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A NOTE ON VECTOR-VALUED EISENSTEIN SERIES OF WEIGHT 3/2

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Abstract. Vector-valued Eisenstein series of weight 3/2 are often not holomorphic. In this paper we prove that, for an even lattice $\underline{L}$, if there exists an odd prime p such that $\underline{L}$ is local p-maximal and the determinant of $\underline{L}$ is divisible by p^2, then the Eisenstein series of weight 3/2 attached to the discriminant form of $\underline{L}$ is holomorphic.

1. Introduction

Vector-valued modular forms have played important roles in many mathematical and physical fields. The simplest vector-valued modular forms are Eisenstein series. In [4] Bruinier and Kuss constructed Eisenstein series of weight $k \geq 5/2$ for the Weil representation attached to discriminant forms of lattices and computed their Fourier coefficients. These coefficients involve special values of L-functions up to finite elementary Euler factors. Since their approach of constructing is invalid when the weight $k \leq 2$, Williams in [8] constructed Eisenstein series E_k by the so called *Hecke's trick.* He showed that $E_{3/2}$, the Eisenstein series of weight 3/2 are often non-holomorphic. In this paper we prove that, for an even lattice $\underline{L}$, if there exists an odd prime p such that $\underline{L}$ is local p-maximal and the determinant of $\underline{L}$ is divisible by p^2 then the Eisenstein series of weight 3/2 attached to the discriminant form of $\underline{L}$ is holomorphic.

We now fix some basic notations throughout this paper. For a complex number z and a positive integer c, we put $e_c(z) := e^{\frac{2\pi i z}{c}}$ and in particular write $e(z) := e_1(z)$ simply. Denote by $\sqrt{z}$ the principal branch of square root such that $\arg(\sqrt{z}) \in (-\frac{\pi}{2}, \frac{\pi}{2}]$, and for a half integer k, z^k stands for $\sqrt{z}^{2k}$. For a prime p, a rational number m, we denote by $\nu_p(m)$ the p-adic valuation of m. The bracket $\left(\frac{\cdot}{\cdot}\right)$ is the Kronecker symbol. For an odd integer m, put

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$\epsilon(m) := \sqrt{\left(\frac{-1}{m}\right)}$. Let χ be a Dirichlet character, t a complex number, we put $L(\chi, t) := \sum_{n\geq 1} \frac{\chi(n)}{n^t}$, the Dirichlet L-function associated with the character χ, and, for a prime p, $L_p(\chi, t) := \sum_{\nu\geq 0} \frac{\chi(p^\nu)}{p^{\nu t}}$. For a discriminant $\Delta = \Delta_0 f_\Delta^2$ with Δ_0 its fundamental discriminant, set

$$L_\Delta(t) := \begin{cases} \zeta(2t-1), & \Delta = 0; \\ L\left(\left(\frac{\Delta_0}{\cdot}\right), t\right) \sum_{d\mid f_\Delta} \mu(d) \left(\frac{\Delta_0}{d}\right) d^{-t} \sigma_{1-2t}(f_\Delta/d), & \Delta \neq 0, \end{cases}$$

and

$$L_\Delta^{(p)}(t) := \begin{cases} \dfrac{1}{1-p^{1-2t}}, & \Delta = 0; \\ \dfrac{1}{1-p^{-t}\left(\frac{\Delta_0}{\cdot}\right)} \sum_{d\mid p^{\nu_p(f_\Delta)}} \mu(d) \left(\frac{\Delta_0}{d}\right) d^{-t} \sigma_{1-2t}\left(p^{\nu_p(f_\Delta)}/d\right), & \Delta \neq 0. \end{cases}$$

Here $\sigma_t(n) := \sum_{d\mid n} d^t$.

2. Basic facts

2.1. Lattices, finite quadratic modules

Let $\underline{L} = (L, q)$ be an even lattice, i.e., a free $\mathbb{Z}$-module of finite rank $r_{\underline{L}}$, equipped with a non-degenerate symmetric $\mathbb{Z}$-valued bilinear form q such that $q(x) := q(x,x)/2$ is integer for each $x \in L$. Denote by $(b_{\underline{L}}^+, b_{\underline{L}}^-)$ its signature and define signature difference as $\operatorname{sign}(\underline{L}) := b_{\underline{L}}^+ - b_{\underline{L}}^-$. Denote by $\det(\underline{L})$ the determinant of $\underline{L}$, i.e., the determinant of the Gram matrix corresponding to any $\mathbb{Z}$-basis of L.

We define the dual of lattice $\underline{L}$ as

$$L' := \{y \in L \otimes_{\mathbb{Z}} \mathbb{Q} : q(x,y) \in \mathbb{Z} \quad \text{for all } x \in L\}.$$

Note that $|L'/L| = |\det(\underline{L})|$. For $x \in L'$, let $\omega_{\underline{L}}(x)$ be its order in L'/L. Call $x \in L'$ isotropic if $q(x) \in \mathbb{Z}$. We say that $\underline{L}$ is p-maximal for a prime p if there is no isotropic element x in L' such that $p \mid \omega_{\underline{L}}(x)$.

The discriminant form of $\underline{L}$ is defined as

$$D_{\underline{L}} := (L'/L, x + L \to q(x) + \mathbb{Z}).$$

The discriminant form of an even lattice defines a finite quadratic module. Recall that a *finite quadratic module over* $\mathbb{Z}$ is a finite $\mathbb{Z}$-module M equipped with a non-degenerate $\mathbb{Z}$-bilinear form $q : M \times M \to \mathbb{Q}/\mathbb{Z}$. Call two finite quadratic modules (M_1, q_1) and (M_2, q_2) are isomorphic if there exists a $\mathbb{Z}$-isomorphism ψ from M_1 to M_2 such that $q_1(x_1) = q_2(\psi(x_1))$ for all $x_1 \in M_1$. According to [7, Proposition 2.11], every finite quadratic module is $\mathbb{Z}$-isomorphic to a direct sum of finite quadratic modules of the following forms:

(i) $A_{p^v}^{\varepsilon_{p^\nu}} := \left(\mathbb{Z}/p^v\mathbb{Z}, \dfrac{\varepsilon_{p^\nu} x^2}{p^v}\right)$, p is odd, $\gcd(\varepsilon_{p^\nu}, p) = 1$;

(ii) $A_{2^v}^{\varepsilon_{2^\nu}} := \left(\mathbb{Z}/2^v\mathbb{Z}, \dfrac{\varepsilon_{2^\nu} x^2}{2^{v+1}}\right)$, ε_{2^ν} is odd;

(iii) $B_{2^v} := \left(\mathbb{Z}/2^v\mathbb{Z} \times \mathbb{Z}/2^v\mathbb{Z}, \dfrac{x^2 + xy + y^2}{2^v}\right)$;

(iv) $C_{2^v} := \left(\mathbb{Z}/2^v\mathbb{Z} \times \mathbb{Z}/2^v\mathbb{Z}, \dfrac{xy}{2^v}\right)$.

2.2. Vector valued modular forms

Here we review some basic facts of vector valued modular forms briefly. For further details one can refer [2].

Let $\mathrm{Mp}(2, \mathbb{Z})$ be the metaplectic group, i.e., the double cover of $\mathrm{SL}_2(\mathbb{Z})$ consisting of all pairs (M, ϕ) where $M \in \mathrm{SL}_2(\mathbb{Z})$ and ϕ is a branch of $\sqrt{c\tau + d}$ on the upper half complex plane $\mathbb{H}$. Often we omit the function ϕ. It is well known that $\mathrm{Mp}(2, \mathbb{Z})$ is generated by

$$\widetilde{T} := \left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, 1\right) \quad \text{and} \quad \widetilde{S} := \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \sqrt{\tau}\right).$$

Let $\underline{L} = (L, q)$ be an even lattice, $\{\mathbf{e}_r\}_{r \in L'/L}$ the standard basis of $\mathbb{C}[L'/L]$. The Weil representation $\rho_{\underline{L}}$ associated to the discriminant form $D_{\underline{L}}$ is defined as

$$\rho_{\underline{L}}(\widetilde{T})\mathbf{e}_r = e(q(r))\mathbf{e}_r, \quad \rho_{\underline{L}}(\widetilde{S})\mathbf{e}_r = \frac{\sqrt{i}^{\,-\operatorname{sign}(\underline{L})}}{\sqrt{|\det(\underline{L})|}} \sum_{t \in L'/L} e(-q(r, t))\mathbf{e}_t.$$

We let $\rho_{\underline{L}}^*$ be the dual representation of $\rho_{\underline{L}}$.

Let k be a half integer, f a $\mathbb{C}[L'/L]$-valued function on $\mathbb{H}$. The Petersson slash operator is defined by

$$\left(f|_k^*(M, \phi)\right)(\tau) := \phi(\tau)^{-2k} \rho_{\underline{L}}^*(M, \phi)^{-1} f(M\tau)$$

with $(M, \phi) \in \mathrm{Mp}(2, \mathbb{Z})$. Call $f : \mathbb{H} \to \mathbb{C}[L'/L]$ a modular form of weight k with respect to the representation $\rho_{\underline{L}}^*$ if f is a invariant under the actions of Petersson operators and $f(\tau)$ has the Fourier expansion of form

$$f(\tau) = \sum_{r \in L'/L} \sum_{\substack{n \geq 0 \\ n \equiv q(r) \bmod \mathbb{Z}}} c(n, r) e(n\tau)\mathbf{e}_r.$$

We denote by $M_k(\rho_{\underline{L}}^*)$ the space of modular forms of weight k associate to $\rho_{\underline{L}}^*$.

For the small weight cases we need to consider *harmonic Maass forms.* Denote by $\mathcal{H}_k(\rho_{\underline{L}}^*)$ the space of all real analytical functions $f : \mathbb{H} \to \mathbb{C}[L'/L]$ which are invariants under the Petersson slash operators of weight k, representation $\rho_{\underline{L}}^*$ and are annihilated by the k-Laplacian $\Delta_k := (\tau - \overline{\tau})^2 \partial_{\tau\overline{\tau}} + k(\tau - \overline{\tau})\partial_{\overline{\tau}}$. By [3, Proposition 3.2], for $f \in \mathcal{H}_k(\rho_{\underline{L}}^*)$, $g(\tau) = \xi_k(f)(\tau) := 2\pi y^k \overline{\frac{\partial f}{\partial \overline{\tau}}}(\tau) \in$

$\mathcal{H}_{2-k}(\rho^*_{\underline{L}(-1)})$, where $\underline{L}(-1)$ stands for the lattice $(L,-q)$. We call g the *shadow* of f.

Let $\underline{L}=(L,q)$ be an even lattice, k a half integer such that $2k+\operatorname{sign}(\underline{L})\equiv 0 \bmod 4$. For a complex number s with $2\Im(s)>2-k$, define the Eisenstein series as

$$E_{k,\underline{L}}(\tau,s):=\sum_{(M,\phi)\in\widetilde{\Gamma_\infty}\backslash \mathrm{Mp}(2,\mathbb{Z})}\left(y^s\mathbf{e}_0\right)\Big|_k^*(M,\phi),$$

where $\Gamma_\infty=\{\pm\left[\begin{smallmatrix}1&n\\0&1\end{smallmatrix}\right]:n\in\mathbb{Z}\}$, $y=\Im(\tau)$. The series $E_{k,\underline{L}}(\tau,s)$ transforms like a modular form of weight k and the representation $\rho^*_{\underline{L}}$ for fixed s. We write $E_{k,\underline{L}}(\tau):=E_{k,\underline{L}}(\tau,0)$ simply.

3. Statement and proof

The main goal of this section is proving the following theorem:

Theorem 3.1. *Let $\underline{L}=(L,q)$ be an even lattice satisfying* $\operatorname{sign}(\underline{L})\equiv 1 \bmod 4$. *If there exists an odd prime p such that $\underline{L}$ is p-maximal and $p^2\mid 2\det(\underline{L})$, then $E_{3/2,\underline{L}}(\tau)$ is holomorphic.*

Remark 3.2. For an even lattice $\underline{L}$, the Eisenstein series $E_{3/2,\underline{L}}(\tau)$ is holomorphic if and only if its shadow is zero in $M_{\frac{1}{2}}(\rho^*_{\underline{L}(-1)})$. For example let $\underline{L}=\underline{\mathbb{Z}}(3)\oplus\underline{\mathbb{Z}}(3)\oplus\underline{\mathbb{Z}}(-1)$ where $\underline{\mathbb{Z}}(m):=(\mathbb{Z},mx^2)$ for integer m. Obviously $3^2|\det(\underline{L})$ and $\underline{L}$ is 3-maximal. One can check that $M_{\frac{1}{2}}(\rho^*_{\underline{L}(-1)})=\{0\}$ hence $E_{3/2,\underline{L}}(\tau)$ defines a modular form.

Proof. According to 5 of [8] we can write the Fourier expansion of $E_{3/2,\underline{L}}(\tau)$ as

$$\begin{aligned}E_{3/2,\underline{L}}(\tau)=&\sum_{\substack{r\in L'/L\\ q(r)\in\mathbb{Z}}}\sum_{\substack{n\geq 0\\ n\equiv q(r) \bmod \mathbb{Z}}}c(n,r)e(n\tau)\mathbf{e}_r\\ &+\frac{C}{\sqrt{y}}\sum_{\substack{r\in L'/L\\ q(r)\in\mathbb{Z}}}\prod_{l|2\det(\underline{L})}\lim_{s\to 0}L_l\left(\gamma_{\underline{L},q(r),r},(3+r_{\underline{L}})/2+2s\right)\mathbf{e}_r\\ &+\sum_{r\in L'/L}\sum_{\substack{n<0\\ n\equiv q(r) \bmod \mathbb{Z}\\ \mathcal{N}_r \text{ is square}}}a(n,r)\frac{\beta(4\pi|n|y)}{\sqrt{y}}\\ &\times\prod_{l|2\det(\underline{L})}\lim_{s\to 0}L_l\left(\gamma_{\underline{L},n,r},(3+r_{\underline{L}})/2+2s\right)e(n\tau)\mathbf{e}_r,\end{aligned}$$

where l refers prime, C is a constant, $\beta(t)=\frac{1}{16\pi}\int_1^\infty u^{-3/2}e^{-tu}du$, $\mathcal{N}_r:=-2|\det(\underline{L})|\omega_{\underline{L}}(r)^2n$,

$$L_l(\gamma_{\underline{L},n,r},t):=\sum_{\nu\geq 0}\frac{\gamma_{\underline{L},n,r}(l^\nu)}{l^{\nu t}}$$

with

$$\gamma_{\underline{L},n,r}(l^\nu) := \sum_{\substack{d \bmod l^\nu \\ \gcd(d,l)=1}} \sum_{x \in L/l^\nu L} e_{l^\nu}\left(d(q(x+r)+n)\right). \tag{1}$$

The local series $L_l\left(\gamma_{\underline{L},n,r},(3+r_{\underline{L}})/2+2s\right)$ is holomorphic at $s=0$ for each prime $l \mid 2\det(\underline{L})$. It is sufficient to show that

$$L_p\left(\gamma_{\underline{L},n,r},(3+r_{\underline{L}})/2\right)=0 \quad \text{if } \mathcal{N}_r \text{ is square.} \tag{2}$$

Let $(D_{\underline{L}})_p$ be the p-component of $D_{\underline{L}}$, i.e., the p-component of the Abelian group L'/L equipped with the same quadratic form. We assert that

$$(D_{\underline{L}})_p \approx A_p^1 \oplus A_p^{\varepsilon_p} \quad \text{with } \left(\frac{-\varepsilon_p}{p}\right) = -1. \tag{3}$$

From Section 2.1 one has

$$(D_{\underline{L}})_p \approx \bigoplus_{j>0}^{m} \left(\underbrace{A_{p^j}^1 \oplus \cdots \oplus A_{p^j}^1}_{h_{p^j}} \oplus A_{p^j}^{\varepsilon_{p^j}} \right). \tag{4}$$

Note that

$$\sum_{j=1}^{m}(h_{p^j}+1) = \nu_p\left(2\det(\underline{L})\right).$$

Since $\underline{L}$ is p-maximal, for each prime $p \mid 2\det(\underline{L})$, there is no non-zero isotropic element in $(D_{\underline{L}})_p$. By this one immediately see that $A_{p^j}^{\varepsilon_{p^j}}$ can not occur in (4) unless $j=1$. Moreover if h_p is more than one, then by [6, Chapter 1, Corollary 2], the congruence equation

$$\varepsilon_p x^2 + \underbrace{y_1^2 + \cdots + y_i^2}_{h_p} \equiv 0 \bmod p$$

has nonzero solutions. This implies that $h_p = 1$ and $\left(\frac{-\varepsilon_p}{p}\right) = -1$.

To prove (2) we need calculate $\gamma_{\underline{L},n,r}(p^\nu)$ explicitly. From (3) one can see $\mathcal{N}_r$ is square unless $p \nmid \omega_{\underline{L}}(r)$. Therefore

$$\gamma_{\underline{L},n,r}(p^\nu) = \sum_{\substack{d \bmod p^\nu \\ \gcd(d,p)=1}} e_{p^\nu}\left(d\omega_{\underline{L}}(r)^2 n\right) \sum_{x \in L/p^\nu L} e_{p^\nu}\left(dq(x)\right).$$

For this we replaced d by $\omega_{\underline{L}}(r)^2 d$ in (1). According to [5, Chapter 11, Theorem 2] and related discussions in the proof of Lemma 2.10 of [7], the quadratic form is isomorphic to the following form over $\mathbb{Z}_p$:

$$p(x_1^2+\varepsilon_p x_2^2) + y_1^2 + \cdots + y_{r_{\underline{L}}-3}^2 + \iota_p y_{r_{\underline{L}}-2}^2,$$

where $\iota_p \in \mathbb{Z}$ such that $2^{r_{\underline{L}}}\varepsilon_p\iota_p = \det(\underline{L})/p^2$. Therefore for $\nu \geq 1$ we have

$$\begin{aligned}
&\sum_{x\in L/p^\nu L} e_{p^\nu}(dq(x)) \\
= \ & p^2 \left(\sum_{x \bmod p^{\nu-1}} e_{p^{\nu-1}}\left(dx^2\right)\right)\left(\sum_{x \bmod p^{\nu-1}} e_{p^{\nu-1}}\left(d\varepsilon_p x^2\right)\right) \\
&\times \left(\sum_{x \bmod p^{\nu}} e_{p^{\nu}}\left(dx^2\right)\right)^{r_{\underline{L}}-3}\left(\sum_{x \bmod p^{\nu}} e_{p^{\nu}}\left(d\iota_p x^2\right)\right).
\end{aligned}$$

Applying the identity

$$\sum_{x \bmod p^\nu} e_{p^\nu}\left(dx^2\right) = \epsilon(p^\nu)p^{\frac{\nu}{2}}\left(\frac{d}{p^\nu}\right)$$

(see [1, Chapter 1]) we get

$$\begin{aligned}
&\sum_{x\in L/p^\nu L} e_{p^\nu}(dq(x)) \\
&= p^{\frac{\nu r_{\underline{L}}}{2}+1}\epsilon(p^{\nu-1})^2\epsilon(p^\nu)^{r_{\underline{L}}-2}\left(\frac{\varepsilon_p}{p^{\nu-1}}\right)\left(\frac{\iota_p}{p^\nu}\right)\left(\frac{d}{p^\nu}\right) \\
&= p^{\frac{\nu r_{\underline{L}}}{2}+1}\left(\frac{\varepsilon_p}{p}\right)\epsilon(p^{\nu-1})^2\epsilon(p^\nu)^{r_{\underline{L}}-2}\left(\frac{2\det(\underline{L})d/p^2}{p^\nu}\right) \\
&= p^{\frac{\nu r_{\underline{L}}}{2}+1}\left(\frac{\varepsilon_p}{p}\right)\epsilon(p^{\nu-1})^2\epsilon(p^\nu)^{\operatorname{sign}(\underline{L})+2b^-_{\underline{L}}-2}\left(\frac{2\det(\underline{L})d/p^2}{p^\nu}\right) \\
&= p^{\frac{\nu r_{\underline{L}}}{2}+1}\left(\frac{\varepsilon_p}{p}\right)\epsilon(p^{\nu-1})^2\epsilon(p^\nu)^{2b^-_{\underline{L}}-1}\left(\frac{2\det(\underline{L})d/p^2}{p^\nu}\right).
\end{aligned}$$

One easily check

$$\epsilon(p^{\nu-1})^2\epsilon(p^\nu)^{-1} = \left(\frac{-1}{p}\right)\epsilon(p^\nu),$$

thus we have

$$\begin{aligned}
&\epsilon(p^\nu)^{2b^-_{\underline{L}}}\left(\frac{2\det(\underline{L})/p^2}{p^\nu}\right) \\
&= \epsilon(p^\nu)^{2b^-_{\underline{L}}}\left(\frac{(-1)^{b^-_{\underline{L}}}}{p}\right)\left(\frac{2(-1)^{b^-_{\underline{L}}}|\det(\underline{L})/p^2|}{p^\nu}\right) \\
&= \left(\frac{2|\det(\underline{L})/p^2|}{p^\nu}\right).
\end{aligned}$$

Hence for each integer d with $p \nmid d$,

$$\sum_{x\in L/p^\nu L} e_{p^\nu}(dq(x)) = -p^{\frac{\nu r_{\underline{L}}}{2}+1}\epsilon(p^\nu)\left(\frac{2|\det(\underline{L})/p^2|d}{p^\nu}\right).$$

Finally we write $\gamma_{\underline{L},n,r}(p^\nu)$ as

$$\begin{aligned}\gamma_{\underline{L},n,r}(p^\nu) &= \sum_{\substack{d \bmod p^\nu \\ \gcd(d,p)=1}} e_{p^\nu}\left(d\omega_{\underline{L}}(r)^2 n\right) \sum_{x\in L/p^\nu L} e_{p^\nu}\left(dq(x)\right) \\ &= -p^{\frac{\nu r_{\underline{L}}}{2}+1}\epsilon(p^\nu) \sum_{\substack{d \bmod p^\nu \\ \gcd(d,p)=1}} e_{p^\nu}\left(-d\mathcal{N}_r\right)\left(\frac{d}{p^\nu}\right) \\ &= -p^{\frac{\nu(r_{\underline{L}}-1)}{2}+1} \sum_{\substack{d \bmod p^\nu \\ \gcd(d,p)=1}} e_{p^\nu}\left(-d\mathcal{N}_r\right) \sum_{x \bmod p^\nu} e_{p^\nu}(dx^2) \\ &= -p^{\frac{\nu(r_{\underline{L}}+1)}{2}+1}\left(N_{1,\mathcal{N}_r}(p^\nu) - N_{1,\mathcal{N}_r}(p^{\nu-1})\right),\end{aligned}$$

where for a positive integer c and an integer Δ,

$$\begin{aligned}N_{1,\Delta}(c) &:= \sharp\{x \bmod c : x^2+\Delta \equiv 0 \bmod c\} \\ &= \sharp\{x \bmod 2c : x^2+4\Delta \equiv 0 \bmod 4c\}.\end{aligned}$$

According to [9, Proposition 3], for each prime l, the following holds:

$$1+\sum_{\nu\geq 1} l^{-\nu t}\left(N_{1,\Delta}(l^\nu) - N_{1,\Delta}(l^{\nu-1})\right) = \left(1-l^{-2t}\right) L_{4\Delta}^{(l)}(t). \tag{5}$$

By (5) we get

$$\begin{aligned}&L_p\left(\gamma_{\underline{L},n,r},(3+r_{\underline{L}})/2\right) \\ &= 1+p-p\left(1+\sum_{\nu\geq 0} p^{-2\nu}\left(N_{1,4\mathcal{N}_r/p^2}(p^\nu) - N_{1,4\mathcal{N}_r/p^2}(p^{\nu-1})\right)\right) \\ &= 1+p-p\left(1-p^{-2}\right) L^{(p)}_{4\mathcal{N}_r/p^2}(1).\end{aligned}$$

For $\Delta = f^2 > 0$ one has

$$L_\Delta(1) = \frac{1}{1-p^{-1}}\left(\sigma_1(f) - p\sigma_1(f/p)\right) = \frac{1}{1-p^{-1}}$$

thus the following holds

$$L^{(p)}_{4\mathcal{N}_r/p^2}(1) = \frac{1}{1-p^{-1}}$$

for a perfect square $\mathcal{N}_r$. Therefore

$$L_p\left(\gamma_{\underline{L},n,r},(3+r_{\underline{L}})/2\right) = 1+p-p(1-p^{-2})L_{4\mathcal{N}_r/p^2}(1) = 0.$$

This proves (2).

Now we complete the proof of Theorem 3.1. □

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