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A RANDOM GENERALIZED NONLINEAR IMPLICIT VARIATIONAL-LIKE INCLUSION WITH RANDOM FUZZY MAPPINGS

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Abstract. In this paper, we introduce and study a new class of random generalized nonlinear implicit variational-like inclusion with random fuzzy mappings in a real separable Hilbert space and give its fixed point formulation. Using the fixed point formulation and the proximal mapping technique for strongly maximal monotone mapping, we suggest and analyze a random iterative scheme for finding the approximate solution of this class of inclusion. Further, we prove the existence of solution and discuss the convergence analysis of iterative scheme of this class of inclusion. Our results in this paper improve and generalize several known results in the literature.

1. Introduction

In 1994, Hassouni and Moudafi [9] used the resolvent operator technique for maximal monotone mapping to study a class of mixed type variational inequalities with single-valued mappings which was called variational inclusions

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and developed a perturbed algorithm for finding approximate solutions of the mixed variational inequalities. Since then, many researchers have obtained some important extensions and generalizations of the results given in [9] in different directions (see [1,7,12,16,17]).

In 1965, Zadeh [23] gave the notion of fuzzy sets as an extension of crisp sets, the usual two-valued sets in ordinary set theory, by enlarging the truth value set to the real unit interval $[0, 1]$. Ordinary fuzzy sets are characterized by, and mostly identified with, mapping called 'membership function' into $[0, 1]$. The basic operations and properties of fuzzy sets or fuzzy relations are defined by equations or inequalities between the membership functions. Heilpern [10] initiated the study of fuzzy mappings and established a fuzzy analogue of the Nadler's fixed point theorem [18] for multivalued mappings. Random variational inequality theory is an important part of random functional analysis. These topics have attracted many scholars and experts due to the extensive applications of the random problems (see [4,5,8-11,19,23]).

In 1989, Chang and Zhu [4] initiated the study of a class of variational inequalities with fuzzy mappings. In recent past, various classes of random variational inequalities have been introduced and studied by Chang [2], Chang and Huang [3], Ding [5], Huang [13], Noor [19] and Park and Jeong [21].

Recently, Huang [14] developed an iterative scheme for a class of random variational inclusions with random fuzzy mappings and discuss its convergence criteria in real separable Hilbert space. Very recently, Ahmad and Bazan [1], Ding and Park [6], Kazmi [15], Lan *et al.* [17], Onjai-uea and Kumam [20] and Park and Jeong [22] introduced and studied various generalized classes of random variational inclusions involving random fuzzy mappings in real separable spaces.

Inspired and motivated by recent work in this field, in this paper, we introduce and study a new class of random generalized nonlinear implicit variational-like inclusion with random fuzzy mappings in a real separable Hilbert space and give its fixed point formulation. Using this fixed point formulation and the proximal mapping technique for strongly maximal monotone mapping, we suggest and analyze a random iterative scheme for finding the approximate solution of this class of inclusion. Further, we prove the existence of solution and discuss the convergence analysis of iterative scheme of this class of inclusion. Our results in this paper improve and generalize some known corresponding results (see [1,3,6-9,12-17,20-22]).

2. Preliminaries

Let H be a real separable Hilbert space whose norm and inner product are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$ respectively, let (Ω,Σ) be a measurable space, where Ω is a set and Σ is σ-algebra of subsets of Ω, $\mathcal{B}(H)$ be the class of Borel σ-fields in H, $CB(H)$ denotes the collection of all nonempty, bounded and closed subsets of H and 2^H denotes the power set of H. The Hausdorff metric $\tilde{H}(\cdot,\cdot)$ on $CB(H)$ is defined by

$$\tilde{\mathcal{H}}(A,B) \;=\; \max\Big\{\sup_{x\in A} d(x,B),\; \sup_{y\in B} d(y,A)\Big\},\;\; A,B\in CB(H). \tag{2.1}$$

First, we recall and define the following concepts and known results.

Definition 2.1. ([20]) A mapping $x:\Omega\to H$ is said to be *measurable* if, for any $B\in\mathcal{B}(H)$, $\{t\in\Omega : x(t)\in B\}\in\Sigma$.

Definition 2.2. ([20]) A mapping $f:\Omega\times H\to H$ is said to be *random* if, for any $x\in H$, $f(t,x)=y(t)$ is measurable. A random mapping f is said to be *continuous* (resp. *linear, bounded*) if for any $t\in\Omega$, the mapping $f(t,\cdot):H\to H$ is continuous (resp. *linear, bounded*).

Similarly, we can define a random mapping $a:\Omega\times H\times H\to H$. We will write $f_t(x)=f(t,x(t))$ and $a_t(x,y)=a(t,x(t),y(t))$ for all $t\in\Omega$ and $x(t),y(t)\in H$.

Remark 2.3. ([20]) It is well known that a measurable mapping is necessarily a random mapping.

Definition 2.4. ([20]) A multivalued mapping $G:\Omega\to 2^H$ is said to be *measurable* if, for any $B\in\mathcal{B}(H)$, $G^{-1}(B)=\{t\in\Omega : G(t)\cap B\neq\emptyset\}\in\Sigma$.

Definition 2.5. ([20]) A mapping $u:\Omega\to H$ is said to be *measurable selection* of a multivalued measurable mapping $G:\Omega\to 2^H$ if u is a measurable and for any $t\in\Omega$, $u(t)\in G(t)$.

Definition 2.6. ([20]) A multivalued mapping $F:\Omega\times H\to 2^H$ is said to be *random* if, for any $x\in H$, $F(\cdot,x)$ is measurable. A random multivalued mapping $F:\Omega\times H\to CB(H)$ is said to be $\tilde{\mathcal{H}}$-*continuous* if, for any $t\in\Omega$, $F(t,\cdot)$ is continuous in the Hausdorff metric.

Definition 2.7. ([20]) Let $\mathcal{F}(H)$ be the family of all fuzzy sets over H. A mapping $F:H\to\mathcal{F}(H)$ is called a *fuzzy mapping* over H.

Definition 2.8. ([20]) If F is a fuzzy mapping over H, then $F(x)$ (denoted by F_x in the sequel) is said to be a *fuzzy set* on H, and $F_x(y)$ is the membership function of y in F_x.

Definition 2.9. ([20]) Let $A \in \mathcal{F}(H)$, $\alpha \in [0, 1]$. Then the set

$$(A)_\alpha = \{x \in H : A(x) \geq \alpha\} \tag{2.2}$$

is called a α-*cut* set of fuzzy set A.

Definition 2.10. ([20]) A fuzzy mapping $F : \Omega \to \mathcal{F}(H)$ is called *measurable* if, for any $\alpha \in (0, 1]$, $(F(\cdot))_\alpha : \Omega \to 2^H$ is a measurable multivalued mapping.

Definition 2.11. ([20]) A fuzzy mapping $F : \Omega \times H \to \mathcal{F}(H)$ is said to be a *random fuzzy mapping* if, for any $x \in H$, $F(\cdot, x) : \Omega \to \mathcal{F}(H)$ is a measurable fuzzy mapping.

Remark 2.12. ([20]) We note that the random fuzzy mappings include multivalued mappings, random multivalued mappings and fuzzy mappings as the special cases.

Definition 2.13. ([16]) Let $\eta : H \times H \to H$ be a single-valued mapping. Then a multi-valued mapping $M : H \to 2^H$ is said to be

(i) η-*monotone*, if

$$\langle u - v, \eta(x, y)\rangle \;\geq\; 0,\ \forall\, x, y \in H,\, u \in M(x),\ v \in M(y);$$

(ii) *strictly* η-*monotone*, if

$$\langle u - v, \eta(x, y)\rangle \;>\; 0,\ \forall\, x, y \in H,\, u \in M(x),\ v \in M(y)$$

and equality holds if and only if $x = y$;

(iii) ν-*strongly* η-*monotone*, if there exists a constant $\nu > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \;\geq\; \nu\|x - y\|^2,\ \forall\, x, y \in H,\, u \in M(x),\ v \in M(y);$$

(iv) *maximal*-η-*monotone*, if M is η-monotone and $(I + \rho M)(H) = H$ for any $\rho > 0$, where I stands for identity mapping.

Definition 2.14. ([7,16]) Let $\eta : H \times H \to H$ be a mapping. Then a mapping $P : H \to H$ is said to be

(i) η-*monotone*, if

$$\langle P(x) - P(y),\, \eta(x, y)\rangle \;\geq\; 0,\ \forall\, x, y \in H;$$

(ii) *strictly* η-*monotone*, if

$$\langle P(x) - P(y),\, \eta(x, y)\rangle \;>\; 0,\ \forall\, x, y \in H$$

and equality holds if and only if $x = y$;

(iii) *δ-strongly η-monotone*, if there exists a constant $\delta > 0$ such that

$$\langle P(x) - P(y),\, \eta(x,y)\rangle \;\geq\; \delta\|x-y\|^2,\; \forall\, x,y \in H.$$

Definition 2.15. ([16]) Let $\eta : H \times H \to H$ and $P : H \to H$ be mappings. A multi-valued mapping $M : H \to 2^H$ is said to be *γ-strongly maximal P-η-monotone*, if M is γ-strongly η-monotone and $(P+\rho M)H = H$ for any $\rho > 0$.

The following theorems give some properties of γ-strongly maximal P-η-monotone mappings.

Theorem 2.16. ([16]) *Let $\eta : H \times H \to H$ be a mapping and $P : H \to H$ be a strictly η-monotone mapping. Let $M : H \to 2^H$ be a γ-strongly maximal P-η-monotone multi-valued mapping, then*

(a) $\langle u-v,\, \eta(x,y)\rangle \;\geq\; 0,\; \forall\,(v,y) \in Graph(M)$ *implies* $(u,x) \in Graph(M)$, *where* $Graph(M) := \{(u,x) \in H \times H :\, u \in M(x)\}$*;*

(b) *the mapping* $(P+\rho M)^{-1}$ *is single-valued for all* $\rho > 0$.

By Theorem 2.16, we define strongly P-η-proximal mapping for a γ-strongly maximal η-monotone mapping M as follows:

$$R^M_{P,\eta}(z) \;=\; (P+\rho M)^{-1},\; \forall\, z \in H, \tag{2.3}$$

where $\rho > 0$ is a constant, $\eta : H \times H \to H$ is a mapping and $P : H \to H$ is a strictly η-monotone mapping.

Theorem 2.17. ([16]) *Let $P : H \to H$ be a δ-strongly η-monotone mapping and $\eta : H \times H \to H$ be a τ-Lipschitz continuous mapping. Let $M : H \to 2^H$ be a γ-strongly maximal η-monotone multivalued mapping. Then strongly P-η-proximal mapping $R^M_{P,\eta}$ is $\dfrac{\tau}{\delta+\rho\gamma}$-Lipschitz continuous, that is,*

$$\|R^M_{P,\eta}(x) - R^M_{P,\eta}(y)\| \;\leq\; \frac{\tau}{\delta+\rho\gamma}\,\|x-y\|,\; \forall\, x,y \in H. \tag{2.4}$$

3. Formulation of problem

Let $Q, R, S, Z : \Omega \times H \to \mathcal{F}(H)$ be random fuzzy mappings satisfying the following condition (**C**): there exist mappings $q, r, s, e : H \to (0,1]$ such that

$$(Q_{t,x})_{q(x)},\; (R_{t,x})_{r(x)},\; (S_{t,x})_{s(x)},\; (Z_{t,x})_{e(x)} \in CB(H), \tag{3.1}$$

for all $(t,x) \in \Omega{\times}H$.

By using the random fuzzy mappings Q, R, S and Z, we can define respectively the multi-valued mappings $\tilde{Q}, \tilde{R}, \tilde{S}, \tilde{Z} : \Omega \times H \to CB(H)$ by $\tilde{Q}(t,x) =$

$(Q_{t,x})_{q(x)}$, $\tilde{R}(t,x) = (R_{t,x})_{r(x)}$, $\tilde{S}(t,x) = (S_{t,x})_{s(x)}$, $\tilde{Z}(t,x) = (Z_{t,x})_{e(x)}$, for each $(t,x) \in \Omega \times H$.

In the sequel, $\tilde{Q}, \tilde{R}, \tilde{S}$ and $\tilde{Z}$ are called the random multi-valued mappings induced by the random fuzzy mappings Q, R, S and Z, respectively.

Let $P : H \to H$; $\eta : H \times H \to H$; $N : \Omega \times H \times H \times H \to H$ be single-valued mappings, and let $g, m : \Omega \times H \to H$ be random mappings such that $g \not\equiv 0$. Let $M : \Omega \times H \times H \to 2^H$ be a multi-valued random mapping such that for each $(t,x) \in \Omega \times H$, $M(t,\cdot,x)$ is strongly maximal P-η-monotone and

$$(g-m)(\Omega \times H) \cap \operatorname{domain} M(t,\cdot,x) \neq \emptyset,$$

where

$$(g-m)(t,x) = g(t,x) - m(t,x), \text{ for any } (t,x) \in \Omega \times H.$$

We consider the following random generalized nonlinear implicit variational-like inclusion problem involving random fuzzy mappings (RGNIVLIP): Find measurable mappings $x, u, v, w, z : \Omega \to H$ such that for all $t \in \Omega$, $x(t) \in H$, $Q_{t,x(t)}(u(t)) \geq q(x(t))$, $R_{t,x(t)}(v(t)) \geq r(x(t))$, $S_{t,x(t)}(w(t)) \geq s(x(t))$, $Z_{t,x(t)}(z(t)) \geq e(x(t))$ and

$$0 \in N(t, u(t), v(t), w(t)) + M(t, (g-m)(t, x(t)), z(t)). \tag{3.2}$$

For a suitable choice of the mappings $N, M, P, Q, R, S, Z, g, m, \eta, q, r, s, e$ and the space H, it is easy to check that RGNIVLIP (3.2) contains a number of known classes of random variational inclusions (inequalities) studied by many researchers as special cases (see [1,4-8,13-15,17-22]).

4. Random iterative scheme

First we recall the following useful lemmas.

Lemma 4.1. ([2]) *Let $M : \Omega \times H \to CB(H)$ be a $\tilde{\mathcal{H}}$-continuous random multivalued mapping. Then, for any measurable mapping $w : \Omega \to H$, the multi-valued mapping $M(\cdot, w(\cdot)) : \Omega \to CB(H)$ is measurable.*

Lemma 4.2. ([2]) *Let $M, V : \Omega \times H \to CB(H)$ be two measurable multi-valued mappings, $\epsilon > 0$ be a constant and $v : \Omega \to H$ be a measurable selection of M. Then there exists a measurable selection $w : \Omega \to H$ of V such that, for any $t \in \Omega$,*

$$\|v(t) - w(t)\| \leq (1+\epsilon)\, \tilde{\mathcal{H}}(M(t), V(t)).$$

Now, we give the fixed point formulation of RGNIVLIP (3.2).

Lemma 4.3. *The set of measurable mappings $x, u, v, w, z : \Omega \to H$ is a random solution of RGNIVLIP (3.2) if and only if, for all $t \in \Omega$ the random multivalued mapping $G : \Omega \times H \to 2^H$ defined by*

$$\begin{aligned} &G(t, x(t)) \\ &= \bigcup_{u(t)\in\tilde{Q}(t,x(t))} \bigcup_{v(t)\in\tilde{R}(t,x(t))} \bigcup_{w(t)\in\tilde{S}(t,x(t))} \bigcup_{z(t)\in\tilde{Z}(t,x(t))} \Big[x(t) - (g-m)(t, x(t)) \\ &+ R_{P,\eta}^{M(t,\cdot,z(t))}\Big(P \circ (g-m)(t, x(t)) - \rho(t) N(t, u(t), v(t), w(t))\Big)\Big], \ t \in \Omega, \quad (4.1) \end{aligned}$$

has a fixed point $x = x(t) \in H$, where $\rho : \Omega \to (0, \infty)$ is a measurable function; $P \circ (g - m)$ denotes P composition $(g - m)$; $R_{P,\eta}^{M(t,\cdot,z(t))} \equiv (P + \rho(t)\, M(t, \cdot, z(t)))^{-1}$.

Proof. RGNIVLIP (3.2) has a random solution (x, u, v, w, z) if and only if

$$0 \in N(t, u(t), v(t), w(t)) + M(t, (g - m)(t, x(t)), z(t)),$$

it implies that

$$\begin{aligned} &P \circ (g-m)(t, x(t)) - \rho(t) N(t, u(t), v(t), w(t)) \\ &\in (P + \rho(t) M(t, \cdot, z(t))(g - m)(t, x(t)). \end{aligned}$$

Since for each $(t, z(t)) \in \Omega \times H$, $M(t, \cdot, z(t))$ is strongly maximal P-η-monotone, by definition of strongly P-η-proximal mapping $R_{P,\eta}^{M(t,\cdot,z(t))}$ of $M(t, \cdot, z(t))$, preceding inclusion holds if and only if

$$(g-m)(t, x(t)) = R_{P,\eta}^{M(t,\cdot,z(t))}\Big[P \circ (g-m)(t, x(t)) - \rho(t) N(t, u(t), v(t), w(t))\Big],$$

that is, $x(t) \in G(t, x(t))$. This completes the proof. □

Now, based on Lemma 4.3, we give the following random iterative scheme to compute the approximate random solution of RGNIVLIP (3.2).

Iterative Scheme A: Let $Q, R, S, Z : \Omega \times H \to \mathcal{F}(H)$ be random fuzzy mappings satisfying the condition (**C**). Let $\tilde{Q}, \tilde{R}, \tilde{S}, \tilde{Z} : \Omega \times H \to CB(H)$ be $\tilde{\mathcal{H}}$-continuous random multi-valued mappings induced by Q, R, S, Z, respectively, and let $N : \Omega \times H \times H \times H \to H$ be a continuous random mapping, let $P : H \to H$, $\eta : H \times H \to H$ be single-valued mappings. Let $M : \Omega \times H \times H \to 2^H$ be a random multi-valued mapping such that for each $(t, z) \in \Omega \times H$, $M(t, \cdot, z)$ is γ-strongly maximal P-η-monotone with $(g - m)(\Omega \times H) \cap \text{domain}\, M(t, \cdot, z) \neq \emptyset$. For any given measurable mapping $x_0 : \Omega \to H$, the multi-valued mappings $\tilde{Q}(\cdot, x_0(\cdot)), \tilde{R}(\cdot, x_0(\cdot)), \tilde{S}(\cdot, x_0(\cdot)), \tilde{Z}(\cdot, x_0(\cdot)) : \Omega \to CB(H)$ are measurable by Lemma 4.1. Hence by Himmelberg [11], there exist measurable selections

$u_0 : \Omega \to H$ of $\tilde{Q}(\cdot, x_0(\cdot))$, $v_0 : \Omega \to H$ of $\tilde{R}(\cdot, x_0(\cdot))$, $w_0 : \Omega \to H$ of $\tilde{S}(\cdot, x_0(\cdot))$ and $z_0 : \Omega \to H$ of $\tilde{Z}(\cdot, x_0(\cdot))$. Let

$$x_1(t) = x_0(t) - (g-m)(t, x_0(t)) + R_{P,\eta}^{M(t,\cdot,z_0(t))}\Big[P \circ (g-m)(t, x_o(t)) - \rho(t) N(t, u_0(t), v_0(t), w_0(t))\Big].$$

Then, it is easy to observe that $x_1 : \Omega \to H$ is measurable. By Lemma 4.2, there exist measurable selections $u_1 : \Omega \to H$ of $\tilde{Q}(\cdot, x_1(\cdot))$, $v_1 : \Omega \to H$ of $\tilde{R}(\cdot, x_1(\cdot))$, $w_1 : \Omega \to H$ of $\tilde{S}(\cdot, x_1(\cdot))$ and $z_1 : \Omega \to H$ of $\tilde{Z}(\cdot, x_1(\cdot))$ such that for all $t \in \Omega$,

$$\|u_1(t) - u_0(t)\| \leq (1 + (1+0)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{Q}(t, x_1(t)),\, \tilde{Q}(t, x_0(t))),$$
$$\|v_1(t) - v_0(t)\| \leq (1 + (1+0)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{R}(t, x_1(t)),\, \tilde{R}(t, x_0(t))),$$
$$\|w_1(t) - w_0(t)\| \leq (1 + (1+0)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{S}(t, x_1(t)),\, \tilde{S}(t, x_0(t)))$$

and

$$\|z_1(t) - z_0(t)\| \leq (1 + (1+0)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{Z}(t, x_1(t)),\, \tilde{Z}(t, x_0(t))).$$

Let

$$x_2(t) = x_1(t) - (g-m)(t, x_1(t)) + R_{P,\eta}^{M(t,\cdot,z_1(t))}\Big[P \circ (g-m)(t, x_1(t)) - \rho(t) N(t, u_1(t), v_1(t), w_1(t))\Big].$$

Then $x_2 : \Omega \to H$ is measurable. Continuing the above process inductively, we can define the following random iterative sequences $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$, $\{w_n(t)\}$ and $\{z_n(t)\}$ as follows:

$$x_{n+1}(t) = x_n(t) - (g-m)(t, x_n(t)) + R_{P,\eta}^{M(t,\cdot,z_n(t))}\Big[P \circ (g-m)(t, x_n(t)) - \rho(t) N(t, u_n(t), v_n(t), w_n(t))\Big], \tag{4.2}$$

$u_{n+1}(t) \in \tilde{Q}(t, x_{n+1}(t))$ such that

$$\|u_{n+1}(t) - u_n(t)\| \leq (1 + (1+n)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{Q}(t, x_{n+1}(t)),\, \tilde{Q}(t, x_n(t))),$$

$v_{n+1}(t) \in \tilde{R}(t, x_{n+1}(t))$ such that

$$\|v_{n+1}(t) - v_n(t)\| \leq (1 + (1+n)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{R}(t, x_{n+1}(t)),\, \tilde{R}(t, x_n(t))),$$

$w_{n+1}(t) \in \tilde{S}(t, x_{n+1}(t))$ such that

$$\|w_{n+1}(t) - w_n(t)\| \leq (1 + (1+n)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{S}(t, x_{n+1}(t)),\, \tilde{\mathcal{S}}(t, x_n(t))),$$

$z_{n+1}(t) \in \tilde{Z}(t, x_{n+1}(t))$ such that

$$\|z_{n+1}(t) - z_n(t)\| \leq (1 + (1+n)^{-1})\, \tilde{\mathcal{H}}\,(\tilde{Z}(t, x_{n+1}(t)),\, \tilde{Z}(t, x_n(t))),$$

for any $t \in \Omega$, $n = 0, 1, 2, \ldots$ and $\rho : \Omega \to (0, \infty)$ is a measurable function.

5. Existence of solution and convergence analysis

First, we define the following concepts.

Definition 5.1. A random mapping $g : \Omega \times H \to H$ is said to be

(i) $s(t)$-*strongly monotone*, if there exists a measurable function $s : \Omega \to (0, \infty)$ such that

$$\langle g(t, x_1(t)) - g(t, x_2(t)),\, x_1(t) - x_2(t)\rangle \;\geq\; s(t)\|x_1(t) - x_2(t)\|^2;$$

(ii) $l_g(t)$-*Lipschitz continuous*, if there exists a measurable function $l_g : \Omega \to (0, \infty)$ such that

$$\|g(t, x_1(t)) - g(t, x_2(t))\| \;\leq\; l_g(t)\|x_1(t) - x_2(t)\|,$$

for all $x_1(t), x_2(t) \in H,\ t \in \Omega$.

Definition 5.2. A random multi-valued mapping $T : \Omega \times H \to CB(H)$ is said to $l_T(t)$-$\tilde{\mathcal{H}}$-*Lipschitz continuous*, if there exists a measurable function $l_T : \Omega \to (0, \infty)$ such that

$$\tilde{\mathcal{H}}(T(t, x_1(t)),\, T(t, x_2(t))) \;\leq\; l_T(t)\|x_1(t) - x_2(t)\|,$$

for all $x_1(t), x_2(t) \in H,\ t \in \Omega$.

Definition 5.3. Let $Q, R, S : \Omega \times H \to CB(H)$ be random multi-valued mappings. A random mapping $N : \Omega \times H \times H \times H \to H$ is said to be

(i) $\alpha(t)$-*strongly mixed monotone with respect to Q, R and S*, if there exists a measurable function $\alpha : \Omega \to (0, \infty)$ such that

$$\begin{aligned}&\langle N(t, u_1(t), v_1(t), w_1(t)) - N(t, u_2(t), v_2(t), w_2(t)), x_1(t) - x_2(t)\rangle\\ &\qquad \geq\; \alpha(t)\|x_1(t) - x_2(t)\|^2,\end{aligned}$$

for all $x_i(t) \in H$, $u_i(t) \in Q(t, x_i(t))$, $v_i(t) \in R(t, x_i(t))$, $w_i(t) \in S(t, x_i(t))$, $t \in \Omega$, $i = 1, 2$;

(ii) $(l_{(N,2)}(t), l_{(N,3)}(t), l_{(N,4)}(t))$-*mixed Lipschitz continuous*, if there exist measurable functions $l_{(N,2)}, l_{(N,3)}, l_{(N,4)} : \Omega \to (0, \infty)$ such that

$$\begin{aligned}&\|N(t, x_1(t), y_1(t), z_1(t)) - N(t, x_2(t), y_2(t), z_2(t))\|\\ &\qquad \leq l_{(N,2)}(t)\|x_1(t) - x_2(t)\| + l_{(N,3)}(t)\|y_1(t) - y_2(t)\|\\ &\qquad\quad + l_{(N,4)}(t)\|z_1(t) - z_2(t)\|,\end{aligned}$$

for all $x_i(t), y_i(t), z_i(t) \in H,\ t \in \Omega,\ i = 1, 2$.

Now, we prove the existence of solution and discuss the convergence criteria of iterative sequences generated by the Iterative Scheme A, for RGNIVLIP (3.2).

Theorem 5.4. *Let the mappings $\eta : H \times H \to H$ be τ-Lipschitz continuous and $P : H \to H$ be δ-strongly η-monotone. Let the random mapping $g : \Omega \times H \to H$ be $s(t)$-strongly monotone and $l_g(t)$-Lipschitz continuous, and the random mapping $m : \Omega \times H \to H$ be $l_m(t)$-Lipschitz continuous. Let the random mapping $P \circ g$ be $r(t)$-strongly monotone and $l_{P\circ g}(t)$-Lipschitz continuous, and the random mapping $P \circ m$ be $l_{P\circ m}(t)$-Lipschitz continuous. Let the random fuzzy mappings $Q, R, S, Z : \Omega \times H \to \mathcal{F}(H)$ satisfy the condition* (**C**), *and the random multi-valued mappings $\tilde{Q}, \tilde{R}, \tilde{S}, \tilde{Z} : \Omega \times H \to CB(H)$ be $\tilde{\mathcal{H}}$-Lipschitz continuous with measurable functions $l_{\tilde{Q}}(t), l_{\tilde{R}}(t), l_{\tilde{S}}(t), l_{\tilde{Z}}(t)$, respectively. Let the random mapping $N : \Omega \times H \times H \times H \to H$ be $\alpha(t)$-strongly mixed monotone with respect to $\tilde{Q}, \tilde{R}$ and $\tilde{S}$, and $(L_{(N,2)}(t), L_{(N,3)}(t), L_{(N,4)}(t))$-mixed Lipschitz continuous. Suppose that the random multi-valued mapping $M : \Omega \times H \times H \to 2^H$ is such that for each $(t, z(t)) \in \Omega \times H$, $M(t, \cdot, z(t)) : H \to 2^H$ is γ-strongly maximal P-η-monotone with $(g-m)(\Omega \times H) \cap domain\, M(t, \cdot, z(t)) \neq \emptyset$. Suppose that there exists a measurable function $k : \Omega \to (0, \infty)$ such that*

$$\|R_{P,\eta}^{M(t,\cdot,z_1(t))}(x(t)) - R_{P,\eta}^{M(t,\cdot,z_2(t))}(x(t))\| \leq k(t)\|z_1(t) - z_2(t)\|, \tag{5.1}$$

for all $x(t), z_1(t), z_2(t) \in H$, $t \in \Omega$, and suppose that for a measurable function $\rho : \Omega \to (0, \infty)$, the following condition holds, for all $t \in \Omega$,

$$\theta(t) := q(t) + \frac{\tau}{\delta + \rho(t)\gamma}\Big[p(t) + \sqrt{1 - 2\rho(t)\alpha(t) + \rho^2(t)L_N^2(t)}\ \Big] < 1, \tag{5.2}$$

where

$$q(t) := l_m(t) + k(t)l_{\tilde{Z}}(t) + \sqrt{1 - 2s(t) + l_g^2(t)},$$

$$p(t) := l_{P\circ m}(t) + \sqrt{1 - 2r(t) + l_{P\circ g}^2(t)}$$

and

$$L_N(t) := L_{(N,2)}(t)l_{\tilde{Q}}(t) + L_{(N,3)}(t)l_{\tilde{R}}(t) + L_{(N,4)}(t)l_{\tilde{S}}(t).$$

Then, there exist measurable mappings $x, u, v, w, z : \Omega \to H$ such that (3.2) holds. Moreover, $x_n(t) \to x(t)$, $u_n(t) \to u(t)$, $v_n(t) \to v(t)$, $w_n(t) \to w(t)$, $z_n(t) \to z(t)$, where $\{x_n(t)\}$, $\{u_n(t)\}$, $\{v_n(t)\}$, $\{w_n(t)\}$, $\{z_n(t)\}$ are random sequences generated by Iterative Scheme A.

Proof. From Iterative Scheme A, (5.1) and Theorem 2.17, for any $t \in \Omega$, we have

$$
\begin{aligned}
&\|x_{n+2}(t) - x_{n+1}(t)\| \\
&\leq \|x_{n+1}(t) - x_n(t) - (g-m)(t, x_{n+1}(t)) + (g-m)(t, x_n(t))\| \\
&\quad + \|R_{P,\eta}^{M(t,\cdot,z_{n+1}(t))}[h(t, x_{n+1}(t))] - R_{P,\eta}^{M(t,\cdot,z_n(t))}[h(t, x_{n+1}(t))]\| \\
&\quad + \|R_{P,\eta}^{M(t,\cdot,z_n(t))}[h(t, x_{n+1}(t))] - R_{P,\eta}^{M(t,\cdot,z_n(t))}[P \circ (g-m)(t, x_n(t)) \\
&\quad - \rho(t) N(t, u_n(t), v_n(t), w_n(t))]\|,
\end{aligned}
$$

where

$$
h(t, x_{n+1}(t)) = P \circ (g-m)(t, x_{n+1}(t)) - \rho(t) N(t, u_{n+1}(t), v_{n+1}(t), w_{n+1}(t)).
$$

Hence, we have

$$
\begin{aligned}
&\|x_{n+2}(t) - x_{n+1}(t)\| \\
&\leq \|x_{n+1}(t) - x_n(t) - (g(t, x_{n+1}(t)) - g(t, x_n(t)))\| \\
&\quad + \|m(t, x_{n+1}(t)) - m(t, x_n(t))\| + k(t)\|z_{n+1}(t) - z_n(t)\| \\
&\quad + \frac{\tau}{\delta + \rho(t)\gamma}\Big[\|x_{n+1}(t) - x_n(t) - (P \circ g(t, x_{n+1}(t)) - P \circ g(t, x_n(t)))\| \\
&\quad + \|P \circ m(t, x_{n+1}(t)) - P \circ m(t, x_n(t))\| \\
&\quad + \|x_{n+1}(t) - x_n(t) - \rho(t)(N(t, u_{n+1}(t), v_{n+1}(t), w_{n+1}(t)) \\
&\quad - N(t, u_n(t), v_n(t), w_n(t)))\|\Big].
\end{aligned}
\tag{5.3}
$$

Since g is $s(t)$-strongly monotone and $l_g(t)$-Lipschitz continuous, we have

$$
\begin{aligned}
&\|x_{n+1}(t) - x_n(t) - (g(t, x_{n+1}(t)) - g(t, x_n(t)))\| \\
&\leq \sqrt{1 - 2s(t) + l_g^2(t)}\ \|x_{n+1}(t) - x_n(t)\|.
\end{aligned}
\tag{5.4}
$$

Again since $P \circ g$ is $r(t)$-strongly monotone and $l_{P\circ g}(t)$-Lipschitz continuous, m is $l_m(t)$-Lipschitz continuous, $P \circ m$ is $l_{P\circ m}(t)$-Lipschitz continuous, $\tilde{Z}$ is $l_{\tilde{Z}}(t)$-$\tilde{\mathcal{H}}$-Lipschitz continuous, we have

$$
\begin{aligned}
&\|x_{n+1}(t) - x_n(t) - (P \circ g(t, x_{n+1}(t)) - P \circ g(t, x_n(t)))\| \\
&\leq \sqrt{1 - 2r(t) + l_{P\circ g}^2(t)}\ \|x_{n+1}(t) - x_n(t)\|,
\end{aligned}
\tag{5.5}
$$

$$
\|m(t, x_{n+1}(t)) - m(t, x_n(t))\| \leq l_m(t)\|x_{n+1}(t) - x_n(t)\|, \tag{5.6}
$$

$$
\|P \circ m(t, x_{n+1}(t)) - P \circ m(t, x_n(t))\| \leq l_{P\circ m}(t)\|x_{n+1}(t) - x_n(t)\|, \tag{5.7}
$$

and

$$\begin{aligned}\|z_{n+1}(t)-z_n(t)\| &\le (1+(1+n)^{-1})\tilde{\mathcal{H}}(\tilde{Z}(t,x_{n+1}(t)),\tilde{Z}(t,x_n(t)))\\ &\le (1+(1+n)^{-1})\ l_{\tilde{Z}}(t)\|x_{n+1}(t)-x_n(t)\|. \end{aligned}\tag{5.8}$$

Since for each fixed $t\in\Omega$, $\tilde{Q},\tilde{R},\tilde{S}$ are $\tilde{\mathcal{H}}$-Lipschitz continuous with constants $l_{\tilde{Q}}(t),l_{\tilde{R}}(t),l_{\tilde{S}}(t)$, respectively, N is $\alpha(t)$-strongly mixed monotone with respect to $\tilde{Q},\tilde{R}$ and $\tilde{S}$, and $(L_{(N,2)}(t),L_{(N,3)}(t),L_{(N,4)}(t))$-mixed Lipschitz continuous, we have

$$\begin{aligned}&\|N(t,u_{n+1}(t),v_{n+1}(t),w_{n+1}(t))-N(t,u_n(t),v_n(t),w_n(t))\|\\ &\le\ l_{(N,2)}(t)\|u_{n+1}(t)-u_n(t)\|+l_{(N,3)}(t)\|v_{n+1}(t)-v_n(t)\|\\ &\quad +l_{(N,4)}(t)\|w_{n+1}(t)-w_n(t)\|\\ &\le\ (1+(1+n)^{-1})\ \big(l_{(N,2)}(t)\ \tilde{\mathcal{H}}(\tilde{Q}(t,x_{n+1}(t))\,,\tilde{Q}(t,x_n(t)))\\ &\quad +l_{(N,3)}(t)\ \tilde{\mathcal{H}}(\tilde{R}(t,x_{n+1}(t))\,,\tilde{R}(t,x_n(t)))\\ &\quad +l_{(N,4)}(t)\tilde{\mathcal{H}}(\tilde{S}(t,x_{n+1}(t))\,,\tilde{S}(t,x_n(t)))\big)\\ &\le\ (1+(1+n)^{-1})\big(l_{(N,2)}(t)l_{\tilde{Q}}(t)+l_{(N,3)}(t)l_{\tilde{R}}(t)\\ &\quad +l_{(N,4)}(t)l_{\tilde{S}}(t)\big)\|x_{n+1}(t)-x_n(t)\|\end{aligned}\tag{5.9}$$

and

$$\begin{aligned}&\|x_{n+1}(t)-x_n(t)-\rho(t)(N(t,u_{n+1}(t),v_{n+1}(t),w_{n+1}(t))\\ &\quad -N(t,u_n(t),v_n(t),w_n(t)))\|^2\\ &\le\ \|x_{n+1}(t)-x_n(t)\|^2-2\rho(t)\langle N(t,u_{n+1}(t),v_{n+1}(t),w_{n+1}(t))\\ &\quad -N(t,u_n(t),v_n(t),w_n(t)),x_{n+1}(t)-x_n(t)\rangle\\ &\quad +\rho^2(t)\|N(t,u_{n+1}(t),v_{n+1}(t),w_{n+1}(t))-N(t,u_n(t),v_n(t),w_n(t))\|^2\\ &\le\ (1-2\rho(t)\alpha(t)+\rho^2(t)L_N^2(t))\,\|x_{n+1}(t)-x_n(t)\|^2.\end{aligned}\tag{5.10}$$

From (5.3)-(5.10), it follows that

$$\|x_{n+2}(t)-x_{n+1}(t)\|\ \le\ \theta_n(t)\ \|x_{n+1}(t)-x_n(t)\|,\ \forall\, t\in\Omega,\tag{5.11}$$

where

$$\begin{aligned}\theta_n(t):=\Big\{&\sqrt{1-2s(t)+l_g^2(t)}+l_m(t)+k(t)l_{\tilde{Z}}(t)(1+(1+n)^{-1})\\ &+\frac{\tau}{\delta+\rho(t)\gamma}\Big[l_{P\circ m}(t)+\sqrt{1-2r(t)+l_{P\circ g}^2(t)}\\ &+\sqrt{1-2\rho(t)\alpha(t)+(1+(1+n)^{-1})^2\rho^2(t)L_N^2(t)}\ \Big]\Big\}.\end{aligned}\tag{5.12}$$

Letting $n \to \infty$, we have $\theta_n(t) \to \theta(t)$ for all $t \in \Omega$, where

$$\theta(t) := \Big\{ \sqrt{1 - 2s(t) + l_g^2(t)} + l_m(t) + k(t) l_{\tilde{Z}}(t) + \frac{\tau}{\delta + \rho(t)\gamma} \Big[l_{P\circ m}(t) + \sqrt{1 - 2r(t) + l_{P\circ g}^2(t)} + \sqrt{1 - 2\rho(t)\alpha(t) + \rho^2(t) L_N^2(t)} \,\Big] \Big\}, \tag{5.13}$$

where $L_N(t) := L_{(N,2)}(t) l_{\tilde{Q}}(t) + L_{(N,3)}(t) l_{\tilde{R}}(t) + L_{(N,4)}(t) l_{\tilde{S}}(t)$.

By condition (5.2), $\theta(t) \in (0,1)$ for all $t \in \Omega$. Hence for any $t \in \Omega$, $\theta_n(t) < 1$ for n sufficiently large. Therefore (5.11) implies that $\{x_n(t)\}$ is a Cauchy sequence in H. Since H is complete, there exists a measurable mapping $x : \Omega \to H$ such that $x_n(t) \to x(t)$, for all $t \in \Omega$. Further, it follows from $\tilde{\mathcal{H}}$-Lipschitz continuity of $\tilde{Q}$ and Iterative Scheme A, we have

$$\|u_{n+1}(t) - u_n(t)\| \;\le\; (1 + (1+n)^{-1}) l_{\tilde{Q}}(t) \|x_{n+1}(t) - x_n(t)\|$$

which implies that $\{u_n(t)\}$ is a Cauchy sequence in H. Similarly, we can prove that $\{v_n(t)\}$, $\{w_n(t)\}$, $\{z_n(t)\}$ are Cauchy sequences in H. Hence, there exist measurable mappings $v, w, z : \Omega \to H$ such that $v_n(t) \to v(t)$, $w_n(t) \to w(t)$, $z_n(t) \to z(t)$ as $n \to \infty$, for all $t \in \Omega$.

Furthermore, for any $t \in \Omega$, we have

$$\begin{aligned} d(u(t),\, \tilde{Q}(t, x(t))) &\le\; \|u(t) - u_n(t)\| + d(u_n(t)\,, \tilde{Q}(t, x(t))) \\ &\le\; \|u(t) - u_n(t)\| + \tilde{\mathcal{H}}(\tilde{Q}(t, x_n(t))\,, \tilde{Q}(t, x(t))) \\ &\le\; \|u(t) - u_n(t)\| + l_{\tilde{Q}}(t) \|x_n(t) - x(t)\| \\ &\to 0 \text{ as } n \to \infty. \end{aligned}$$

Hence $u(t) \in \tilde{Q}(t, x(t))$ for all $t \in \Omega$.

Similarly we can prove that $v(t) \in \tilde{R}(t, x(t))$, $w(t) \in \tilde{S}(t, x(t))$, $z(t) \in \tilde{Z}(t, x(t))$, for all $t \in \Omega$. Thus, it follows from Iterative Scheme A, and Lipschitz continuity of $g, m, P \circ g, P \circ m, R_{P,\eta}^{M(t,\cdot,z(t))}, N, M$, that $x(t)$ is a fixed point of random multi-valued mapping $G(t, x(t))$ defined by (4.1). Hence, by Lemma 4.3, it follows that the set $\{x(t), u(t), v(t), w(t), z(t)\}$ is a random solution of RGNIVLIP (3.2). This completes the proof. □

Remark 5.5. For all $t \in \Omega$, and measurable functions $\rho, k : \Omega \to (0, \infty)$, it is clear that $2r(t) < 1 + l_{P\circ g}^2(t)$, $2s(t) < 1 + l_g^2(t)$, $2\rho(t)\alpha(t) < 1 + \rho^2(t) L_N^2(t)$, where $L_N(t) := L_{(N,2)}(t) l_{\tilde{Q}}(t) + L_{(N,3)}(t) l_{\tilde{R}}(t) + L_{(N,4)}(t) l_{\tilde{S}}(t)$. Further, $\theta \in$

$(0, 1)$ and condition (5.2) of Theorem 5.4 holds for some suitable values of constants.

Remark 5.6. Since the RGNIVLIP (3.2) includes many known classes of parametric generalized variational inclusions (inequalities) as special cases, so the technique utilized in this paper can be used to extend and advance the theorems given by many researchers (see [1,3,6-9,12-18,20-22]).

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