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EXISTENCE OF THREE WEAK SOLUTIONS FOR A CLASS OF NONLINEAR OPERATORS INVOLVING $p(x)$-LAPLACIAN WITH MIXED BOUNDARY CONDITIONS

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Abstract. In this paper, we consider a mixed boundary value problem to a class of nonlinear operators containing $p(x)$-Laplacian. More precisely, we consider the problem with the Dirichlet condition on a part of the boundary and the Steklov boundary condition on an another part of the boundary. We show the existence of at least three weak solutions under some hypotheses on given functions and the values of parameters.

1. Introduction

In this paper, we consider the following nonlinear problem:

$$\begin{cases} -\mathrm{div}\,[S_t(x,|\nabla u|^2)\nabla u] = \lambda f(x,u) \ \text{ in } \Omega, \\ u=0 \ \text{ on } \Gamma_1, \\ S_t(x,|\nabla u|^2)\frac{\partial u}{\partial \boldsymbol{n}} = \mu g(x,u) \ \text{ on } \Gamma_2, \end{cases} \tag{1.1}$$

where $\Omega \subset \mathbb{R}^N$ $(N \geq 2)$ is a bounded domain with a $C^{0,1}$-boundary Γ, and Γ_1 and Γ_2 are disjoint open subsets of Γ such that $\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma$, and $\boldsymbol{n}$ denotes the unit, outer, normal vector to Γ. Thus we impose the mixed boundary conditions, that is, the Dirichlet condition on Γ_1 and the Steklov condition on Γ_2. The given data $f : \Omega \times \mathbb{R} \to \mathbb{R}$ and $g : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ are Carathéodory functions and λ, μ are parameters. The function $S(x,t)$ is a

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Carathéodory function on $\Omega \times [0, \infty)$ satisfying some structure conditions associated with an anisotropic exponent function $p(x)$ and $S_t = \partial S/\partial t$. Then $\operatorname{div}[S_t(x, |\nabla u|^2)\nabla u]$ is a more general operator containing $p(x)$-Laplacian $\Delta_{p(x)} u = \operatorname{div}(|\nabla u|^{p(x)-2}\nabla u)$, where $p(x) > 1$. This generality brings about difficulties and requires more general conditions.

The study of such type of differential equations with $p(x)$-growth conditions is a very interesting topic recently. Studying such problem stimulated its application in mathematical physics, in particular, in elastic mechanics (Zhikov [27]), in electrorheological fluids (Diening [10], Halsey [16], Mihăilescu and Rădulescu [19], Růžička [21]).

Over the last two decades, there are many articles on the existence of weak solutions for the Dirichlet boundary condition, that is, in the case $\Gamma_2 = \emptyset$ in (1.1), (for example, see Fan [12], Fan and Zhang [13], Avci [6], Yücedağ̂ [23]). On the other hand, for the Steklov boundary condition, that is, $\Gamma_1 = \emptyset$, for example, see Wei and Chen [22], Yücedağ̂ [24], Allaoui et al [1], Ayoujil [7], Deng [9].

However, since we can not find any problem with the mixed boundary condition in variable exponent Sobolev space as in (1.1). We are convinced of the reason for existence of this paper.

Throughout this paper, we assume that Γ_1 and Γ_2 are disjoint open subsets of Γ such that

$$\overline{\Gamma_1} \cup \overline{\Gamma_2} = \Gamma \text{ and } \Gamma_1 \neq \emptyset. \tag{1.2}$$

When $p(x) = p = \text{const.}$, Zeidler [25] considered the following mixed boundary value problem:

$$\begin{cases} \operatorname{div} \boldsymbol{j} = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ \boldsymbol{j} \cdot \boldsymbol{n} = h & \text{on } \Gamma_2, \end{cases} \tag{1.3}$$

where $\boldsymbol{j}$ is the current density, $f(x), g(x)$ and $h(x)$ are given functions. If $\boldsymbol{j}$ is of the form

$$\boldsymbol{j} = -\alpha(|\nabla u|^2)\nabla u, \tag{1.4}$$

problem (1.3) corresponds to many physical problems, for example, hydrodynamics, gas dynamics, electrostatics, heat conduction, elasticity and plasticity. If $\alpha \equiv 1$, then the problem (1.3) becomes

$$\begin{cases} -\Delta u = f & \text{in } \Omega, \\ u = g & \text{on } \Gamma_1, \\ -\frac{\partial u}{\partial \boldsymbol{n}} = h & \text{on } \Gamma_2. \end{cases} \tag{1.5}$$

From the mathematical point of view, this is a mixed boundary value problem for the Poisson equation. If $\alpha(|\nabla u|^2) = |\nabla u|^{p-2}$, the problem (1.3) corresponds to the p-Laplacian equation. Of course if $\Gamma_2 = \emptyset$ (resp. $\Gamma_1 = \emptyset$), then the system (1.3) becomes the first (resp. second) boundary value problem, respectively. In order to have an intuitive picture at hand, let $N = 3$ and regard $u(x)$ as the temperature of a body Ω at the point x. Then $\boldsymbol{j}$ in (1.4) is the current density vector of stationary heat flow in Ω, f describes outer heat source, and the boundary conditions means the prescription of the temperature on Γ_1 and heat flow through Γ_2. System (1.3) represents a constitutive law which depends on the specific properties of the material. If α is a positive constant, α represents the heat conductivity and (1.3) is called heat conductivity.

In this paper, we use the direct method of variational calculus. Under some assumptions on f and g in (1.1), we show the existence of three weak solutions using the three critical points theorem of Ricceri [20].

The paper is organized as follows. Section 2 consists of four subsections. In subsection 2.1, we recall some results on variable exponent Lebesgue-Sobolev spaces. In subsection 2.2, we introduce a Carathéodory function $S(x, t)$ satisfying the structure conditions and some properties. In subsection 2.3, we set the problem (1.1) rigorously. In subsection 2.4, we examine the properties of associated functionals. Section 3 is devoted to existence theorems of three weak solutions and their proofs.

2. Preliminaries

Throughout this paper, we only consider vector spaces of real valued functions over $\mathbb{R}$. For any space B, we denote B^N by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector-valued functions, and we denote the standard inner product of vectors $\boldsymbol{a} = (a_1, \dots, a_N)$ and $\boldsymbol{b} = (b_1, \dots, b_N)$ in $\mathbb{R}^N$ by $\boldsymbol{a} \cdot \boldsymbol{b} = \sum_{i=1}^N a_i b_i$ and $|\boldsymbol{a}| = (\boldsymbol{a} \cdot \boldsymbol{a})^{1/2}$.

2.1. Basic properties of variable exponent Lebesgue-Sobolev spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$.

In this subsection, we recall some results on variable exponent Lebesgue-Sobolev spaces. See [13], Diening et al. [11], Kovăčik and Răkosnic [18] and references therein for more detail.

Let Ω be a bounded domain in $\mathbb{R}^N$ $(N \geq 2)$ with a $C^{0,1}$-boundary Γ. Write $C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) : p(x) > 1 \text{ for all } x \in \overline{\Omega}\}$, and let

$$p^+ = \max_{x\in\overline{\Omega}} p(x) \text{ and } p^- = \min_{x\in\overline{\Omega}} p(x)\ (> 1) \text{ for } p \in C_+(\overline{\Omega}).$$

The variable exponent Lebesgue space is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : u : \Omega \to \mathbb{R} \text{ is a measurable function satisfying } \int_\Omega |u(x)|^{p(x)} dx < \infty \right\}.$$

We introduce the Luxemburg norm on $L^{p(\cdot)}(\Omega)$ by

$$\|u\|_{L^{p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1\right\}.$$

Then $(L^{p(\cdot)}(\Omega), \|\cdot\|_{L^{p(\cdot)}})$ becomes a Banach space. The dual space of $L^{p(\cdot)}(\Omega)$ is identified with $L^{p'(\cdot)}(\Omega)$, where $p'(x)$ is the conjugate exponent of $p(x)$, that is, $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. A modular on $L^{p(\cdot)}(\Omega)$ which is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$ defined by

$$\rho_{p(\cdot)}(u) = \int_\Omega |u(x)|^{p(x)} dx \text{ for } u \in L^{p(\cdot)}(\Omega).$$

The following four propositions are well known (see Fan et al. [15], [22], Fan and Zhao [14], Zhao et al. [23], [26]).

Proposition 2.1. *Let $u, u_n \in L^{p(\cdot)}(\Omega)$ $(n = 1, 2, \ldots)$. Then we have the following properties.*

(i) $\|u\|_{L^{p(\cdot)}(\Omega)} < 1 (= 1, > 1)$ *if and only if* $\rho_{p(\cdot)}(u) < 1 (= 1, > 1)$.

(ii) *If* $\|u\|_{L^{p(\cdot)}(\Omega)} > 1$, *then* $\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$.

(iii) *If* $\|u\|_{L^{p(\cdot)}(\Omega)} < 1$, *then* $\|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)}$.
Hence $\|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \wedge \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)} \le \rho_{p(\cdot)}(u) \le \|u\|^{p^-}_{L^{p(\cdot)}(\Omega)} \vee \|u\|^{p^+}_{L^{p(\cdot)}(\Omega)}$, *where* $a \wedge b = \min\{a, b\}$ *and* $a \vee b = \max\{a, b\}$ *for any real numbers* a *and* b.

(iv) $\|u_n - u\|_{L^{p(\cdot)}(\Omega)} \to 0$ *as* $n \to \infty$ *if and only if* $\rho_{p(\cdot)}(u_n - u) \to 0$ *as* $n \to \infty$.

(v) $\|u_n\|_{L^{p(\cdot)}(\Omega)} \to \infty$ *as* $n \to \infty$ *if and only if* $\rho_{p(\cdot)}(u_n) \to \infty$ *as* $n \to \infty$.

Let $q \in C_+(\Gamma) := \{q \in C(\Gamma) : q(x) > 1 \text{ on } \Gamma\}$ and denote the surface measure on Γ induced from the Lebesgue measure dx on Ω by $d\sigma$. We define

$$Ł^{q(\cdot)}(\Gamma) = \Big\{u : u : \Gamma \to \mathbb{R} \text{ is a measurable function with respect to } d\sigma \\ \text{satisfying } \int_\Gamma |u(x)|^{q(x)} d\sigma < \infty\Big\}$$

and the norm is defined by

$$\|u\|_{L^{q(\cdot)}(\Gamma)} = \inf\left\{\lambda > 0 : \int_\Gamma \left|\frac{u(x)}{\lambda}\right|^{q(x)} d\sigma \le 1\right\},$$

and we also define a modular on $L^{q(\cdot)}(\Gamma)$ by

$$\rho_{q(\cdot),\Gamma}(u) = \int_\Gamma |u(x)|^{q(x)} d\sigma.$$

Proposition 2.2. *We have the following properties.*

(i) *If* $|u|_{q(x),\Gamma} \geq 1$, *then* $\|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)} \leq \rho_{q(\cdot),\Gamma}(u) \leq \|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)}$.

(ii) *If* $\|u\|_{L^{q(\cdot)}(\Gamma)} < 1$, *then* $\|u\|^{q^+}_{L^{q(\cdot)}(\Gamma)} \leq \rho_{q(\cdot),\Gamma}(u) \leq \|u\|^{q^-}_{L^{q(\cdot)}(\Gamma)}$.

The following proposition is a generalized Hölder inequality.

Proposition 2.3. *Let* $p \in C_+(\overline{\Omega})$. *For* $u \in L^{p(\cdot)}(\Omega)$ *and* $v \in L^{p'(\cdot)}(\Omega)$, *we have*

$$\begin{aligned}\left|\int_\Omega uv dx\right| &\leq \left(\frac{1}{p^-} + \frac{1}{(p')^-}\right) \|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)} \\ &\leq 2\|u\|_{L^{p(\cdot)}(\Omega)} \|v\|_{L^{p'(\cdot)}(\Omega)}. \end{aligned} \tag{2.1}$$

Moreover, if $p_j \in C_+(\overline{\Omega})$ $(j = 1, 2, 3)$ *satisfy*

$$\frac{1}{p_1(x)} + \frac{1}{p_2(x)} + \frac{1}{p_3(x)} = 1,$$

then for all $u \in L^{p_1(\cdot)}(\Omega), v \in L^{p_2(\cdot)}(\Omega), w \in L^{p_3(\cdot)}(\Omega)$,

$$\left|\int_\Omega uvw dx\right| \leq \left(\frac{1}{p_1^-} + \frac{1}{p_2^-} + \frac{1}{p_3^-}\right) \|u\|_{L^{p_1(\cdot)}(\Omega)} \|v\|_{L^{p_2(\cdot)}(\Omega)} \|w\|_{L^{p_3(\cdot)}(\Omega)}. \tag{2.2}$$

Since $L^{p(\cdot)}(\Omega) \subset L^1_{\text{loc}}(\Omega)$, every function in $L^{p(\cdot)}(\Omega)$ has a distributional (weak) derivatives. The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : \boldsymbol{\nabla} u \in \boldsymbol{L}^{p(\cdot)}(\Omega)\},$$

where $\boldsymbol{\nabla}$ is the gradient operator, equipped with the norm

$$\|u\|_{W^{1,p(\cdot)}(\Omega)} = \inf\left\{\lambda > 0 : \int_\Omega \left(\left|\frac{u(x)}{\lambda}\right|^{p(x)} + \left|\frac{\boldsymbol{\nabla} u(x)}{\lambda}\right|^{p(x)}\right) dx \leq 1\right\}.$$

Define

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N, \end{cases}$$

and

$$p^\partial(x) = \begin{cases} \frac{(N-1)p(x)}{N-p(x)} & \text{if } p(x) < N, \\ \infty & \text{if } p(x) \geq N. \end{cases}$$

Proposition 2.4. (i) *The spaces $L^{p(\cdot)}(\Omega)$ and $W^{1,p(\cdot)}(\Omega)$ are separable, reflexive and uniformly convex Banach spaces.*

(ii) *If $q(x) \in C_+(\overline{\Omega})$ satisfies $q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Omega)$ is compact and continuous.*

(iii) *If $q(x) \in C_+(\Gamma)$ satisfies $q(x) < p^\partial(x)$ for all $x \in \Gamma$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{q(\cdot)}(\Gamma)$ is well defined and compact and continuous. In particular, if $p \in C_+(\overline{\Omega})$, then the trace mapping $W^{1,p(\cdot)}(\Omega) \to L^{p(\cdot)}(\Gamma)$ is compact and continuous and there exists a constant $C > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Gamma)} \leq C\|u\|_{W^{1,p(\cdot)}(\Omega)} \text{ for } u \in W^{1,p(\cdot)}(\Omega).$$

For $p \in C_+(\overline{\Omega})$, define

$$L^{p(\cdot)}(\Gamma_1) = \{v : v : \Gamma_1 \to \mathbb{R} \text{ is measurable with respect to } d\sigma \\ \text{and there exists } u \in L^{p(\cdot)}(\Gamma) \text{ such that } u = v \text{ on } \Gamma_1\}$$

with the norm

$$\|v\|_{L^{p(\cdot)}(\Gamma_1)} = \inf\{\|u\|_{L^{p(\cdot)}(\Gamma)} : u \in L^{p(\cdot)}(\Gamma) \text{ and } u = v \text{ on } \Gamma_1\}.$$

Clearly, the restriction mapping $L^{p(\cdot)}(\Gamma) \to L^{p(\cdot)}(\Gamma_1)$ is continuous, so the embeddings

$$W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Gamma) \hookrightarrow L^{p(\cdot)}(\Gamma_1)$$

are continuous and there exists a constant $C > 0$ such that

$$\|v\|_{L^{p(x)}(\Gamma_1)} \leq \|v\|_{L^{p(\cdot)}(\Gamma)} \leq C\|v\|_{W^{1,p(\cdot)}(\Omega)} \text{ for all } v \in W^{1,p(\cdot)}(\Omega).$$

Define a space

$$X = \{v \in W^{1,p(\cdot)}(\Omega) : v = 0 \text{ on } \Gamma_1\}.$$

Then it is clear to see that X is a closed subspace of $W^{1,p(\cdot)}(\Omega)$, so X is a reflexive and separable, uniformly convex Banach space. We show the following Poincaré type inequality.

Lemma 2.5. *There exists a constant $C = C(\Omega, N, p(\cdot)) > 0$ such that*

$$\|u\|_{L^{p(\cdot)}(\Omega)} \leq C\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for all } u \in X.$$

Proof. If the conclusion is false, then there exists a sequence $\{u_n\} \subset X$ such that $\|u_n\|_{L^{p(\cdot)}(\Omega)} = 1$ and $1 > n\|\nabla u_n\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}$. Since $\|u_n\|_{L^{p(\cdot)}(\Omega)} = 1$ and $\nabla u_n \to \mathbf{0}$ strongly in $\boldsymbol{L}^{p(\cdot)}(\Omega)$, $\{u_n\}$ is bounded in $W^{1,p(\cdot)}(\Omega)$. Therefore, by the fact that X is a reflexive Banach space, there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ and $u \in X$ such that $u_{n'} \to u$ weakly in $W^{1,p(\cdot)}(\Omega)$ and in $L^{p(\cdot)}(\Omega)$. Thus $u_{n'} \to u$ in $\mathcal{D}(\Omega)$, so $\nabla u_{n'} \to \nabla u$ in $\boldsymbol{\mathcal{D}}'(\Omega)$. Since $\nabla u_{n'} \to \mathbf{0}$

in $\boldsymbol{L}^{p(\cdot)}(\Omega)$, $\boldsymbol{\nabla} u = \boldsymbol{0}$ in $\boldsymbol{\mathcal{D}}'(\Omega)$. Therefore $u = \text{const.}$ (cf. Boyer and Fabrie [8, Lemma II.2.44]). As $u = 0$ on $\Gamma_1 (\neq \emptyset)$, we have $u = 0$. Thus $u_{n'} \to 0$ weakly in $W^{1,p(\cdot)}(\Omega)$. Since $p(x) < p^*(x)$ for all $x \in \overline{\Omega}$, the embedding mapping $W^{1,p(\cdot)}(\Omega) \hookrightarrow L^{p(\cdot)}(\Omega)$ is compact, so $u_{n'} \to 0$ strongly in $L^{p(\cdot)}(\Omega)$. This contradicts $\|u_{n'}\|_{L^{p(\cdot)}(\Omega)} = 1$. □

Thus we can define the norm on X so that

$$\|v\|_X = \|\boldsymbol{\nabla} v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \text{ for } v \in X,$$

which is equivalent to $\|v\|_{W^{1,p(\cdot)}(\Omega)}$.

2.2. **A Carathéodory function.** Let $p \in C_+(\overline{\Omega})$ be fixed. Let $S(x,t)$ be a Carathéodory function on $\Omega \times [0,\infty)$, and assume that for a.e. $x \in \Omega$, $S(x,t) \in C^2((0,\infty)) \cap C([0,\infty))$ satisfies the following structure conditions: there exist positive constants $0 < s_* \le s^* < \infty$ such that for a.e. $x \in \Omega$

$$S(x,0) = 0 \text{ and } s_* t^{(p(x)-2)/2} \le S_t(x,t) \le s^* t^{(p(x)-2)/2} \text{ for } t > 0. \tag{2.3a}$$

$$s_* t^{(p(x)-2)/2} \le S_t(x,t) + 2tS_{tt}(x,t) \le s^* t^{(p(x)-2)/2} \text{ for } t > 0. \tag{2.3b}$$

$$S_{tt}(x,t) < 0 \text{ when } 1 < p(x) < 2$$
$$\text{and } S_{tt}(x,t) \ge 0 \text{ when } p(x) \ge 2 \text{ for } t > 0, \tag{2.3c}$$

where $S_t = \partial S/\partial t$ and $S_{tt} = \partial^2 S/\partial t^2$. We note that from (2.3a), we have

$$\frac{2}{p(x)} s_* t^{p(x)/2} \le S(x,t) \le \frac{2}{p(x)} s^* t^{p(x)/2} \text{ for } t \ge 0. \tag{2.4}$$

We introduce two examples. When $S(x,t) = \nu(x)\frac{1}{p(x)} t^{p(x)/2}$, where ν is a measurable function in Ω satisfying $0 < \nu_* \le \nu(x) \le \nu^* < \infty$ for a.e. in Ω, the function $S(x,t)$ satisfies (2.3a)-(2.3c). This example corresponds to the $p(x)$-Laplacian. As an another example, we can take

$$g(t) = \begin{cases} ae^{-1/t} + a & \text{for } t > 0, \\ a & \text{for } t = 0, \end{cases}$$

where $a > 0$ is a constant. Then we can see that $S(x,t) = \nu(x) g(t) \frac{1}{p(x)} t^{p(x)/2}$ satisfies (2.3a)-(2.3c) if $p(x) \ge 2$ for all $x \in \overline{\Omega}$, (cf. Aramaki [4]).

We have the following strict monotonicity of S_t.

Lemma 2.6. ([5, Lemma 3.6]) *There exists a constant $c > 0$ depending only on s_* and p^+ such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^N$,*

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big) \cdot (\boldsymbol{a} - \boldsymbol{b}) \ge \begin{cases} c|\boldsymbol{a}-\boldsymbol{b}|^{p(x)} & \textit{when } p(x) \ge 2, \\ c(|\boldsymbol{a}| + |\boldsymbol{b}|)^{p(x)-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \textit{when } 1 < p(x) < 2. \end{cases}$$

In particular,

$$\big(S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}\big)\cdot(\boldsymbol{a}-\boldsymbol{b})>0 \ \textit{if}\ \boldsymbol{a}\neq\boldsymbol{b}.$$

Lemma 2.7. ([3]) *There exists a constant $C>0$ depending only on s^* and p^- such that for any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^N$,*

$$|S_t(x,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,|\boldsymbol{b}|^2)\boldsymbol{b}|\leq\begin{cases} C|\boldsymbol{a}-\boldsymbol{b}|^{p(x)-1} & \textit{when } 1<p(x)<2,\\ C(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p(x)-2}|\boldsymbol{a}-\boldsymbol{b}| & \textit{when } p(x)\geq 2.\end{cases}$$

Lemma 2.8. *The function $T(x,t)=\frac{1}{2}S(x,t^2)$ defined in $\Omega\times[0,\infty)$ is uniformly convex, that is, for any $\varepsilon>0$, there exists a constant $\delta>0$ such that*

$$|t-s|\leq\varepsilon\max\{t,s\}$$

or

$$T\left(x,\frac{t+s}{2}\right)\leq(1-\delta)\frac{T(x,t)+T(x,s)}{2}$$

for a.e. $x\in\Omega$ and all $t,s\geq 0$. In particular, the function $T(x,t)$ is convex with respect to $t\in[0,\infty)$.

Proof. Without loss of generality, we can assume that $t\geq s\geq 0$. Hence it suffieces to show that for any $0<\varepsilon<1$, there exits $0<\delta<1$ such that if $t-s>\varepsilon t$, then

$$\begin{aligned}&\frac{1}{2}\left(T(x,t)-T\left(x,\frac{t+s}{2}\right)\right)-\frac{1}{2}\left(T\left(x,\frac{t+s}{2}\right)-T(s)\right)\\ &\geq\delta\frac{T(x,t)+T(x,s)}{2}.\end{aligned}\tag{2.5}$$

Since for a.e. $x\in\Omega$, $T(x,t)$ is of class C^2 in $(0,\infty)$ with respect ot t, we have, from (2.3b),

$$\begin{aligned}T_{tt}(x,t)&=S_t(x,t^2)+2t^2S_{tt}(x,t^2)\\ &\geq s_*t^{p(x)-2}.\end{aligned}\tag{2.6}$$

Hence, using the mean value theorem,

$$\begin{aligned}&\frac{1}{2}\left(T(x,t)-T\left(x,\frac{t+s}{2}\right)\right)-\frac{1}{2}\left(T\left(x,\frac{t+s}{2}\right)-T(s)\right)\\ &=\frac{1}{2}\cdot\frac{t-s}{2}\int_0^1\left(T_t\left(x,\frac{t+s}{2}+\theta\frac{t-s}{2}\right)-T_t\left(x,s+\theta\frac{t-s}{2}\right)\right)d\theta\\ &=\frac{1}{2}\left(\frac{t-s}{2}\right)^2\int_0^1\int_0^1 T_{tt}\left(x,s+(\theta+\tau)\frac{t-s}{2}\right)d\theta d\tau\\ &\geq\frac{1}{2}\left(\frac{t-s}{2}\right)^2 s_*\int_0^1\int_0^1\left(s+(\theta+\tau)\frac{t-s}{2}\right)^{p(x)-2}d\theta d\tau.\end{aligned}\tag{2.7}$$

When $p(x) \geq 2$, we have

$$\begin{aligned}
&\int_0^1 \int_0^1 \left(s + (\theta+\tau)\frac{t-s}{2}\right)^{p(x)-2} d\theta d\tau \\
&\geq \int_0^1 \int_0^1 (\theta+\tau)^{p(x)-2} d\theta d\tau \left(\frac{t-s}{2}\right)^{p(x)-2} \\
&\geq \left(\frac{1}{2}\right)^{p^+-2} \frac{1}{(p^+-1)p^+}(2^{p^-}-2)(t-s)^{p(x)-2} \\
&\geq \left(\frac{1}{2}\right)^{p^+-2} \frac{1}{(p^+-1)p^+}(2^{p^-}-2)\varepsilon^{p^+-2} t^{p(x)-2}.
\end{aligned}$$

When $1 < p(x) < 2$, since

$$s + (\theta+\tau)\frac{t-s}{2} \leq s + 2\frac{t-s}{2} = t,$$

we have

$$\int_0^1 \int_0^1 \left(s + (\theta+\tau)\frac{t-s}{2}\right)^{p(x)-2} d\theta d\tau \geq t^{p(x)-2}.$$

Thus we have

$$\begin{aligned}
&\frac{1}{2}\left(T(x,t) - T\left(x, \frac{t+s}{2}\right)\right) - \frac{1}{2}\left(T\left(x, \frac{t+s}{2}\right) - T(s)\right) \\
&\geq \begin{cases} \frac{s_*}{8}\left(\frac{1}{2}\right)^{p^+-2} \frac{1}{(p^+-1)p^+}(2^{p^-}-2)\varepsilon^{p^+} t^{p(x)} & \text{if } p(x) \geq 2, \\ \frac{s_*}{8}\varepsilon^2 t^{p(x)} & \text{if } 1 < p(x) < 2. \end{cases}
\end{aligned}$$

On the other hand, since $s < (1-\varepsilon)t$, using (2.4), we have

$$\frac{T(x,t) + T(x,s)}{2} \leq \frac{s^*}{p(x)}(t^{p(x)} + s^{p(x)}) \leq \frac{s^*}{p^-}\big(1 + (1-\varepsilon)^{p^-}\big)t^{p(x)}.$$

If we choose $0 < \delta < 1$ so that

$$\delta\frac{s^*}{p^-}\big(1 + (1-\varepsilon)^{p^-}\big) \leq \min\left\{\frac{s_*}{8}\left(\frac{1}{2}\right)^{p^+-2} \frac{1}{(p^+-1)p^+}(2^{p^-}-2)\varepsilon^{p^+}, \frac{s_*}{8}\varepsilon^2\right\},$$

then we can see that (2.5) holds. □

2.3. **Setting of the problem.** We consider the system (1.1). From now on we suppose the following conditions.

(f_0) $f : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies

$$|f(x,t)| \leq C_1 + C_2|t|^{\alpha(x)-1} \text{ for a.e } x \in \Omega \text{ and all } t \in \mathbb{R},$$

where C_1 and C_2 are non-negative constants, $\alpha \in C_+(\overline{\Omega})$ and $\alpha(x) < p^*(x)$ for all $x \in \overline{\Omega}$.

(g_0) $g : \Gamma_2 \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function and satisfies

$$|g(x,t)| \le D_1 + D_2|t|^{\beta(x)-1} \text{ for a.e } x \in \Gamma_2 \text{ and all } t \in \mathbb{R},$$

where D_1 and D_2 are non-negative constants, $\beta \in C_+(\overline{\Gamma_2})$ and $\beta(x) < p^{\partial}(x)$ for all $x \in \overline{\Gamma_2}$.

Define

$$F(x,t) = \int_0^t f(x,s)ds \text{ for } (x,t) \in \Omega \times \mathbb{R}, \tag{2.8}$$

$$G(x,t) = \int_0^t g(x,s)ds \text{ for } (x,t) \in \Gamma_2 \times \mathbb{R}. \tag{2.9}$$

We introduce the notion of weak solutions for the problem (1.1).

Definition 2.9. We say $u \in X$ is a weak solution of (1.1), if

$$\int_\Omega S_t(x,|\nabla u|^2)\nabla u \cdot \nabla v dx = \lambda \int_\Omega f(x,u)v dx + \mu \int_{\Gamma_2} g(x,u)v d\sigma \text{ for all } v \in X. \tag{2.10}$$

We solve the problem (1.1) by the direct method of variation. For this purpose, we consider the functional on X defined by

$$I(u) = \Phi(u) - \lambda J(u) - \mu K(u),$$

where, for $u \in X$,

$$\Phi(u) = \frac{1}{2}\int_\Omega S(x,|\boldsymbol{\nabla} u|^2)dx, \tag{2.11}$$

$$J(u) = \int_\Omega F(x,u)dx, \tag{2.12}$$

$$K(u) = \int_{\Gamma_2} G(x,u)d\sigma. \tag{2.13}$$

From Lemma 2.8, $T(x,t) = \frac{1}{2}S(x,t^2)$ is continuous and uniformly convex on $\Omega \times [0,\infty)$ and it follows from (2.4) that the function T is a generalized N-function, that is, for a.e. $x \in \Omega$,

$$\lim_{t\to 0}\frac{T(x,t)}{t} = 0 \text{ and } \lim_{t\to\infty}\frac{T(x,t)}{t} = \infty.$$

Hence Φ is a positive and uniformly convex modular (cf. [9, Theorem 2.4.11]), that is, for any $\varepsilon > 0$, there exists a constant $\delta > 0$ such that

$$\Phi\left(\frac{u-v}{2}\right) \le \varepsilon\frac{\Phi(u)+\Phi(v)}{2} \quad \text{or} \quad \Phi\left(\frac{u+v}{2}\right) \le (1-\delta)\frac{\Phi(u)+\Phi(v)}{2}$$

for all $u, v \in X$. The modular space and the Luxemburg norm associated with Φ are defined by

$$X_\Phi = \{u \in X : \lim_{\tau \to 0} \Phi(\tau u) = 0\}$$

and

$$\|u\|_\Phi = \inf \left\{ \tau > 0 : \Phi\left(\frac{u}{\tau}\right) \leq 1 \right\} \text{ for } u \in X_\Phi.$$

Clearly we see that $X_\Phi = X$.

Lemma 2.10. *There exist positive constants c and C depending only on s_*, s^*, p^- and p^+ such that*

$$c\|u\|_X \leq \|u\|_\Phi \leq C\|u\|_X \textit{ for all } u \in X.$$

Proof. By (2.4), we have

$$\int_\Omega \frac{s_*}{p(x)} \left|\frac{\nabla u}{\tau}\right|^{p(x)} dx \leq \Phi\left(\frac{u}{\tau}\right) = \frac{1}{2}\int_\Omega S\left(x, \left|\frac{\nabla u}{\tau}\right|^2\right) dx \leq \int_\Omega \frac{s^*}{p(x)} \left|\frac{\nabla u}{\tau}\right|^{p(x)} dx,$$

so

$$\frac{s_*}{p^+}\int_\Omega \left|\frac{\nabla u}{\tau}\right|^{p(x)} dx \leq \Phi\left(\frac{u}{\tau}\right) \leq \frac{s^*}{p^-}\int_\Omega \left|\frac{\nabla u}{\tau}\right|^{p(x)} dx.$$

Therefore, there exists $0 < c < 1$ and $C > 1$ such that

$$c\int_\Omega \left|\frac{\nabla u}{\tau}\right|^{p(x)} dx \leq \Phi\left(\frac{u}{\tau}\right) \leq C\int_\Omega \left|\frac{\nabla u}{\tau}\right|^{p(x)} dx. \tag{2.14}$$

Since $p(x) > 1$, we have $c^{p(x)} \leq c$ and $C \leq C^{p(x)}$. Thus we have

$$c\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)} \leq \|u\|_\Phi \leq C\|\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}.$$

□

Lemma 2.11. *If $u_n \to u$ weakly in X and $\Phi(u_n) \to \Phi(u)$ as $n \to \infty$, then $u_n \to u$ strongly in X.*

Proof. If $u_n \to u$ weakly in X, then clearly $u_n \to u$ weakly in X_Φ. Then it follows from [9, Lemma 2.4.17] that

$$\Phi\left(\frac{u_n - u}{2}\right) \to 0 \text{ as } n \to \infty.$$

From Lemma 2.10, $u_n \to u$ strongly in X. □

2.4. **The properties of the functionals Φ, J and K.** In this subsection, we give some basic properties of the functionals Φ, J and K defined by (2.11), (2.12) and (2.13).

Proposition 2.12. *Let $p \in C_+(\overline{\Omega})$. Assume that functions f and g satisfy (f_0) and (g_0), respectively. Then we can see that the following properties are satisfied.*

(i) *We can see that $\Phi, J, K \in C^1(X, \mathbb{R})$.*
(ii) *The functional Φ is a uniformly convex modular on X, sequentially weakly lower semi-continuous, coercive on X, that is,*

$$\lim_{\|u\|_X \to \infty} \frac{\Phi(u)}{\|u\|_X} = \infty,$$

and bounded on every bounded subset of X. The mapping $\Phi' : X \to X^*$ is a strictly monotone, bounded on each bounded subset of X, homeomorphism and of (S_+)-type, that is, if $u_n \to u$ weakly in X and $\limsup_{n\to\infty} \langle \Phi'(u_n), u_n - u\rangle \le 0$, then $u_n \to u$ strongly in X.
(iii) *The mappings $J', K' : X \to X^*$ are sequentially weakly-strongly continuous, namely, if $u_n \to u$ weakly in X, then $J'(u_n) \to J'(u)$ and $K'(u_n) \to K'(u)$ strongly in X^*, so the functionals $J, K : X \to \mathbb{R}$ are sequentially weakly continuous,*

Proof. (i) Clearly Φ is Gâteau differentiable at every $u \in X$ and for any $v \in X$, the Gâteau differential $d\Phi$ is written by

$$d\Phi(u)(v) = \int_\Omega S_t(x, |\nabla u|^2)\nabla u \cdot \nabla v dx.$$

We show the continuity of $d\Phi$. Let $u_n \to u$ in X. By Lemma 2.7, we have

$$\begin{aligned} |\big(d\Phi(u_n) - d\Phi(u)\big)(v)| &= \left|\int_\Omega \big(S_t(x, |\nabla u_n|^2)\nabla u_n - S_t(x, |\nabla u|^2)\nabla u\big) \cdot \nabla v dx\right| \\ &\le C(I_1 + I_2), \end{aligned}$$

where

$$\begin{aligned} I_1 &= \int_{\Omega_1} |\nabla u_n - \nabla u|^{p(x)-1} |\nabla v| dx, \\ I_2 &= \int_{\Omega_2} (|\nabla u_n| + |\nabla u|^{p(x)-2}) |\nabla u_n - \nabla u| |\nabla v| dx, \end{aligned}$$

$\Omega_1 = \{x \in \Omega : 1 < p(x) \le 2\}$ and $\Omega_2 = \{x \in \Omega : p(x) > 2\}$. By the Hölder inequality (2.1),

$$I_1 \le 2\| |\nabla u_n - \nabla u|^{p(x)-1} \|_{L^{p'(\cdot)}(\Omega_1)} \|v\|_X.$$

Here we note that from Proposition 2.1,

$$\begin{aligned}
&\||\nabla u_n-\nabla u|^{p(x)-1}\|_{L^{p'(\cdot)}(\Omega_1)}\\
&\le \rho_{p'(\cdot)}(|\nabla u_n-\nabla u|^{p(x)-1})^{1/p^-}\vee\rho_{p'(\cdot)}(|\nabla u_n-\nabla u|^{p(x)-1})^{1/p^+}\\
&=\rho_{p(\cdot)}(|\nabla u_n-\nabla u|)^{1/p^-}\vee\rho_{p(\cdot)}(|\nabla u_n-\nabla u|)^{1/p^+}\\
&\to 0 \text{ as } n\to\infty.
\end{aligned}$$

Since $\frac{1}{p(x)/(p(x)-2)}+\frac{1}{p(x)}+\frac{1}{p(x)}$ on Ω_2, we use the Hölder inequality (2.2). Thus we have

$$I_2\le 3\|(|\nabla u_n|+|\nabla u|)^{p(x)-2}\|_{L^{p(x)/(p(x)-2)}(\Omega_2)}\|\nabla u_n-\nabla u\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)}\|v\|_X.$$

Here from [9, Corollary 2.1.15],

$$\|(|\nabla u_n|+|\nabla u|)^{p(x)-2}\|_{L^{p(x)/(p(x)-2)}(\Omega_2)}\le\int_\Omega(|\nabla u_n|+|\nabla u|)^{p(x)}dx+1.$$

Since from Proposition 2.1 (v), the right-hand side of the above inequality is bounded. Summing up the above inequalities, we have $\|d\Phi(u_n)-d\Phi(u)\|_{X^*}\to 0$ as $n\to\infty$. That is, $d\Phi$ is continuous, so Φ is Fréchet differentiable and has continuous derivative $\Phi'=d\Phi$. That $J,K\in C^1(X,\mathbb{R})$ follows from [13] or Ji [17, Proposition 2.5].

(ii) We already showed that Φ is a uniformly convex modular. From (2.14) with $\tau=1$, we can see that Φ is coercive and bounded on every bounded subset of X.

Since $[0,\infty)\ni t\mapsto S(x,t^2)$ is convex from (2.3b). the functional Φ is convex. Since Φ is continuous and uniformly convex, the functional Φ is sequentially weakly lower semi-continuous on X.

We show that $\Phi':X\to X^*$ is bounded on every bouded subset of X. Let $\|u\|_X\le M$. Then $\rho_{p(\cdot)}(|\nabla u|)\le M_1$ for some constant M_1. By the Hölder inequality (2.1),

$$\begin{aligned}
|\langle\Phi'(u),v\rangle|&=\left|\int_\Omega S_t(x,|\nabla u|^2)\nabla u\cdot\nabla v dx\right|\\
&\le 2\|S_t(x,|\nabla u|^2)\nabla u\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}\|\nabla v\|_{\boldsymbol{L}^{p(\cdot)}(\Omega)},\forall v\in X,
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality bracket between X^* and X. Hence

$$\begin{aligned}
\|\Phi'(u)\|_{X^*}&\le 2\|S_t(x,|\nabla u|^2)\nabla u\|_{\boldsymbol{L}^{p'(\cdot)}(\Omega)}\\
&\le 2\rho_{p'(\cdot)}(S_t(x,|\nabla u|^2)\nabla u)^{1/(p')^-}\vee\rho_{p'(\cdot)}(S_t(x,|\nabla u|^2)\nabla u)^{1/(p')^+}.
\end{aligned}$$

Here it suffices to note that

$$\begin{aligned}\rho_{p'(\cdot)}(S_t(x,|\nabla u|^2)\nabla u) &= \int_\Omega |S_t(x,|\nabla u|^2)\nabla u|^{p'(x)}dx \\ &\le \int_\Omega \big(s^*|\nabla u|^{p(x)-1}\big)^{p'(x)}dx \\ &\le \big((s^*)^{(p')^-}\vee(s^*)^{(p')^+}\big)\int_\Omega |\nabla u|^{p(x)}dx \\ &\le \big((s^*)^{(p')^-}\vee(s^*)^{(p')^+}\big)M_1.\end{aligned}$$

According to Aramaki [2, Proposition 2.9], Φ' is of (S_+)-type.

From Lemma 2.6, we can see that Φ' is strictly monotone. Since

$$\langle\Phi'(u),u\rangle = \int_\Omega S_t(x,|\nabla u|^2)|\nabla u|^2dx \ge s_*\int_\Omega |\nabla u|^{p(x)}dx \ge s_*\|u\|_X^{p^-}$$

for $\|u\|_X > 1$ and $p^- > 1$, we have

$$\frac{\langle\Phi'(u),u\rangle}{\|u\|_X}\to\infty \text{ as } \|u\|_X\to\infty.$$

Thus Φ' is coercive.

We show that $\Phi' : X \to X^*$ is a homeomorphism. The mapping Φ' is coercive and clearly hemi-continuous, that is, for any $u,v,w\in X$, $[0,1]\ni\tau\mapsto\langle\Phi'(u+\tau v),w\rangle$ is continuous. Since Φ' is strictly monotone, $\Phi' : X\to X^*$ is injective. By the Minty-Browder theorem, we see that Φ' is surjective. Thus $(\Phi')^{-1} : X^*\to X$ exists. Let $f_n\to f$ in X^*. Define $u_n = (\Phi')^{-1}f_n, u = (\Phi')^{-1}f$. Then $\Phi'(u_n) = f_n, \Phi'(u) = f$. We derive that $\{u_n\}$ is bounded in X. In fact, if $\{u_n\}$ is unbounded, then there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\|u_{n'}\|_X\to\infty$ as $n'\to\infty$. Hence, for some constant $C > 0$,

$$\langle\Phi'(u_{n'}),u_{n'}\rangle = \langle f_{n'},u_{n'}\rangle \le \|f_{n'}\|_{X^*}\|u_{n'}\|_X \le C\|u_{n'}\|_X.$$

This contradicts the coerciveness of Φ'.

Since $\{u_n\}$ is bounded and X is reflexive Banach space, there exists a subsequence $\{u_{n''}\}$ of $\{u_n\}$ such that $u_{n''}\to u_0$ weakly in X for some $u_0\in X$. Therefore, we have

$$\begin{aligned}\lim_{n''\to\infty}\langle\Phi'(u_{n''})-\Phi'(u_0),u_{n''}-u_0\rangle &= \lim_{n''\to\infty}\langle\Phi'(u_{n''}),u_{n''}-u_0\rangle \\ &= \lim_{n''\to\infty}\langle f_{n''},u_{n''}-u_0\rangle = 0.\end{aligned}$$

Since Φ' is of (S_+)-type, we can see that $u_{n''}\to u_0$ strongly in X. Since Φ' is continuous, $\Phi'(u_{n''}) = f_{n''}\to f = \Phi'(u_0)$. Thus $\Phi'(u) = \Phi'(u_0)$. Since Φ' is injective, $u = u_0$. By the convergent principle, the full sequence $u_n\to u$ strongly in X, that is, $(\Phi')^{-1}f_n\to(\Phi')^{-1}f$ as $n\to\infty$. Hence $(\Phi')^{-1}$ is continuous.

(iii) All the other properties on J and K are well known (cf. [13], [17, Proposition 2.5]). $\square$

Since I is Fréchet differentiable at every $u \in X$ and

$$\langle I'(u), v\rangle = \int_\Omega S_t(x, |\boldsymbol{\nabla} u|^2)\boldsymbol{\nabla} u \cdot \boldsymbol{\nabla} v dx - \lambda \int_\Omega f(x,u)v dx \\ - \mu \int_{\Gamma_2} g(x,u)v d\sigma \text{ for } v \in X. \tag{2.15}$$

Thus if $u \in X$ is a critical point of I, that is, $I'(u) = 0$, then u satisfies (2.10), so u is a weak solution of (1.1).

3. Main results on the existence of three weak solutions

In this section, we derive the existence of three weak solutions to problem (1.1). In order to do so, we define a class of functionals needed in the proposition 3.2 below.

In general, if X is a real Banach space, we denote $\mathcal{W}_X$ by the class of all functionals $\Phi : X \to \mathbb{R}$ possessing the following property: if a sequence $\{u_n\}$ satisfies that $u_n \to u$ weakly in X and $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$, then the sequence $\{u_n\}$ has a subsequence converging strongly to u in X.

We show that the functional Φ defined by (2.11) belongs to $\mathcal{W}_X$.

Lemma 3.1. *If $p \in C_+(\overline{\Omega})$, then the functional Φ defined by* (2.11) *belongs to $\mathcal{W}_X$.*

Proof. Let $u_n \to u$ weakly in X and $\liminf_{n\to\infty} \Phi(u_n) \le \Phi(u)$. Since Φ is sequentially weakly lower semi-continuous, we have $\Phi(u) \le \liminf_{n\to\infty} \Phi(u_n)$. Thus $\liminf_{n\to\infty} \Phi(u_n) = \Phi(u)$. Hence there exists a subsequence $\{u_{n'}\}$ of $\{u_n\}$ such that $\lim_{n'\to\infty} \Phi(u_{n'}) = \liminf_{n\to\infty} \Phi(u_n) = \Phi(u)$. Since Φ is a uniformly convex modular, it follows from [9, Lemma 2.4.17] that $\Phi\left(\frac{u_{n'}-u}{2}\right) \to 0$ as $n' \to \infty$. It follows from Lemma 2.11 that $\|u_{n'} - u\|_X \to 0$ as $n' \to \infty$. $\square$

We apply the following result of [20, Theorem 2].

Proposition 3.2. *Let X be a separable, reflexive and real Banach space. Assume that a functional $\Phi : X \to \mathbb{R}$ is coercive, sequentially weakly lower semi-continuous, of C^1-functional belonging to $\mathcal{W}_X$, bounded on each bounded subset of X and the derivative $\Phi' : X \to X^*$ admits a continuous inverse on X^*. Moreover, assume that $J : X \to \mathbb{R}$ is a C^1-functional with compact derivative, and assume that Φ has a strictly local minimum u_0 with $\Phi(u_0) = J(u_0) = 0$. Finally, put*

$$\alpha = \max\left\{0, \limsup_{\|u\|\to\infty} \frac{J(u)}{\Phi(u)}, \limsup_{u\to u_0} \frac{J(u)}{\Phi(u)}\right\},$$

$$\beta = \sup_{u \in \Phi^{-1}((0,\infty))} \frac{J(u)}{\Phi(u)},$$

and assume that $\alpha < \beta$. *Then for each compact interval* $[a,b] \subset \left(\frac{1}{\beta}, \frac{1}{\alpha}\right)$ (*with the conventions* $\frac{1}{0} = \infty, \frac{1}{\infty} = 0$), *there exists* $r > 0$ *with the following property: for every* $\lambda \in [a,b]$ *and every* C^1 *functional* $K : X \to \mathbb{R}$ *with compact derivative, there exists* $\delta > 0$ *such that for each* $\mu \in [0,\delta]$, *the equation* $\Phi'(u) = \lambda J'(u) + \mu K'(u)$ *has at least three solutions whose norms are less than* r.

We can obtain the following main theorem.

Theorem 3.3. *Let* Ω *be a bounded domain with a* $C^{0,1}$*-boundary* Γ *satisfying* (1.2) *and* $p \in C_+(\overline{\Omega})$ *verifying*

$$p^+ - p^- < \frac{p^+ p^-}{N} \quad \text{if } p^- < N. \tag{3.1}$$

Assume that a function f *satisfies* (f_0) *and define the function* F *by* (2.8). *Moreover, suppose that*

$$\max\left\{\limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F(x,t)}{|t|^{p^+}}, \limsup_{|t|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} F(x,t)}{|t|^{p^-}}\right\} \le 0 \tag{3.2}$$

and

$$\sup_{u\in X} \int_\Omega F(x,u(x))dx > 0. \tag{3.3}$$

Set

$$\theta = \inf\left\{\frac{\frac{1}{2}\int_\Omega S(x,|\nabla u(x)|^2)dx}{\int_\Omega F(x,u(x))dx} : u \in X \text{ with } \int_\Omega F(x,u(x))dx > 0\right\}.$$

Then for each compact interval $[a,b] \subset (\theta,\infty)$, *there exists* $r > 0$ *with the following property: for every* $\lambda \in [a,b]$ *and every function* g *satisfying* (g_0), *there exists* $\delta > 0$ *such that for each* $\mu \in [0,\delta]$, *problem* (1.1) *has at least three weak solutions whose norms are less than* r.

Proof. We apply Proposition 3.2. Define the functionals Φ and J on X by (2.11) and (2.12). According to Proposition 2.12, we see that $\Phi, J \in C^1(X,\mathbb{R})$, Φ is coercive and bounded on each bounded subset of X. Moreover, Φ is sequentially weakly lower semi-continuous and $\Phi' : X \to X^*$ admits a continuous inverse, and $J' : X \to X^*$ is compact and continuous. From Lemma 3.1, we can see that $\Phi \in \mathcal{W}_X$. Clearly, Φ has a strictly local minimum at $u_0 = 0$ with $\Phi(0) = J(0) = 0$.

Fix $\varepsilon > 0$. From (3.2), there exist ρ_1 and ρ_2 with $0 < \rho_1 < 1 < \rho_2$ such that

$$F(x,t) \le \varepsilon |t|^{p^+} \text{ for all } (x,t) \in \Omega \times [-\rho_1, \rho_1], \tag{3.4}$$

$$F(x,t) \le \varepsilon |t|^{p^-} \text{ for all } (x,t) \in \Omega \times \big(\mathbb{R} \setminus [-\rho_2, \rho_2]\big). \tag{3.5}$$

Thus we have

$$F(x,t) \le \varepsilon |t|^{p^+} \text{ for all } (x,t) \in \Omega \times \big(\mathbb{R} \setminus ([-\rho_2, -\rho_1] \cup [\rho_1, \rho_2])\big).$$

On the other hand, since f satisfies (f_0), we have

$$|F(x,t)| \le C_1|t| + \frac{C_2}{\alpha(x)}|t|^{\alpha(x)} \le C_1|t| + \frac{C_2}{\alpha^-}|t|^{\alpha(x)}.$$

Hence F is bounded on each bounded subset of $\Omega \times \mathbb{R}$. The hypothesis (3.1) means that

$$p^+ < \frac{Np^-}{N-p^-} \le \frac{Np(x)}{N-p(x)} = p^*(x) \;\text{ if }\; p(x) < N.$$

If we choose $q \in \mathbb{R}$ such that $p^+ < q < p^*(x)$ for all $x \in \overline{\Omega}$, then we have

$$F(x,t) \le \varepsilon |t|^{p^+} + c|t|^q \text{ for all } (x,t) \in \Omega \times \mathbb{R} \tag{3.6}$$

for some constant $c > 0$. Since the embeddings $X \hookrightarrow L^{p^+}(\Omega)$ and $X \hookrightarrow L^q(\Omega)$ are continuous, there exist positive constants $C_{p(\cdot),p^+}$ and $C_{p(\cdot),q}$ such that

$$\left(\int_\Omega |u|^{p^+} dx\right)^{1/p^+} \le C_{p(\cdot),p^+}\|u\|_X \text{ and } \left(\int_\Omega |u|^q dx\right)^{1/q} \le C_{p(\cdot),q}\|u\|_X$$

for all $u \in X$. Thus, there exists a constant $c_1 > 0$ such that

$$J(u) = \int_\Omega F(x,u)dx \le \varepsilon \int_\Omega |u|^{p^+} dx + c\int_\Omega |u|^q dx \le C^{p^+}_{p(\cdot),p^+}\varepsilon \|u\|_X^{p^+} + c_1\|u\|_X^q.$$

When $\|u\|_X = ||\nabla u||_{\boldsymbol{L}^{p(\cdot)}(\Omega)} < 1$, it follows from Proposition 2.1 that

$$\frac{J(u)}{\Phi(u)} \le \frac{C^{p^+}_{p(\cdot),p^+}\varepsilon\|u\|_X^{p^+} + c_1\|u\|_X^q}{\frac{s_*}{p^+}\|u\|_X^{p^+}}.$$

Hence, since $q > p^+$, we have

$$\limsup_{u \to 0} \frac{J(u)}{\Phi(u)} \le \frac{p^+}{s_*} C^{p^+}_{p(\cdot),p^+}\varepsilon. \tag{3.7}$$

On the other hand, since the embedding $X \hookrightarrow L^{p^-}(\Omega)$ is continuous, there exists a constant $C_{p(\cdot),p^-} > 0$ such that

$$\left(\int_\Omega |u|^{p^-} dx\right)^{1/p^-} \le C_{p(\cdot),p^-}\|u\|_X, \;\forall\, u \in X.$$

Since F is bounded on each bounded subset of $\Omega \times \mathbb{R}$, if $\|u\|_X > 1$, then it follows from (3.5) that there exists a constant $C_1 > 0$ such that

$$\begin{aligned}
\frac{J(u)}{\Phi(u)} &= \frac{\int_{\{x\in\Omega;|u(x)|\le\rho_2\}} F(x,u)dx + \int_{\{x\in\Omega;|u(x)|>\rho_2\}} F(x,u)dx}{\frac{1}{2}\int_\Omega S(x,|\nabla u|^2)dx} \\
&\le \frac{p^+}{s_*}\frac{C_1 + \varepsilon\int_\Omega |u|^{p^-}dx}{\|u\|_X^{p^-}} \\
&\le \frac{p^+}{s_*}\frac{C_1 + \varepsilon C_{p(\cdot),p^-}^{p^-}\|u\|_X^{p^-}}{\|u\|_X^{p^-}}.
\end{aligned}$$

Hence

$$\limsup_{\|u\|_X\to\infty}\frac{J(u)}{\Phi(u)} \le \frac{p^+}{s_*}C_{p(x),p^-}^{p^-}\varepsilon. \tag{3.8}$$

Since $\varepsilon > 0$ is arbitrary, it follows from (3.7) and (3.8) that

$$\max\left\{\limsup_{u\to 0}\frac{J(u)}{\Phi(u)}, \limsup_{\|u\|\to\infty}\frac{J(u)}{\Phi(u)}\right\} \le 0.$$

Therefore, we have $\alpha = 0$ in Proposition 3.2. By the hypothesis (3.3), we have

$$\beta = \sup_{u\in\Phi^{-1}((0,\infty))}\frac{J(u)}{\Phi(u)} > 0.$$

Thus all the hypotheses of Proposition 3.2 hold. If we put $\theta = 1/\beta$, then the conclusion of this theorem holds. $\square$

Remark 3.4. In [17], the author insisted that there exists $q \in \mathbb{R}$ such that (3.6) holds. However, in general, (3.6) does not hold without the hypothesis (3.1).

Corollary 3.5. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ satisfying* (1.2) *and $p \in C_+(\overline{\Omega})$ satisfy* (3.1)*, and assume that a Carathéodory function f satisfies (f_0) with $\alpha^+ < p^-$ and*

$$\lim_{t\to 0}\frac{f(x,t)}{|t|^{p^+-1}} = 0 \quad \text{uniformly in} \quad x \in \Omega.$$

Moreover, assume that f satisfies $f(x,t) > 0$ for a.e $x \in \Omega$ and $0 < t \le \delta_0$ for some $\delta_0 > 0$. Then the conclusion of Theorem 3.3 *holds, that is, problem* (1.1) *has at least three weak solutions.*

Proof. For any $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x,t)| \leq \varepsilon |t|^{p^+-1}$ for $|t| < \delta$. Thus

$$F(x,t) = \int_0^t f(x,s)ds \leq \frac{\varepsilon}{p^+}|t|^{p^+} \text{ for } |t| < \delta.$$

Therefore, we have

$$\limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F(x,t)}{|t|^{p^+}} \leq \frac{\varepsilon}{p^+}.$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Omega} F(x,t)}{|t|^{p^+}} \leq 0.$$

On the other hand, since the function f satisfies (f_0),

$$F(x,t) \leq C_1|t| + \frac{C_2}{\alpha^-}|t|^{\alpha(x)} \leq C_3|t|^{\alpha^+} \text{ for a.e } x \in \Omega \text{ and } |t| > 1.$$

Since $\alpha^+ < p^-$, we have

$$\limsup_{|t|\to \infty} \frac{\operatorname{ess\,sup}_{x\in\Omega} F(x,t)}{|t|^{p^-}} \leq 0.$$

Therefore, the condition (3.2) holds.

If we choose $\phi \in C_0^\infty(\Omega)$ such that $0 \leq \phi(x) \leq \delta_0$ with $\phi \not\equiv 0$, then $\phi \in X$ and

$$F(x,\phi(x)) = \int_0^{\phi(x)} f(x,s)ds.$$

Therefore, we have $\int_\Omega F(x,\phi(x))dx > 0$, so (3.3) holds. This completes the proof. $\square$

If we exchange f for g, then we can derive the following theorem.

Theorem 3.6. *Let Ω be a bounded domain with a $C^{0,1}$-boundary Γ satisfying* (1.2) *and $p \in C_+(\overline{\Omega})$ satisfies* (3.1). *Assume that a function g satisfies (g_0) and define G by* (2.9). *Moreover, suppose that*

$$\max\left\{\limsup_{t\to 0} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G(x,t)}{|t|^{p^+}}, \limsup_{|t|\to\infty} \frac{\operatorname{ess\,sup}_{x\in\Gamma_2} G(x,t)}{|t|^{p^-}}\right\} \leq 0,$$

$$\sup_{u\in X} \int_{\Gamma_2} G(x,u(x))d\sigma > 0.$$

Set

$$\theta' = \inf\left\{\frac{\frac{1}{2}\int_\Omega S(x,|\nabla u|^2)dx}{\int_{\Gamma_2} G(x,u(x))d\sigma} : u \in X \text{ with } \int_{\Gamma_2} G(x,u(x))d\sigma > 0\right\}.$$

Then for each compact interval $[c,d] \subset (\theta', \infty)$*, there exists* $r > 0$ *with the following property: for every* $\mu \in [c,d]$ *and every function* f *satisfying* (f_0)*, there exists* $\delta > 0$ *such that for each* $\lambda \in [0,\delta]$*, problem* (1.1) *has at least three weak solutions whose norms are less than* r*.*

Corollary 3.7. *Let* Ω *be a bounded domain with a* $C^{0,1}$*-boundary* Γ *satisfying* (1.2) *and* $p \in C_+(\overline{\Omega})$ *satisfy* (3.1)*, and assume that the Carathéodory function* g *satisfies* (g_0) *with* $\beta^+ < p^-$ *and*

$$\lim_{t\to 0} \frac{g(x,t)}{|t|^{p^+-1}} = 0 \quad \textit{uniformly in} \quad x \in \Gamma_2.$$

Moreover, assume that g *satisfies* $g(x,t) > 0$ *for a.e* $x \in \Gamma_2$ *and* $0 < t \le \delta_0$ *for some* $\delta_0 > 0$*. Then the conclusion of Theorem* 3.6 *holds, that is, problem* (1.1) *has at least three weak solutions.*

Remark 3.8. In [1], the authors considered the case $\lambda = 0$. They supposed different conditions on g which seems to be more restrictive.

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