration and Ishikawa-iteration schemes.

Example 4.1. Let $X = \mathbb{R}$ and define $T : [0,1] \to [0,1]$ as

$$Tx = \frac{x}{2}.$$

Then we can see that 0 is the fixed point of T. Take $\alpha_n = \frac{1}{n}$, $\beta_n = \frac{1}{n^2}$ and $\gamma_n = \frac{n}{n+1}$ in SP-iteration, Mann-iteration and Ishikawa-iteration schemes.

Now we present the iterative values of different methods in following table:

iteration	SP			Mann			Ishikawa		
number	$x_1 = .25$	$x_1 = .5$	$x_1 = 1$	$x_1 = .25$	$x_1 = .5$	$x_1 = 1$	$x_1 = .25$	$x_1 = .5$	$x_1 = 1$
1	.25	.5	1	.25	.5	1	.25	.5	1
5	.0071	.0142	.0284	.0781	.1563	.3125	.0382	.0764	.1529
10	.0000	.0000	.0000	.0049	0098	.0195	.0003	.0005	.0010
20	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000	.0000

Table 1. Convergence table of SP, Mann and Ishikawa iteration schemes for initial values .25, .5 and 1.

From Table 1 and Figure 1 we can see that the SP-iteration scheme converge to fixed point of T faster than Mann and Ishikawa iteration schemes.

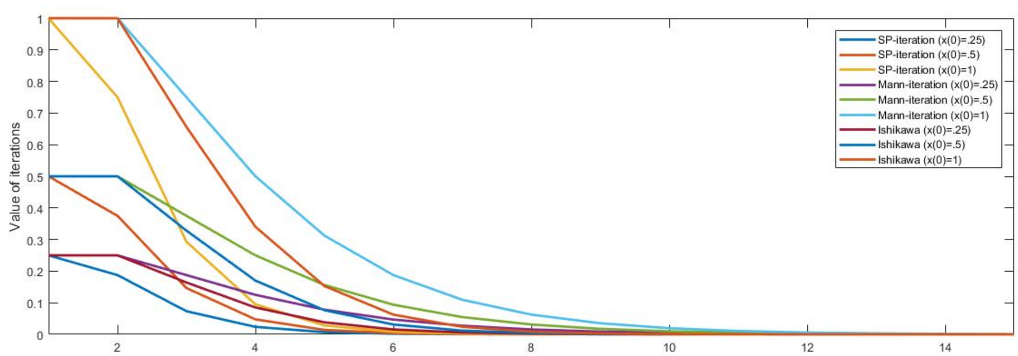

Figure 1

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