Bull. Korean Math. Soc. **58** (2021), No. 2, pp. 497–505
https://doi.org/10.4134/BKMS.b200368
pISSN: 1015-8634 / eISSN: 2234-3016

GORENSTEIN PROJECTIVE DIMENSIONS OF COMPLEXES UNDER BASE CHANGE WITH RESPECT TO A SEMIDUALIZING MODULE

Chunxia Zhang

Abstract. Let $R \to S$ be a ring homomorphism. The relations of Gorenstein projective dimension with respect to a semidualizing module of homologically bounded complexes between $U \otimes_R^{\mathbf{L}} X$ and X are considered, where X is an R-complex and U is an S-complex. Some sufficient conditions are given under which the equality $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X) = \mathcal{GP}_C\text{-pd}_R(X)$ holds. As an application it is shown that the Auslander-Buchsbaum formula holds for G_C-projective dimension.

1. Introduction

The classical theory of homological dimensions is very important to commutative algebra. In particular, it is useful that there are a number of finiteness conditions on these dimensions which characterize regular rings. For example, if the projective dimension of each finitely generated R-module is finite, then R is a regular ring.

Semidualizing modules (cf. Definition 6) have been considered by many authors (see, for example, [4,8,9,12–15]). For any commutative noetherian ring R, any semidualizing R-module C and any complex Z with bounded and finitely generated homology, Christensen introduced the dimension $G\text{-dim}_C Z$ in [4], and developed a satisfactory theory for this new invariant, which characterized Cohen-Macaulay rings in a way one could hope for. However, Christensen's $G\text{-dim}_C(-)$ only works when the argument has bounded and finitely generated homology. To circumvent this shortcoming, Holm and Jϕrgensen proposed to study a homological dimension based on a larger class of complexes: $\mathcal{GP}_C$-projective dimension of X, $\mathcal{GP}_C\text{-pd}_R X$, for every homologically right-bounded complex X (see [8]). It was already known from [8] that for complexes with

Received April 21, 2020; Revised July 26, 2020; Accepted September 23, 2020.
2010 *Mathematics Subject Classification.* Primary 13D25, 16E65.
Key words and phrases. Semidualizing module, G_C-projective module, G_C-projective dimension, ring homomorphism, depth of complex.
This work was financially supported by National Natural Science Foundation of China (11871125) and Natural Science Foundation of Chongqing (cstc 2017jcyjAX0298).

bounded and finitely generated homology, the $\mathcal{GP}_C$-pd$_R(-)$ agrees with Christensen's G-dim$_C(-)$.

Transfer of homological properties along ring homomorphisms is a classical field of study (see, for instance, [1,2,5,6,10,16]). The main goal of this paper is to study the properties of $\mathcal{GP}_C$-projective dimensions for complexes over ring homomorphisms.

In this paper, all rings are commutative, unital, and noetherian.

2. Ring homomorphisms and G_C-projective dimensions

In this section, the Gorenstein projective dimension of complexes with respect to a semidualizing module is considered. First, we recall the following definitions for later use.

Definition 1. Let $\varphi : R \to S$ be a ring homomorphism. φ is said to be of *finite flat dimension* if flat dimension of S is finite as an R-module. We say φ is *faithfully flat* if S is a faithfully flat R-module (that is, S_R satisfies the condition that $0 \to A \to B$ is an exact sequence of R-modules if and only if $0 \to S \otimes_R A \to S \otimes_R B$ is exact). We call φ *finite* if it makes S a finite R-module, and we say that φ is *local* if R and S are local rings and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$, where $\mathfrak{m}$ and $\mathfrak{n}$ are the maximal ideals of R and S.

Definition 2. An *R-complex* X is a sequence of R-modules X_i and R-linear maps $\partial_i^X : X_i \to X_{i-1}$, $i \in \mathbb{Z}$. If $X_i = 0$ for $i \neq 0$ we identify X with the module in degree 0, and an R-module M is thought of as a complex $0 \to M \to 0$, with M in degree 0. The homological position of a complex is captured by the numbers *supremum* and *infimum* defined by $\sup X = \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$ and $\inf X = \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$. By convention $\sup X = -\infty$ and $\inf X = \infty$ if $X \simeq 0$.

The category of R-complexes is denoted by $\mathcal{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote boundedness conditions. For example, $\mathcal{C}_\square(R)$ is the full subcategory of $\mathcal{C}(R)$ of bounded complexes.

Definition 3. The *derived category* of the category of R-modules is the category of R-complexes localized at the class of all quasi-isomorphisms, it is denoted by $\mathcal{D}(R)$. The symbol "$\simeq$" is used to designate isomorphisms in $\mathcal{D}(R)$ and quasi-isomorphisms in $\mathcal{C}(R)$, and we use subscripts $\sqsubset$, $\sqsupset$ and $\square$ to denote homological boundedness conditions. Superscript "f" signifies that the homology is degreewise finitely generated. Thus, $\mathcal{D}^f_\sqsupset(R)$ denotes the full subcategory of $\mathcal{D}(R)$ of homologically right-bounded complexes with finitely generated homology modules.

Definition 4. The *left derived functor* of the tensor product functor of R-complexes is denoted by $- \otimes^{\mathbf{L}}_R -$, and $\mathbf{R}\mathrm{Hom}_R(-,-)$ denotes the *right derived functor* of the homomorphism functor of complexes. For $X, Y \in \mathcal{D}(R)$ and $i \in \mathbb{Z}$, we set $\mathrm{Tor}_i^R(X,Y) = \mathrm{H}_i(X \otimes^{\mathbf{L}}_R Y)$ and $\mathrm{Ext}^i_R(X,Y) = \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X,Y))$.

For modules X and Y this agrees with the notation of classical homological algebra.

Definition 5. A complex $X \in \mathcal{D}_\square(R)$ is said to be of *finite projective* (*or flat*) *dimension* if $X \simeq U$, where U is a complex of projective (or flat) modules and $U_i = 0$ for $|i| \gg 0$. By $\mathbf{P}(R)$ and $\mathbf{F}(R)$ we denote the full subcategories of $\mathcal{D}_\square(R)$ whose objects are complexes of finite projective and flat dimension, respectively. Note that $\mathbf{P}_0(R)$ and $\mathbf{F}_0(R)$ are equivalent, respectively, to the full subcategories of modules of finite projective or flat dimension. We use two-letter abbreviations pd, fd for the homological dimensions.

Definition 6. A finitely generated R-module C is *semidualizing* if

(a) The natural homothety morphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism,

(b) $\mathrm{Ext}_R^{\geq 1}(C, C) = 0$.

Let C be a semidualizing R-module. Set

$\mathcal{P}_C(R) =$ the subcategory of modules $C \otimes_R P$ where P is R-projective,

$\mathcal{F}_C(R) =$ the subcategory of modules $C \otimes_R F$ where F is R-flat.

Modules in $\mathcal{P}_C(R)$ and $\mathcal{F}_C(R)$ are called *C-projective* and *C-flat*, respectively.

A free R-module of rank one is semidualizing. If R admits a dualizing module D, then D is semidualizing.

Setting $C = R$ in the definition above we see that $\mathcal{P}_R(R)$ and $\mathcal{F}_R(R)$ are the classes of ordinary projective and flat R-modules, which we usually denote $\mathcal{P}(R)$ and $\mathcal{F}(R)$, respectively.

Definition 7. Let $\mathcal{X}$ be a class of R-modules and M an R-module. An *$\mathcal{X}$-resolution* of M is a complex of R-modules in $\mathcal{X}$ of the form

$$X = \cdots \to X_n \to X_{n-1} \to \cdots \to X_1 \to X_0 \to 0$$

such that $\mathrm{H}_0(X) \cong M$ and $\mathrm{H}_n(X) = 0$ for $n \geq 1$. The *$\mathcal{X}$-projective dimension* of M is the quantity

$$\mathcal{X}\text{-}\mathrm{pd}_R(M) = \inf\{\sup\{n \geq 0 \mid X_n \neq 0\} \mid X \text{ is an } \mathcal{X}\text{-resolution of } M\}.$$

In particular, one has $\mathcal{X}\text{-}\mathrm{pd}_R(0) = -\infty$. The modules of $\mathcal{X}$-projective dimension 0 are the nonzero modules of $\mathcal{X}$.

The $\mathcal{P}_C$-projective dimension and $\mathcal{F}_C$-projective dimension of M are defined as above in [13], which are called *C-projective* and *C-flat dimension* of M, respectively.

Lemma 2.1 ([7, Lem. 3.2])**.** *Let $\varphi : R \to S$ be a ring homomorphism of finite flat dimension and C a semidualizing R-module. Then $\widetilde{C} = C \otimes_R S$ is a semidualizing S-module.*

Definition 8 ([15])**.** Let C be a semidualizing R-module.

A *complete $\mathcal{P}\mathcal{P}_C$-resolution* is a complex X of R-modules satisfying the following:

(1) X is exact and $\mathrm{Hom}_R(-,\mathcal{P}_C(R))$-exact, and

(2) X_i is projective if $i \geq 0$ and X_i is C-projective if $i < 0$.

An R-module M is G_C*-projective* if there exists a complete $\mathcal{PP}_C$-resolution X such that $M \cong \mathrm{Coker}\partial_1^X$, in which case X is a complete $\mathcal{PP}_C$-resolution of M.

We set

$$\mathcal{GP}_C(R) = \text{the subcategory of } G_C\text{-projective } R\text{-modules}.$$

In the special case $C = R$, we set $\mathcal{GP}_R(R) = \mathcal{GP}(R)$, and $\mathcal{GP}_R(R)\text{-pd}_R(-) = \mathrm{Gpd}_R(-)$.

Example 2.2 ([8, Exam. 2.8])**.** Projective and C-projective R-modules are G_C-projective.

Remark 2.3 ([8]). An R-module M is G_C-projective if and only if

(P1) $\mathrm{Ext}_R^{\geq 1}(M, C \otimes_R P) = 0$ for any projective R-module P, and

(P2) there exist projective R-modules P_{-1}, P_{-2}, $\cdots$ together with an exact sequence:

$$X = 0 \to M \to C \otimes_R P_{-1} \to C \otimes_R P_{-2} \to \cdots$$

such that this sequence stays exact when we apply the functor $\mathrm{Hom}_R(-, C \otimes_R P)$ to it for any projective R-module P (i.e., M admits a proper $\mathcal{P}_C(R)$-coresolution).

By Example 2.2, there exists for every homologically bounded below complex X a bounded below complex A of G_C-projective R-modules with $A \simeq X$ in $\mathcal{D}(R)$ (as one could take A to be a projective resolution of X). Every such A is called a G_C-projective resolution of X.

We proceed by recalling the definition of G_C-projective dimensions from [17].

Definition 9. The G_C*-projective dimension*, $\mathcal{GP}_C\text{-pd}_R(X)$, of $X \in \mathcal{D}_\sqsupset(R)$ is defined as

$$\mathcal{GP}_C\text{-pd}_R(X) = \inf\{\sup\{l \in \mathbb{Z} \mid A_l \neq 0\} \mid X \simeq A \in \mathcal{C}_\sqsupset^{\mathcal{GP}_C}(R)\}.$$

For modules, this dimension above agree with Definition 7, see [17].

The following result is one of the main results in this paper.

Theorem 2.4. *Let $\varphi : R \to S$ be a ring homomorphism of finite flat dimension. Assume that $X \in \mathcal{D}_\square(R)$. If U is a complex of finite projective dimension, i.e., $U \in \mathbf{P}(S)$, then*

$$\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(U \otimes_R^{\mathbf{L}} X) \leq \mathcal{GP}_C\text{-pd}_R(X) + \mathrm{pd}_S U$$

provided $\mathbf{F}_0(S) \subseteq \mathbf{P}_0(R)$.

Proof. If $U \simeq 0$ or $X \simeq 0$, the $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(U \otimes_R^{\mathbf{L}} X) = -\infty$ and so the result is clear. If $\mathcal{GP}_C\text{-pd}_R(X) = \infty$, then there is nothing to do. So we assume that $U \not\simeq 0$ and $X \not\simeq 0$ and $\mathcal{GP}_C\text{-pd}_R(X) < \infty$. Denote $\mathcal{GP}_C\text{-pd}_R(X) = g \in \mathbb{Z}$. Then there exists a complex $A \in \mathcal{C}_\square^{\mathcal{GP}_C}(R)$ which is equivalent to X in $\mathcal{D}(R)$

and has $A_l = 0$ for $l > g$ by [17, Thm. 3.5]. Since $U \in \mathbf{P}(S)$, there exists a bounded complex P of projective S-modules such that $U \simeq P$ and $P_l = 0$ when $l < v = \inf U$ or $l > u = \mathrm{pd}_S U$. It is easy to see that U and P are quasi-isomorphisms as complexes of R-modules.

Note that $U \otimes_R^{\mathbf{L}} X$ is represented by the complex $P \otimes_R A$ by [17, Cor. 2.14] and for any $l \in \mathbb{Z}$,

$$(2.1) \qquad (P \otimes_R A)_l = \oplus_{t\in\mathbb{Z}} P_t \otimes_R A_{l-t} = \oplus_{v\leq t\leq u, l-t\leq g} P_t \otimes_R A_{l-t}$$

is a $G_{\widetilde{C}}$-projective S-module by [7, Prop. 4.12], and direct sums of $G_{\widetilde{C}}$-projective S-modules are $G_{\widetilde{C}}$-projective by [15, Prop. 2.4]. So $P \otimes_R A \in \mathcal{C}^{\mathcal{GP}_{\tilde{C}}}(S)$. Furthermore, it is easy to see that $P \otimes_R A$ is bounded: by (2.1), we have $(P \otimes_R A)_l = 0$ for $g + u < l < g + v$. That is, $P \otimes_R A \in \mathcal{C}_{\square}^{\mathcal{GP}_{\tilde{C}}}(S)$, and therefore, $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(U \otimes_R^{\mathbf{L}} X) \leq g + u = \mathcal{GP}_C\text{-pd}_R(X) + \mathrm{pd}_S U$ as desired. □

Corollary 2.5. *Let $\varphi : R \to S$ be a ring homomorphism of finite flat dimension, and assume that* $\dim R$ *is finite. For every $X \in \mathcal{D}_{\square}(R)$, there is an inequality*

$$\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X) \leq \mathcal{GP}_C\text{-pd}_R(X).$$

Proof. Note that under the condition that $\varphi : R \to S$ is a ring homomorphism of finite flat dimension and $\dim R$ is finite, one has every S-module of finite flat dimension is of finite projective dimension over R via φ. Now the result follows from Theorem 2.4. □

Next, we consider when the equality in Corollary 2.5 holds. To this end we need the next two lemmas.

Lemma 2.6 ([16, Lem. 3.2])**.** *Let $\varphi : R \to S$ be a faithfully flat finite ring homomorphism. If P is a projective R-module, then it is a direct summand* (*as an R-module*) *of the projective S-module $S \otimes_R P$.*

Lemma 2.7. *Let $\varphi : R \to S$ be a faithfully flat ring homomorphism. Assume that* $\dim R$ *is finite. Then an R-module M is G_C-projective if and only if $S \otimes_R M$ is a $G_{\widetilde{C}}$-projective S-module and* $\mathrm{Ext}_R^i(M, C \otimes_R P) = 0$ *for all $i > 0$ and all projective R-modules P.*

Proof. The necessity follows from Remark 2.3 and [7, Prop. 4.12(3)]. The sufficiency follows from [18, Thm. 3.10, Cor. 3.11]. □

Note that if $\varphi : R \to S$ is a faithfully flat ring homomorphism and $\dim S$ is finite, one has $\dim R$ is finite. Then we have:

Theorem 2.8. *Let $\varphi : R \to S$ be a faithfully flat finite ring homomorphism. If* $\dim S$ *is finite, then for every $X \in \mathcal{D}_{\square}(R)$, there is an equality*

$$\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X) = \mathcal{GP}_C\text{-pd}_R(X).$$

Proof. By Corollary 2.5, it is enough to show that

$$\mathcal{GP}_C\text{-pd}_R(X) \leq \mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X).$$

Assume that $\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S\otimes_R^{\mathbf{L}}X) = g < \infty$. Then by [17, Thm. 3.5], $\sup(S\otimes_R^{\mathbf{L}} X) \leq g$ and for every bounded complex $A \simeq S \otimes_R^{\mathbf{L}} X$ of $G_{\widetilde{C}}$-projective S-modules, the module C_g^A is $G_{\widetilde{C}}$-projective.

Consider a G_C-projective resolution $G \xrightarrow{\simeq} X$ over R. Then by [17, Cor. 2.14], $S \otimes_R^{\mathbf{L}} X \simeq S \otimes_R G$. Clearly, $S \otimes_R G$ is a complex of $G_{\widetilde{C}}$-projective S-modules by [7, Cor. 4.17]. Then $S \otimes_R G$ is a $G_{\widetilde{C}}$-projective resolution of $S \otimes_R X$, and so $\sup(S \otimes_R G) \leq g$. Hence the sequence

$$\cdots \to S \otimes_R G_{g+2} \to S \otimes_R G_{g+1} \to S \otimes_R G_g$$

is exact. Clearly, it is exact as a sequence of R-modules. Since S is a faithfully flat R-module, the sequence

$$\cdots \to G_{g+2} \to G_{g+1} \to G_g$$

is exact. Consequently, one has $\sup G \leq g$ and so $\sup X \leq g$.

Next, we prove that C_g^G is G_C-projective. For $i > g$, one has $\mathrm{H}_i(S\otimes_R G) = 0$. Right-exactness of the functor $S \otimes_R -$ yields an isomorphism $\mathrm{Coker}\partial_n^{S\otimes_R G} \cong S\otimes_R \mathrm{Coker}\partial_n^G$ for each n. Set $K = \mathrm{C}_g^G$. By [17, Thm. 3.5], one has the S-module $\mathrm{C}_g^{S\otimes_R G} \cong S \otimes_R K$ is $G_{\widetilde{C}}$-projective. For every projective R-module P, one has P is a direct summand of a projective S-module Q by Lemma 2.6. Let $\mathbb{P}$ be a projective resolution of K. For all $i \geq 1$, one has $\widetilde{C} \otimes_S Q \cong (C \otimes_R S) \otimes_S Q \cong C \otimes_R Q$, then we have

$$\begin{aligned}
\mathrm{Ext}_R^i(K, C \otimes_R Q) &= \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathbb{P}, C \otimes_R Q)) \\
&= \mathrm{H}_{-i}(\mathrm{Hom}_R(\mathbb{P}, \mathrm{Hom}_S(S, \widetilde{C} \otimes_S Q))) \\
&= \mathrm{H}_{-i}(\mathrm{Hom}_S(S \otimes_R \mathbb{P}, \widetilde{C} \otimes_S Q)) \\
&= \mathrm{Ext}_S^i(S \otimes_R K, \widetilde{C} \otimes_S Q) \\
&= 0.
\end{aligned}$$

Therefor, one has $\mathrm{Ext}_R^i(K, C\otimes_R P) = 0$ and so K is a G_C-projective R-module by Lemma 2.7. It follows from [17, Thm. 3.5] that $\mathcal{GP}_C\text{-pd}_R(X) < \infty$.

To prove the equality, using [17, Thm. 3.5], choose a projective R-module Q such that $\mathcal{GP}_C\text{-pd}_R(X) = -\inf \mathbf{R}\mathrm{Hom}_R(X, C \otimes_R Q)$. Since Q is a direct summand of a projective S-module $\overline{Q}$ by Lemma 2.6, hence one has

$$\begin{aligned}
\mathcal{GP}_{\widetilde{C}}\text{-pd}_S(S \otimes_R^{\mathbf{L}} X) &\geq -\inf \mathbf{R}\mathrm{Hom}_S(S \otimes_R^{\mathbf{L}} X, \widetilde{C} \otimes_S \overline{Q}) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_S(S, \widetilde{C} \otimes_S \overline{Q})) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(X, \widetilde{C} \otimes_S \overline{Q}) \\
&\geq -\inf \mathbf{R}\mathrm{Hom}_R(X, \widetilde{C} \otimes_S Q) \\
&= -\inf \mathbf{R}\mathrm{Hom}_R(X, C \otimes_R Q)
\end{aligned}$$

$$= \mathcal{GP}_C\text{-pd}_R(X).$$

The first step is by [17, Thm. 3.5], the second one is from Hom-tensor adjointness, the fourth one follows from Q is a direct summand of a projective S-module $\overline{Q}$ and the last one comes from the choice of Q. This completes the proof. $\square$

3. An application

Let $(R, \mathfrak{m}, k)$ be a local ring. Recall that the depth of an R-complex X is defined as

$$\text{depth}_R X = -\sup \mathbf{R}\text{Hom}_R(k, X).$$

The following equality is well-known as the Auslander-Buchsbaum formula: for any $X \in \mathbf{P}^f(R)$, there is an equality

$$\text{pd}_R X = \text{depth} R - \text{depth}_R X. \tag{3.1}$$

For homologically bounded complex with finite homology, for finite modules in particular, the G_C-projective dimension coincides with Christensen's notion of G_C-dimension; see [8, Prop. 3.1]. Then we have the next equality, which is the Auslander-Buchsbaum formula of G_C-projective dimension.

Theorem 3.1. *Let R be local and $X \in \mathcal{D}^f_\square(R)$. If $G_C\text{-dim}_R X$ is finite, then there is an equality*

$$G_C\text{-dim}_R X = \text{depth} R - \text{depth}_R X. \tag{3.2}$$

Proof. By [8, Thm. 2.6], $G_C\text{-dim}_R X = \text{Gpd}_{R\propto C} X$, where $R \propto C$ is the trivial extension of R by C. On the other hand $\text{Gpd}_{R\propto C} X = \text{depth} R \propto C - \text{depth}_{R\propto C} X$ by [4, Thm. 3.14] since $\text{Gpd}_{R\propto C} X < \infty$. Note that

$$\text{depth} R \propto C = \min\{\text{depth} R, \text{depth}_R C\} = \text{depth} R$$

since $\text{depth}_R C = \text{depth} R$ by [12, Thm. 2.2.6] and $\text{depth}_{R\propto C} X = \text{depth}_R X$ by [3, Exercise 1.2.26]. $\square$

Then we have the following result for modules, and which recovers [11, Thm. 3.12] and [14, Thm. 2.5].

Corollary 3.2. *Let R be a local ring. Then for every finitely generated R-module $M \neq 0$ of finite G_C-dimension, there is an equality*

$$G_C\text{-dim}_R M = \text{depth} R - \text{depth}_R M.$$

Corollary 3.3. *Let $\varphi : R \to S$ be a local ring homomorphism of finite flat dimension. Assume that $X \in \mathcal{D}^f_\square(R)$ with $G_C\text{-dim}_R X$ finite and $U \in \mathbf{P}^f(S)$, then the following equality holds*

$$G_C\text{-dim}_S(U \otimes^{\mathbf{L}}_R X) = G_C\text{-dim}_R X + \text{pd}_S U.$$

Proof. By Theorem 2.4, one has $G_C\text{-}\dim_S(U\otimes^{\mathbf{L}}_R X)$ is finite. By hypothesis, it is not hard to see that $U\otimes^{\mathbf{L}}_R X \in \mathcal{D}^f_\square(R)$ and $U \in \mathbf{P}(R)$. Since $G_C\text{-}\dim_R X$ is finite, the complex X is homologically bounded above. Now the first equality in the computation below follows from (3.2), the second one follows by [5, Thm. 6.2(i)] and the last one follows from (3.2) and the Auslander-Buchsbaum formula (3.1).

$$\begin{aligned} G_C\text{-}\dim_S(U \otimes^{\mathbf{L}}_R X) &= \mathrm{depth} S - \mathrm{depth}_S(U \otimes^{\mathbf{L}}_R X) \\ &= \mathrm{depth} S - \mathrm{depth}_S U - \mathrm{depth}_R X + \mathrm{depth} R \\ &= G_C\text{-}\dim_R X + \mathrm{pd}_S U. \end{aligned}$$
$\square$

Acknowledgements. The authors would like to thank the referees for the very helpful suggestions and corrections that have improved this article.

References

[1] L. L. Avramov and H.-B. Foxby, *Gorenstein local homomorphisms*, Bull. Amer. Math. Soc. (N.S.) **23** (1990), no. 1, 145–150. https://doi.org/10.1090/S0273-0979-1990-15921-X

[2] ———, *Ring homomorphisms and finite Gorenstein dimension*, Proc. London Math. Soc. (3) **75** (1997), no. 2, 241–270. https://doi.org/10.1112/S0024611597000348

[3] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics, 39, Cambridge University Press, Cambridge, 1993.

[4] L. W. Christensen, *Semi-dualizing complexes and their Auslander categories*, Trans. Amer. Math. Soc. **353** (2001), no. 5, 1839–1883. https://doi.org/10.1090/S0002-9947-01-02627-7

[5] L. W. Christensen and H. Holm, *Ascent properties of Auslander categories*, Canad. J. Math. **61** (2009), no. 1, 76–108. https://doi.org/10.4153/CJM-2009-004-x

[6] L. W. Christensen and S. Sather-Wagstaff, *Transfer of Gorenstein dimensions along ring homomorphisms*, J. Pure Appl. Algebra **214** (2010), no. 6, 982–989. https://doi.org/10.1016/j.jpaa.2009.09.007

[7] Z. Di and X. Yang, *Transfer properties of Gorenstein homological dimension with respect to a semidualizing module*, J. Korean Math. Soc. **49** (2012), no. 6, 1197–1214. https://doi.org/10.4134/JKMS.2012.49.6.1197

[8] H. Holm and P. Jørgensen, *Semi-dualizing modules and related Gorenstein homological dimensions*, J. Pure Appl. Algebra **205** (2006), no. 2, 423–445. https://doi.org/10.1016/j.jpaa.2005.07.010

[9] H. Holm and D. White, *Foxby equivalence over associative rings*, J. Math. Kyoto Univ. **47** (2007), no. 4, 781–808. https://doi.org/10.1215/kjm/1250692289

[10] Z. Liu and W. Ren, *Transfer of Gorenstein dimensions of unbounded complexes along ring homomorphisms*, Comm. Algebra **42** (2014), no. 8, 3325–3338. https://doi.org/10.1080/00927872.2013.783039

[11] M. Salimi, *On relative Gorenstein homological dimensions with respect to a dualizing module*, Mat. Vesnik **69** (2017), no. 2, 118–125.

[12] S. Sather-Wagstaff, *Semidualizing modules*, URL: http://www.ndsu.edu/pubweb/~ssatherw/Accessed August 25, 2013.

[13] R. Takahashi and D. White, *Homological aspects of semidualizing modules*, Math. Scand. **106** (2010), no. 1, 5–22. https://doi.org/10.7146/math.scand.a-15121

[14] E. Tavasoli, M. Salimi, and S. Yassemi, *Reflexive modules with finite Gorenstein dimension with respect to a semidualizing module*, Proc. Indian Acad. Sci. Math. Sci. **125** (2015), no. 1, 21–28. https://doi.org/10.1007/s12044-015-0218-7

[15] D. White, *Gorenstein projective dimension with respect to a semidualizing module*, J. Commut. Algebra **2** (2010), no. 1, 111–137. `https://doi.org/10.1216/JCA-2010-2-1-111`

[16] D. Wu, *Gorenstein dimensions over ring homomorphisms*, Comm. Algebra **43** (2015), no. 5, 2005–2028. `https://doi.org/10.1080/00927872.2014.881836`

[17] C. Zhang, L. Wang, and Z. Liu, *Gorenstein homological dimensions of complexes with respect to a semidualizing module*, Comm. Algebra **42** (2014), no. 6, 2684–2703. `https://doi.org/10.1080/00927872.2013.772185`

[18] ———, *Gorenstein homological dimensions and Auslander categories with respect to a semidualizing module*, J. Math. Res. Appl. **33** (2013), no. 3, 297–311.

Chunxia Zhang
School of Mathematical Sciences
Chongqing Normal University
Chongqing 401331, P. R. China
Email address: `cxzhang@cqnu.edu.cn`