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SOME COINCIDENCE POINT THEOREMS FOR PREŠIĆ-ĆIRIĆ TYPE CONTRACTIONS

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Abstract. In this paper, we prove some coincidence point theorems for mappings satisfying nonlinear Prešić-Ćirić type contraction in complete metric spaces as well as in ordered metric spaces. As a consequence, we deduce corresponding fixed point theorems. Further, we give some examples to substantiate the utility of our results.

1. Introduction

The fundamental fixed point result, called Banach contraction principle, is due to Polish mathematician Banach [3] in 1922. This classical result states:

Theorem 1.1. ([3]) *Let (X, d) be a complete metric space and $T : X \to X$ be a mapping. If there exists $\lambda \in (0, 1)$ such that*

$$d(T(x), T(y)) \leq \lambda d(x, y)$$

for all $x, y \in X$, then T has a unique fixed point.

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There are many generalizations of Banach contraction principle, like as [1, 2, 6, 7, 11, 14, 15, 16, 17, 18]. One of the most generalizations is given by Prešić [12] in 1965.

Theorem 1.2. ([12]) *Let (X, d) be a complete metric space and $T : X^k \to X$ be a mapping. If there exist constants $\lambda_1, \lambda_2, ..., \lambda_k \in (0, 1)$ satisfying $\lambda_1 + \lambda_2 + ... + \lambda_k < 1$ such that*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_k, x_{k+1})) \leq \sum_{i=1}^{k} \lambda_i d(x_i, x_{i+1})$$

for all $x_1, x_2, ..., x_{k+1} \in X$, then T has a unique fixed point, that is, there exists a unique $x \in X$ such that $T(x, x, ..., x) = x$.

The result of Prešić is very important because this theorem can be used to investigate the existence of solution for several linear and nonlinear difference equations. For instance, consider k-th order nonlinear difference equations:

$$x_{n+k} = T(x_n, x_{n+1}, ..., x_{n+k-1}),\ n \in \mathbb{N}_0, \tag{1.1}$$

with initial value $x_0, x_1, ..., x_k \in X$, where (X, d) is a metric space, $k \in \mathbb{N}_0$ and $T : X^k \to X$. The equation (1.1) can be studied by means of fixed point theory in view of the fact that $x^* \in X$ is a solution of (1.1) if and only if x^* is a fixed point of T, that is,

$$x^* = T(x^*, x^*, ..., x^*).$$

Afterward, some generalizations of Theorem 1.2 were established (See [4, 13, 15] and references therein). In this continuation, Ćirić and Prešić [4] extended Theorem 1.2 as follows:

Theorem 1.3. ([4]) *Let (X, d) be a complete metric space and $T : X^k \to X$ be a mapping. If there exists $\lambda \in (0, 1)$ such that*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_k, x_{k+1})) \leq \lambda \max_{1 \leq i \leq k} d(x_i, x_{i+1})$$

for all $x_1, x_2, ..., x_{k+1} \in X$, then T has a fixed point, that is, there exists a $x \in X$ such that $T(x, x, ..., x) = x$.

If in addition, we suppose that on the diagonal $\triangle \subset X^k$,

$$d(T(u, u, ..., u), T(v, v, ..., v)) < d(gu, gv) \tag{1.2}$$

holds for all $v, u \in X$ with $g(u) \neq g(v)$, then T has a unique fixed point.

In this paper, firstly, we prove a coincidence point theorem for mappings satisfying nonlinear Prešić-Ćirić type contraction in complete metric spaces which is a generalization of some existing fixed point results. Then we prove a coincidence point theorem in the context of ordered metric spaces for g-increasing mappings satisfying nonlinear Prešić-Ćirić type contraction. Further, we give some examples to substantiate the utility of our results.

2. Preliminaries

In this section, we give some basic definitions which will be required to prove our main results. Throughout this paper, we denote $\mathbb{N} \cup \{0\}$ as $\mathbb{N}_0$ and $g(x)$ as gx for some places.

Definition 2.1. ([5]) Two mappings $T : X^k \to X$ and $g : X \to X$ are said to be commuting if for $x_1, x_2, ..., x_k \in X$,

$$T(gx_1, gx_2, ...gx_k) = g(T(x_1, x_2, ..., x_k)).$$

Definition 2.2. ([5]) Let $T : X^k \to X$ and $g : X \to X$ be two mappings. A point $x \in X$ is called a coincidence point of T and g if

$$T(x, x, ..., x) = g(x).$$

Definition 2.3. ([5]) Let $T : X^k \to X$ and $g : X \to X$ be two mappings. A point $x \in X$ is called a common fixed point of T and g if

$$T(x, x, ..., x) = g(x) = x.$$

Let Φ denote all functions $\varphi : [0, \infty) \to [0, \infty)$ satisfying

(i) φ is continuous and increasing,
(ii) $\sum_{i=1}^{\infty} \varphi^i(t) < \infty$ for all $t \in (0, \infty)$.

Lemma 2.4. ([9]) *Suppose that $\varphi : [0, \infty) \to [0, \infty)$ is increasing. Then for every $t > 0$, $\lim_{n\to\infty} \varphi^n(t) = 0$ implies $\varphi(t) < t$.*

The property (ii) of φ implies $\lim_{n\to\infty} \varphi^n(t) = 0$ for every $t > 0$. Therefore, by Lemma 2.4, if $\varphi \in \Phi$ then $\varphi(t) < t$.

Now we are well equipped to establish our results.

3. Main results

In this section, we prove a coincidence point theorem for a nonlinear Prešić-Ćirić type contraction in a complete metric space.

Theorem 3.1. *Let (X, d) be a complete metric space and $T : X^k \to X$ and $g : X \to X$ be two mappings such that the following conditions hold:*

(a) $T(X^k) \subseteq g(X)$,
(b) T and g commuting pair,
(c) g is continuous,

(*d*) *there exists* $\varphi \in \Phi$

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq \varphi(\max_{1 \leq i \leq k}\{d(gx_i, gx_{i+1})\}) \tag{3.1}$$

for all $x_1, x_2, ..., x_{k+1} \in X$.

Then T *and* g *have a coincidence point.*

If in addition to the above hypothesis, we consider the following condition:

(*e*) *On the diagonal* $\triangle \subset X^k$,

$$d(T(u, u, ..., u), T(v, v, ..., v)) < d(gu, gv) \tag{3.2}$$

holds for all $u, v \in X$ *with* $g(u) \neq g(v)$,

then T *and* g *have unique common fixed point.*

Proof. Let $x_1, x_2, ..., x_k$ be k arbitrary points in X. Using these points and condition (a) define a sequence $\{gx_n\}_{n \in \mathbb{N}}$ as follows:

$$g(x_{n+k}) = T(x_n, x_{n+1}, ..., x_{n+k-1}). \tag{3.3}$$

Suppose $\alpha = \max\{d(gx_1, gx_2), d(gx_2, gx_3), ..., d(gx_k, gx_{k+1})\}$. Now if $gx_1 = gx_2 = ... = gx_k = gx_{k+1} = x$, then we are done. Otherwise, we may assume that $gx_1, gx_2, ..., gx_k, gx_{k+1}$ are not all equal, then we know that $\alpha > 0$. By assumption (d), (3.3) and Lemma 2.4 we have,

$$\begin{aligned} d(gx_{k+1}, gx_{k+2}) &= d(T(x_1, x_2, ..., x_k), T(x_2, x_3, .., x_{k+1})) \\ &\leq \varphi(\max\{d(gx_1, gx_2), d(gx_2, gx_3), ..., d(gx_k, gx_{k+1})\}) \\ &\leq \varphi(\alpha) < \alpha, \end{aligned}$$

$$\begin{aligned} d(gx_{k+2}, gx_{k+3}) &= d(T(x_2, x_3, ..., x_{k+1}), T(x_3, x_4, ..., x_{k+2})) \\ &\leq \varphi(\max\{d(gx_2, gx_3), d(gx_3, gx_4), ..., d(gx_{k+1}, x_{k+2})\}) \\ &\leq \varphi(\max\{\alpha, \varphi(\alpha)\}) < \alpha, \end{aligned}$$

$$\begin{aligned} d(gx_{2k}, gx_{2k+1}) &= d(T(x_k, x_{k+1}, ..., x_{2k-1}), T(x_{k+1}, x_{k+2}, ..., x_{2k})) \\ &\leq \varphi(\max\{d(gx_k, gx_{k+1}), d(gx_{k+1}, gx_{k+2}), ..., d(gx_{2k-1}, gx_{2k})\}) \\ &\leq \varphi(\max\{\alpha, \varphi(\alpha), ..., \varphi(\alpha)\}) = \varphi(\alpha) < \alpha, \end{aligned}$$

$$\begin{aligned} d(gx_{2k+1}, gx_{2k+2}) &= d(T(x_{k+1}, x_{k+2}, ..., x_{2k}), T(x_{k+2}, x_{k+3}, ..., x_{2k+1})) \\ &\leq \varphi(\max\{d(gx_{k+1}, gx_{k+2}), d(gx_{k+2}, gx_{k+3}), ..., \\ &\qquad d(gx_{2k}, gx_{2k+1})\}) \\ &\leq \varphi(\max\{\varphi(\alpha), \varphi(\alpha), ..., \varphi(\alpha)\}) = \varphi^2(\alpha) < \alpha \end{aligned}$$

and so on

$$d(gx_{nk+1}, gx_{nk+2}) \quad \leq \quad \varphi^n(\alpha), \quad n \geq 1$$

and

$$d(gx_{n+1}, gx_{n+2}) \leq \varphi^{[\frac{n}{k}]}(\alpha), \quad n \geq k. \tag{3.4}$$

By the property (ii) of φ and (3.4), we have

$$\lim_{n\to\infty} d(gx_{n+1}, gx_{n+2}) = 0. \tag{3.5}$$

For any $n, m \in \mathbb{N}$, $n > k$, we have,

$$\begin{aligned} d(gx_n, gx_{n+m}) &\leq d(gx_n, gx_{n+1}) + d(gx_{n+1}, gx_{n+2}) + ... \\ &\quad + d(gx_{n+m-1}, gx_{n+m}) \\ &\leq \varphi^{[\frac{n-1}{k}]}(\alpha) + \varphi^{[\frac{n}{k}]}(\alpha) + ... + \varphi^{[\frac{n+m-2}{k}]}(\alpha). \end{aligned} \tag{3.6}$$

Assume $l = [\frac{n-1}{k}]$ and $m' = [\frac{n+m-2}{k}]$. Then $l \leq m'$. It follows from (3.6) that

$$\begin{aligned} d(gx_n, gx_{n+m}) \leq & \underbrace{\varphi^{l}(\alpha) + \varphi^{l}(\alpha) + ... + \varphi^{l}(\alpha)}_{\text{k times}} \\ & + \underbrace{\varphi^{l+1}(\alpha) + \varphi^{l+1}(\alpha) + ... + \varphi^{l+1}(\alpha)}_{\text{k times}} \\ & \vdots \\ & + \underbrace{\varphi^{m'}(\alpha) + \varphi^{m'}(\alpha) + ... + \varphi^{m'}(\alpha)}_{\text{k times}}. \end{aligned}$$

So,

$$d(gx_n, gx_{n+m}) \leq k \sum_{i=l}^{m'} \varphi^{i}(\alpha). \tag{3.7}$$

By the property (ii) of φ, we have

$$\lim_{l\to\infty} \sum_{i=l}^{\infty} \varphi^{i}(t) = 0$$

and in view of (3.7), we have $d(gx_n, gx_{n+m}) \to 0$ as $n \to \infty$. This means that $\{gx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x. \tag{3.8}$$

Using assumption (c) and (3.8), we have

$$\lim_{n\to\infty} g(g(x_n)) = g(x). \tag{3.9}$$

By using (3.3) and commutativity of T and g, we get

$$\begin{aligned} g(gx_{n+k}) &= g(T(x_n, x_{n+1}, ..., x_{n+k-1})) \\ &= T(gx_n, gx_{n+1}, ..., gx_{n+k-1}). \end{aligned} \tag{3.10}$$

By using triangular inequality and (3.10), we get

$$\begin{aligned} d(gx, T(x, x, ..., x)) &\leq d(gx, g(gx_{n+k})) + d(g(gx_{n+k}), T(x, x, ..., x)) \\ &= d(gx, g(gx_{n+k})) \\ &\quad + d(T(gx_n, gx_{n+1}, ..., gx_{n+k-1}), T(x, x, ..., x)) \end{aligned}$$

which gives

$$\begin{aligned} &d(gx, T(x, x, ..., x)) \\ &\leq d(gx, g(gx_{n+k})) + d(T(gx_n, gx_{n+1}, ..., gx_{n+k-1}), T(gx_{n+1}, ..., gx_{n+k-1}, x)) \\ &\quad + d(T(gx_{n+1}, ..., gx_{n+k-1}, x), T(gx_{n+2}, ..., gx_{n+k-1}, x, x)) \\ &\quad \vdots \\ &\quad + d(T(gx_{n+k-1}, x, ..., x), T(x, x, ..., x)). \end{aligned}$$

Therefore, by assumption (d), we have

$$\begin{aligned} &d(gx, T(x, x, ..., x)) \\ &\leq d(gx, g(gx_{n+k})) + \varphi(\max\{d(g(gx_n), g(gx_{n+1})), ..., d(g(gx_{n+k-1}), g(x))\}) \\ &\quad + \varphi(\max\{d(g(gx_{n+1}), g(gx_{n+2})), ..., d(g(gx_{n+k-1}), g(x)), d(g(x), g(x))\}) \\ &\quad \vdots \\ &\quad + \varphi(\max\{d(g(gx_{n+k-1}), g(x)), d(g(x), g(x)), \cdots, d(g(x), g(x))\}). \end{aligned}$$

Taking $n \to \infty$ and using (3.8), (3.9), (3.5) and properties of φ, we have

$$d(gx, T(x, x, ..., x)) \leq 0,$$

that is,

$$d(gx, T(x, x, ..., x)) = 0$$

which gives

$$g(x) = T(x, x, ..., x).$$

Hence, x is a coincidence point of T and g.

Now, suppose assumption (e) holds. We show that T and g have unique common fixed point. Let x and y be the two coincidence points of T and g then

$$T(x, x, ..., x) = g(x) = \overline{x} \tag{3.11}$$

and

$$T(y, y, ..., y) = g(y) = \overline{y}. \tag{3.12}$$

Then we shall show that

$$\overline{x} = \overline{y}. \tag{3.13}$$

On contrary, suppose that $\overline{x} \neq \overline{y}$, then by using assumption (e), (3.11) and (3.12), we get

$$d(T(x, x, ..., x), T(y, y, ..., y)) < d(gx, gy)$$

so that

$$d(\overline{x}, \overline{y}) < d(\overline{x}, \overline{y})$$

which is a contradiction yielding that (3.13) holds.

Again since T and g are commuting pair, from (3.11) we get

$$\begin{aligned} g(\overline{x}) &= g(T(x, x, ..., x)) \\ &= T(gx, gx, ..., gx) \\ &= T(\overline{x}, \overline{x}, ..., \overline{x}) \end{aligned}$$

so that

$$g(\overline{x}) = T(\overline{x}, \overline{x}, ..., \overline{x}), \tag{3.14}$$

which implies that $\overline{x}$ is also coincidence point of T and g.
Using (3.13) and (3.14), we get

$$T(\overline{x}, \overline{x}, ..., \overline{x}) = g(\overline{x}) = \overline{x},$$

which yields that $\overline{x}$ is a common fixed point of T and g.

Suppose that x^* is another common fixed point of T and g. Then again by using assumption (3.13), we get

$$x^* = g(x^*) = g(\overline{x}) = \overline{x}.$$

Hence, T and g have a unique common fixed point. □

Corollary 3.2. *Let (X, d) be a complete metric space and $T : X^k \to X$ and $g : X \to X$ be two mappings such that the following conditions are satisfied:*

- (a) $T(X^k) \subseteq g(X)$,
- (b) *T and g is a commuting pair,*
- (c) *g is continuous,*
- (d) *There exists $\lambda \in (0, 1)$ such that*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_k, x_{k+1})) \leq \lambda \max_{1 \leq i \leq k} d(gx_i, gx_{i+1})$$

for all $x_1, x_2, ..., x_{k+1} \in X$.

Then T and g have a coincidence point.

If in addition to the above hypothesis we consider the following condition:

(e) *On the diagonal* $\triangle\subset X^k$,

$$d(T(u,u,...,u),T(v,v,...,v)) < d(gu,gv)$$

holds for all $u,v\in X$ *with* $g(u)\neq g(v)$,

then T *and* g *have unique common fixed point.*

Proof. In Theorem 3.1, taking $\varphi(t)=\lambda t$ for all $t\in[0,\infty)$ with $\lambda\in(0,1)$ we obtain Corollary 3.2. □

Remark 3.3. Some of existing results are deducible from our newly proved results, as given below:

(1) In Theorem 3.1, taking g as identity map and considering $\varphi(t)=\lambda t$ for all $t\in[0,\infty)$ with $\lambda\in(0,1)$ we obtain Theorem 1.3.
(2) If we take g as identity map with $\varphi(t)=\lambda t$ for all $t\in[0,\infty)$ with $\lambda\in(0,1)$ and consider the map T on X in Theorem 3.1, then we obtain Theorem 1.1.
(3) Contractive condition of Theorem 1.2 implies contractive conditions of Corollary 3.2. So, by considering the map g as identity in Corollary 3.2 we obtain Theorem 1.2.
(4) Theorem 3.1 improves other fixed point results given by Luong and Thuan(Theorem 2.2) [8].

4. Results in ordered metric spaces

In this section, we prove a coincidence point theorem for g-increasing mappings satisfying nonlinear Prešić-Ćirić type contraction in an ordered complete metric space.

Let $(X,\preceq)$ be a partially ordered set. We endow X^k, $k\in\mathbb{N}$ with the following partial order:

$(x_1,x_2,...,x_k)\sqsubseteq(y_1,y_2,...,y_k)$ if and only if $x_1\preceq y_1,x_2\preceq y_2,...,x_k\preceq y_k$.

Definition 4.1. ([10]) Let X be a nonempty set with partial order $\preceq$ and $T:X^k\to X$ and $g:X\to X$ be two mappings. Now,

(a) A sequence $\{x_n\}_{n\in\mathbb{N}}$ is said to be increasing with respect to $\preceq$ if

$$x_1\preceq x_2\preceq ...\preceq x_n\preceq ...,$$

(b) T is said to be increasing with respect to $\preceq$ if for any finite increasing sequence $\{x_n\}_{n=1}^{k+1}$ we have,

$$T(x_1,x_2,...,x_k)\preceq T(x_2,x_3,...,x_{k+1}),$$

(c) T is said to be g-increasing with respect to $\preceq$ if for any finite increasing sequence $\{gx_n\}_{n=1}^{k+1}$ we have,

$$T(x_1, x_2, ..., x_k) \preceq T(x_2, x_3, ..., x_{k+1}).$$

Theorem 4.2. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let k be a positive integer and the mapping $T : X^k \to X$ be g-increasing. Suppose the following conditions hold:*

(a) *$T(X^k) \subseteq g(X)$,*
(b) *T and g commuting pair,*
(c) *T is continuous or if $\{x_n\}$ is an increasing sequence with $x_n \to x$ then $x_n \preceq x$ for all n,*
(d) *g is continuous,*
(e) *there exist k elements $x_1, x_2, ..., x_k \in X$ such that*

$$gx_1 \preceq gx_2 \preceq ... \preceq gx_k \text{ and } gx_k \preceq T(x_1, x_2, ..., x_k),$$

(f) *there exists $\varphi \in \Phi$ such that*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq \varphi(\max_{1\leq i\leq k}\{d(gx_i, gx_{i+1})\}) \tag{4.1}$$

for all $x_1, x_2, ..., x_{k+1} \in X$ with $gx_1 \preceq gx_2 \preceq ... \preceq gx_{k+1}$.

Then T and g have a coincidence point.

If in addition to the above hypothesis we consider the following condition:

(g) *On the diagonal $\triangle \subset X^k$,*

$$d(T(u, u, ..., u), T(v, v, ..., v)) < d(gu, gv) \tag{4.2}$$

holds for all $u, v \in X$ with $g(u) \neq g(v)$,

then T and g have unique common fixed point.

Proof. By assumption (e) there exist k elements $x_1, x_2, ..., x_k \in X$ such that

$$gx_1 \preceq gx_2 \preceq ... \preceq gx_k \text{ and } gx_k \preceq T(x_1, x_2, ..., x_k).$$

Using assumption (a) we can define a sequence $\{gx_n\}_{n\in\mathbb{N}}$ such that

$$g(x_{n+k}) = T(x_n, x_{n+1}, ..., x_{n+k-1}). \tag{4.3}$$

Now

$$gx_{k+1} = T(x_1, x_2, ..., x_k) \succeq gx_k,$$

$$gx_{k+2} = T(x_2, x_3, ..., x_{k+1}) \succeq T(x_1, x_2, ..., x_k) = gx_{k+1}.$$

Continuing this process, we can show

$$gx_1 \preceq gx_2 \preceq ... \preceq gx_n \preceq ... \tag{4.4}$$

Proceeding in the same way as in Theorem 3.1, we can prove that $\{gx_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence. Since X is complete, there exists $x \in X$ such that

$$\lim_{n\to\infty} g(x_n) = x. \tag{4.5}$$

By using assumption (d) and (4.5), we have

$$\lim_{n\to\infty} g(g(x_n)) = g(x). \tag{4.6}$$

Using (4.3) and commutativity of T and g, we get

$$\begin{aligned} g(gx_{n+k}) &= g(T(x_n, x_{n+1}, ..., x_{n+k-1})) \\ &= T(gx_n, gx_{n+1}, ..., gx_{n+k-1}). \end{aligned} \tag{4.7}$$

Now suppose that assumption (c) holds, i.e., T is continuous.
Using continuity of T, (4.5),(4.6) and (4.7), we get

$$\begin{aligned} g(x) &= \lim_{n\to\infty} g(gx_{n+k}) \\ &= T(g(x_n, gx_{n+1}, ..., gx_{n+k-1})) \\ &= T(x, x, ..., x). \end{aligned}$$

Hence, x is a coincidence point of T and g. Alternately, suppose that if $\{x_n\}$ is an increasing sequence with $x_n \to x$, then $x_n \preceq x$ for all n. Since $\{gx_n\}_{n\in\mathbb{N}}$ is increasing, we have

$$gx_n \preceq x \text{ for all } n \in \mathbb{N}. \tag{4.8}$$

Using triangular inequality and (4.7), we get

$$\begin{aligned} d(gx, T(x, x, ..., x)) &\le d(gx, g(gx_{n+k})) + d(g(gx_{n+k}), T(x, x, ..., x)) \\ &= d(gx, g(gx_{n+k})) \\ &\quad + d(T(gx_n, gx_{n+1}, ..., gx_{n+k-1}), T(x, x, ..., x)), \end{aligned}$$

which gives

$$\begin{aligned} &d(gx, T(x, x, ..., x)) \\ &\le d(gx, g(gx_{n+k})) + d(T(gx_n, gx_{n+1}, ..., gx_{n+k-1}), T(gx_{n+1}, ..., gx_{n+k-1}, x)) \\ &\quad + d(T(gx_{n+1}, ..., gx_{n+k-1}, x), T(gx_{n+2}, ..., gx_{n+k-1}, x, x)) \\ &\quad \vdots \\ &\quad + d(T(gx_{n+k-1}, x, ..., x), T(x, x, ..., x)). \end{aligned}$$

Therefore, in view of (4.8) and assumption (f), we have

$$\begin{aligned} &d(gx, T(x, x, ..., x)) \\ &\le d(gx, g(gx_{n+k})) + \varphi(\max\{d(g(gx_n), g(gx_{n+1})), ..., d(g(gx_{n+k-1}), g(x))\}) \\ &\quad + \varphi(\max\{d(g(gx_{n+1}), g(gx_{n+2})), ..., d(g(gx_{n+k-1}), g(x)), d(g(x), g(x))\}) \\ &\quad \vdots \\ &\quad + \varphi(\max\{d(g(gx_{n+k-1}), g(x)), d(g(x), g(x)), ..., d(g(x), g(x))\}). \end{aligned}$$

Now, following the lines of the proof of Theorem 3.1, we can show that x is a coincidence point of T and g. The proof of existence of unique common fixed point is similar to Theorem 3.1. □

Corollary 4.3. *Let $(X, \preceq)$ be a partially ordered set and suppose there is a metric d on X such that (X, d) is a complete metric space. Let k be a positive integer and the mapping $T : X^k \to X$ be g-increasing. Suppose the following conditions hold:*

(a) $T(X^k) \subseteq g(X)$,

(b) *T and g commuting pair,*

(c) *T is continuous or if $\{x_n\}$ is an increasing sequence with $x_n \to x$ then $x_n \preceq x$ for all n,*

(d) *g is continuous,*

(e) *there exist k elements $x_1, x_2, ..., x_k \in X$ such that*

$$gx_1 \preceq gx_2 \preceq ... \preceq gx_k \text{ and } gx_k \preceq T(x_1, x_2, ..., x_k),$$

(f) *there exists $\lambda \in (0, 1)$ such that*

$$d(T(x_1, x_2, ..., x_k), T(x_2, x_3, ..., x_{k+1})) \leq \lambda \max_{1 \leq i \leq k} \{d(gx_i, gx_{i+1})\}$$

for all $x_1, x_2, ..., x_{k+1} \in X$ with $gx_1 \preceq gx_2 \preceq ... \preceq gx_{k+1}$.

Then T and g have a coincidence point.

If in addition to the above hypothesis we consider the following condition:

(g) *On the diagonal $\triangle \subset X^k$,*

$$d(T(u, u, ..., u), T(v, v, ..., v)) < d(gu, gv)$$

holds for all $u, v \in X$ with $g(u) \neq g(v)$,

then T and g have a unique common fixed point.

Proof. In Theorem 4.2, taking $\varphi(t) = \lambda t$ for all $t \in [0, \infty)$ with $\lambda \in (0, 1)$ we obtain Corollary 4.3. □

Remark 4.4. In Theorem 4.2, the contractive condition need not to hold on the whole space. Therefore, Theorem 4.2 is more general than Theorem 3.1.

5. Illustrative examples

Now we give examples to support our results.

Example 5.1. Consider $X = [0, 1]$ with usual metric d. Let $T : X^2 \to X$ and $g : X \to X$ be mappings given by

$$T(x_1, x_2) = \frac{x_1^2 + 2x_2^2}{7} \quad \text{and} \quad g(x_1) = x_1^2.$$

Then for any $x_1, x_2, x_3 \in X$, we have

$$\begin{aligned} d(T(x_1,x_2),T(x_2,x_3)) &= \left|\frac{x_1^2+2x_2^2}{7} - \frac{x_2^2+2x_3^2}{7}\right| \\ &= \left|\frac{x_1^2}{7} + \frac{2x_2^2}{7} - \frac{x_2^2}{7} - \frac{2x_3^2}{7}\right| \\ &= \left|\frac{(x_1^2-x_2^2)}{7} + \frac{2}{7}(x_2^2-x_3^2)\right| \\ &\le \frac{1}{7} \mid x_1^2 - x_2^2 \mid + \frac{2}{7} \mid x_2^2 - x_3^2 \mid \\ &\le \frac{2}{7} \mid x_1^2 - x_2^2 \mid + \frac{2}{7} \mid x_2^2 - x_3^2 \mid \\ &= \frac{2}{7}[d(gx_1,gx_2) + d(gx_2,gx_3)] \\ &\le \frac{4}{7}\max\{d(gx_1,gx_2), d(gx_2,gx_3)\}. \end{aligned}$$

Also,

$$\begin{aligned} d(T(x_1,x_1),T(x_2,x_2)) &= \left|\frac{3x_1^2}{7} - \frac{3x_2^2}{7}\right| \\ &= \frac{3}{7} \mid x_1^2 - x_2^2 \mid \\ &< \mid x_1^2 - x_2^2 \mid \\ &= d(gx_1,gx_2). \end{aligned}$$

Therefore, all the conditions of Theorem 3.1 are satisfied with $\varphi(t) = \frac{4}{7}t$. Hence, T and g have unique common fixed point, that is,

$$T(0,0) = g(0) = 0.$$

Example 5.2. Let $X = \{0,1,2\}$ with usual metric $d(x,y) = \mid x - y \mid$ for all $x, y \in X$. Then (X,d) is a complete metric space. Consider the partial order on X:

$$x, y \in X,\ x \preceq y \quad \Longleftrightarrow \quad x, y \in \{0,1\} \text{ and } x \le y,$$

where $\le$ is usual order. Then X has the property: if $\{x_n\}$ is increasing sequence, $x_n \to x$ then $x_n \preceq x$ for all n. Define $T : X^2 \to X$ as follows:

$$T(0,0) = T(0,1) = T(1,1) = T(1,0) = T(2,2) = T(0,2) = 0,$$

$$T(2,1) = 1,\ T(1,2) = T(2,0) = 2,$$

and $g : X \to X$ as follows:

$$g(0) = 0,\ g(1) = 2,\ g(2) = 1.$$

Then, obviously, T is g-increasing. Let $\varphi : [0,\infty) \to [0,\infty)$ be given by $\varphi(t) = \frac{t}{2}$ for all $t \in [0,\infty)$. If $y_1, y_2, y_3 \in X$ with $gy_1 \preceq gy_2 \preceq gy_3$, then $gy_1 = gy_2 = gy_3 = 0$ or $gy_1 = gy_2 = gy_3 = 1$ or $gy_1 = gy_2 = 0, gy_3 = 1$ or $gy_1 = 0, gy_2 = gy_3 = 1$. In all cases, we have $d(T(y_1,y_2),T(y_2,y_3)) = 0$, so

$$d(T(y_1,y_2),T(y_2,y_3)) \leq \varphi(\max\{d(gy_1,gy_2),d(gy_2,gy_3)\}).$$

Also, $d(T(0,0),T(1,1)) = 0 < 2 = d(g(0),g(1))$, $d(T(1,1),T(2,2)) = 0 < 1 = d(g(1),g(2))$ and $d(T(0,0),T(2,2)) = 0 < 1 = d(g(0),g(2))$. Therefore, all the conditions of Theorem 4.2 are satisfied. Applying Theorem 4.2, we can conclude that T has a unique common fixed point which is 0. However,

$$d(T(1,1),T(1,2)) = 2 > 1 > \varphi(1) = \varphi(\max\{d(g(1),g(1)),d(g(1),g(2))\})$$

for every $\varphi\in \Phi$. Hence, the contractive condition of Theorem 3.1 is not satisfied by the mapping. Therefore, we cannot apply this example to Theorem 3.1.

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