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BERGMAN SPACES, BLOCH SPACES AND INTEGRAL MEANS OF p-HARMONIC FUNCTIONS

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Abstract. In this paper, we investigate the properties of Bergman spaces, Bloch spaces and integral means of p-harmonic functions on the unit ball in $\mathbb{R}^n$. Firstly, we offer some Lipschitz-type and double integral characterizations for Bergman space $\mathcal{A}^k_\gamma$. Secondly, we characterize Bloch space $\mathcal{B}^\alpha_\omega$ in terms of weighted Lipschitz conditions and BMO functions. Finally, a Hardy-Littlewood type theorem for integral means of p-harmonic functions is established.

1. Introduction

For $n \geq 2$, let $\mathbb{R}^n$ denote the usual real vector space of dimension n. For two column vectors $x, y \in \mathbb{R}^n$, we use $\langle x, y\rangle$ to denote the inner product of x and y. The ball in $\mathbb{R}^n$ with center a and radius r is denoted by $\mathbb{B}(a, r)$. In particular, we write $\mathbb{B} = \mathbb{B}(0, 1)$ and $r\mathbb{B} = \mathbb{B}(0, r)$. Let dv be the normalized volume measure on $\mathbb{B}$ and $d\sigma$ the normalized surface measure on the unit sphere $\mathbb{S} = \partial\mathbb{B}$.

The purpose of this paper is to consider the p-harmonic functions whose definition is as follows.

Definition 1.1. Let $p > 1$ and Ω be a domain in $\mathbb{R}^n$. A continuous function $u \in W^{1,p}_{loc}(\Omega)$ is *p-harmonic* if

$$-\mathrm{div}\Big(|\nabla u|^{p-2}\nabla u\Big) = 0$$

in the weak sense, i.e.,

$$\int_\Omega \langle |\nabla u|^{p-2}\nabla u, \nabla\eta\rangle dv(x) = 0$$

for each $\eta \in C^\infty_0(\Omega)$.

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Obviously, when $p = 2$, u is the classical harmonic function. By [15], we know that p-harmonic functions are differentiable. For each $p > 1$, we denote by $\mathcal{H}(\mathbb{B})$ the set of all p-harmonic functions in $\mathbb{B}$.

Let $\gamma > -1$ and $0 < k < \infty$, the weighted *Bergman space* $\mathcal{A}^k_\gamma$ consists of all $u \in \mathcal{H}(\mathbb{B})$ such that

$$\|u\|^k_{\mathcal{A}^k_\gamma} = \int_{\mathbb{B}} |u(x)|^k dv_\gamma(x) < \infty,$$

where $dv_\gamma(x) = (1 - |x|)^\gamma dv(x)$. In particular, if $\gamma = 0$, then we simply write $\mathcal{A}^k$ for $\mathcal{A}^k_\gamma$.

A continuous increasing function $\omega : [0, \infty) \to [0, \infty)$ with $\omega(0) = 0$ is called a *majorant* if $\omega(t)/t$ is non-increasing for $t > 0$. Given a subdomain Ω of $\mathbb{R}^n$, a function $u : \Omega \to \mathbb{R}$ is said to belong to the *Lipschitz space* $\Lambda_\omega(\Omega)$ if there is a positive constant C such that

$$|u(x) - u(y)| \le C\omega(|x - y|)$$

for all $x, y \in \Omega$ (cf. [9]).

For $\alpha > 0$ and a given majorant ω, the *harmonic ω-α-Bloch space* $\mathcal{B}^\alpha_\omega$ consists of all functions $u \in \mathcal{H}(\mathbb{B})$ such that

$$\|u\|_{\omega,\alpha} = |u(0)| + \sup_{x\in\mathbb{B}} \omega\big((1 - |x|)^\alpha\big)|\nabla u(x)| < \infty.$$

In particular, when $\omega(t) = t$, the space $\mathcal{B}^\alpha_\omega$ is the α-Bloch space $\mathcal{B}^\alpha$ (cf. [24,27]).

In the theory of function spaces, characterizations and operator theory of (weighted) Bergman and Bloch spaces have been studied extensively in recent years (see [2,3,10,12,21–25,27,28]). Wulan and Zhu [26] characterized weighted holomorphic Bergman space in the unit disk in terms of Lipschitz type conditions with respect to pseudo-hyperbolic, hyperbolic and Euclidean metrics. In [16, 17] the authors studied further and offered several kinds of double integral characterizations for standard weighted Bergman spaces in the unit ball of $\mathbb{C}^n$. Recently, Cho and Park [6] extended these results to Bergman space with exponential type weight. For generalizations of these results in the setting of harmonic functions, we refer to [7, 8, 20, 25]. As the first aim of this paper, we consider similar results of the above type in the setting of $\mathcal{H}(\mathbb{B})$.

Theorem 1.1. *Let $\gamma > -1$, $0 < k < \infty$ and $u \in \mathcal{H}(\mathbb{B})$. Then the following statements are equivalent.*

(a) $u \in \mathcal{A}^k_\gamma$;

(b) *There exists a positive continuous function $g \in L^k(\mathbb{B}, dv_\gamma)$ such that*

$$|u(x) - u(y)| \le \sigma(x, y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$;

(c) *There exists a positive continuous function $h \in L^k(\mathbb{B}, dv_\gamma)$ such that*

$$|u(x) - u(y)| \le \rho(x, y)\big(h(x) + h(y)\big)$$

for all $x, y \in \mathbb{B}$;

(d) *There exists a positive continuous function* $\tau \in L^k(\mathbb{B}, dv_{\gamma+k})$ *such that*

$$|u(x) - u(y)| \leq |x - y|\big(\tau(x) + \tau(y)\big)$$

for all $x, y \in \mathbb{B}$.

Theorem 1.2. *Suppose that* $\gamma > -1, 0 < k < \infty$ *and* $u \in \mathcal{H}(\mathbb{B})$. *If* s *and* t *are real parameters satisfying the following*

$$s + t = \gamma + k - n$$

and

$$-1 < s < k - n, \ -1 < t < k - n,$$

then $u \in \mathcal{A}^k_\gamma$ *if and only if*

$$I = \int_{\mathbb{B}} \int_{\mathbb{B}} \frac{|u(x) - u(y)|^k}{[x, y]^k} dv_s(x) dv_t(y) < \infty. \tag{1}$$

Remark 1.1. Theorem 1.1 and Theorem 1.2 are generalizations of [26, Theorem 1.1] and [16, Theorm 5], respectively, in the setting of holomorphic functions in the unit disk.

Let $\mu, \nu \geq 0$ and f be a continuous function in $\mathbb{B}$. If there exists a constant C such that

$$(1 - |x|)^\mu (1 - |y|)^\nu |f(x) - f(y)| \leq C|x - y|$$

for any $x, y \in \mathbb{B}$, then we say that f is a *weighted Lipschitz function* of indices (μ, ν) (cf. [22]).

Holland and Walsh [13] used the weighted Lipschitz condition to give an equivalent characterization of the analytic Bloch space. In [22, 23], Ren and Kähler generalized Holland and Walsh's result to the setting of (hyperbolic) harmonic functions ([22, Theorem 1.1], [23, Theorem 1.2]). In [19], Muramoto characterized Bloch space in terms of BMO functions. Recently, Chen and Rasila generalized these results to the solutions of certain elliptic PDEs, see [3–5].

For nonnegative quantities X and Y, $X \lesssim Y$ means that X is dominated by Y times some inessential positive constant. We write $X \approx Y$ if $Y \lesssim X \lesssim Y$. As the second aim of this paper, we discuss the corresponding problems in the setting of p-harmonic functions and obtain the following:

Theorem 1.3. *Let* ω *be a majorant,* $u \in \mathcal{H}(\mathbb{B})$ *and* $0 < \beta < 1, \ \beta \leq \alpha < 1 + \beta$, $0 \leq \theta \leq 1$. *Then* $u \in \mathcal{B}^\alpha_\omega$ *if and only if for all* $x, y \in \mathbb{B}$,

$$|u(x) - u(y)| \lesssim \frac{[x, y]^\theta |x - y|^{1-\theta}}{\omega\big((1 - |x|)^\beta (1 - |y|)^{\alpha-\beta}\big)}. \tag{2}$$

Theorem 1.4. *Suppose* $\alpha > 0$, $q \geq 1$, $\alpha q \in [1, 2)$, ω *is a majorant and* $u \in \mathcal{H}(\mathbb{B})$. *Then* $u \in \mathcal{B}^\alpha_\omega$ *if and only if for all* $r \in (0, 1 - |x|]$

$$\Big(\fint_{\mathbb{B}(x,r)} |u(x) - u(y)|^q dv(y)\Big)^{\frac{1}{q}} \lesssim \frac{r}{\omega(r^\alpha)}. \tag{3}$$

Remark 1.2. If we take $\alpha = 1, \beta = \frac{1}{2}, \theta = 0, q = 1$ and $\omega(t) = t$ in Theorems 1.3 and 1.4, then Theorem 1.3 and Theorem 1.4 coincide with [23, Theorem 1.2] and [5, Theorem 2], respectively.

Let $0 < s < \infty$ and $u \in \mathcal{H}(\mathbb{B})$, the *integral mean* of u is defined as

$$M_s(r,u) = \Big(\int_{\mathbb{S}} |u(r\xi)|^s d\sigma(\xi)\Big)^{\frac{1}{s}},\ 0 < r < 1.$$

The famous Hardy-Littlewood Theorem for integral means of analytic functions asserts that:

Theorem A ([11,18])**.** *Suppose that $\alpha \in (1,\infty)$ and f is an analytic function in the unit disk $\mathbb{D}$. Then the following two statements are equivalent.*

(1) $M_s(r,f') = O\Big(\frac{1}{(1-r)^{\alpha}}\Big)$ *as* $r \to 1$;

(2) $M_s(r,f) = O\Big(\frac{1}{(1-r)^{\alpha-1}}\Big)$ *as* $r \to 1$.

In our final result, we prove an analogue of Hardy-Littlewood Theorem for integral means in the setting of p-harmonic functions.

Theorem 1.5. *Let ω be a given majorant, $\alpha \in (0,\infty)$ and $u \in \mathcal{H}(\mathbb{B})$.*

(I) *If for all $\rho \in (0,1)$, $\psi(\rho) = (1-\rho)^{\alpha}\big(1-\ln(1-\rho)\big)^{\beta}$, $\alpha \geq \beta$, $s \geq 1$ and $M_s(\rho,\nabla u) \leq C/\omega\big(\psi(\rho)\big)$, then*

$$M_s(r,u) \leq |u(0)| + C\int_0^1 \frac{r}{\psi(rt)}dt.$$

(II) *If for all $\rho \in (0,1)$, $s\alpha > 1$, and $M_s(\rho,u) \leq C/\omega\big((1-\rho)^{\alpha}\big)$, then for all $q \in (0,\infty)$*

$$M_q(r,\nabla u) = O\Big(\frac{1}{(1-r)^{\alpha+1+\frac{n-1}{s}}}\Big)\ \textit{as}\ r \to 1.$$

The organization of this paper is as follows. In Section 2, some necessary terminology and notation will be introduced. In Section 3, we shall prove Theorems 1.1 and 1.2 and discuss boundedness of the symmetric lifting operator by using Theorem 1.1. The proofs of Theorems 1.3 and 1.4 will be presented in Section 4. The final Section 5 is devoted to the proof of Theorem 1.5. Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other.

2. Preliminaries

In this section, we introduce notation and collect some preliminary results that involve Möbius transformations and p-harmonic functions.

Let $a \in \mathbb{R}^n$, we write a in polar coordinate by $a = |a|a'$. For $a, b \in \mathbb{R}^n$, let

$$[a,b] = \Big||a|b - a'\Big|.$$

The symmetric lemma shows

$$[a,b]=[b,a].$$

For any $a\in\mathbb{B}$, denote by ϕ_a the Möbius transformation in $\mathbb{B}$. It's an involution of $\mathbb{B}$ such that $\phi_a(0)=a$ and $\phi_a(a)=0$, which is of the form

$$\phi_a(x)=\frac{|x-a|^2a-(1-|a|^2)(x-a)}{[x,a]^2},\ x\in\mathbb{B}.$$

An elementary computation gives

$$|\phi_a(x)|=\frac{|x-a|}{[x,a]}.$$

In terms of ϕ_a, the *pseudo-hyperbolic metric* σ and the *hyperbolic metric* ρ in $\mathbb{B}$ can be given by

$$\sigma(a,b)=|\phi_a(b)|,\ a,b\in\mathbb{B}$$

and

$$\rho(a,b)=\ln\frac{1+\sigma(a,b)}{1-\sigma(a,b)},$$

respectively.

For $a\in\mathbb{B}$ and $r\in(0,1)$, the *pseudo-hyperbolic ball* with center a and radius r is defined as

$$E(a,r)=\{x\in\mathbb{B}:\sigma(a,x)<r\}.$$

However, $E(a,r)$ is also a Euclidean ball with center c_a and radius r_a given by

$$c_a=\frac{(1-r^2)a}{1-|a|^2r^2}\quad\text{and}\quad r_a=\frac{r(1-|a|^2)}{1-|a|^2r^2},$$

respectively (cf. [1,23]).

Lemma 2.1 ([23])**.** *Let $a\in\mathbb{B}$, $r\in(0,1)$ and $x\in E(a,r)$. Then*

$$1-|a|^2\approx 1-|x|^2\approx[a,x]\quad\textit{and}\ |E(a,r)|\approx(1-|a|^2)^n,$$

where $|E(a,r)|$ denotes the Euclidean volume of $E(a,r)$.

The following standard estimate will be needed in the proofs of our main results, see [22,23].

Lemma 2.2. *Let $\alpha>-1$ and $\beta\in\mathbb{R}$. Then for any $x\in\mathbb{B}$,*

$$\int_{\mathbb{B}}\frac{(1-|y|^2)^\alpha}{[x,y]^{n+\alpha+\beta}}dv(y)\approx\begin{cases}(1-|x|^2)^{-\beta}, & \beta>0,\\ \log\dfrac{1}{1-|x|^2}, & \beta=0,\\ 1, & \beta<0.\end{cases}$$

We end this section with some useful inequalities concerning p-harmonic functions (cf. [14]). For convenience, we denote

$$⨍_{\mathbb{B}(x,r)}u(y)dv(y)=\frac{1}{|\mathbb{B}(x,r)|}\int_{\mathbb{B}(x,r)}u(y)dv(y).$$

Lemma 2.3. *Assume that $u \in \mathcal{H}(\mathbb{B})$. Then we have the following inequalities.*

(1) *For each $\delta > 1$, there is a positive constant C such that*

$$\int_{\mathbb{B}(x,r)} |\nabla u(y)|^p dv(y) \le \frac{C}{r^p} \int_{\mathbb{B}(x,\delta r)} |u(y)|^p dv(y),$$

whenever $\mathbb{B}(x,\delta r) \subset \mathbb{B}$.

(2) *For each $\delta > 1$ and $0 < s < q$, there is a positive constant C such that*

$$\Big(\fint_{\mathbb{B}(x,r)} |u(y)|^q dv(y)\Big)^{\frac{1}{q}} \le C\Big(\fint_{\mathbb{B}(x,\delta r)} |u(y)|^s dv(y)\Big)^{\frac{1}{s}},$$

whenever $\mathbb{B}(x,\delta r) \subset \mathbb{B}$.

(3) *For each $\delta > 1$ and $0 < q \le p$, there is a positive constant C such that*

$$\Big(\fint_{\mathbb{B}(x,r)} |\nabla u(y)|^p dv(y)\Big)^{\frac{1}{p}} \le C\Big(\fint_{\mathbb{B}(x,\delta r)} |\nabla u(y)|^q dv(y)\Big)^{\frac{1}{q}},$$

whenever $\mathbb{B}(x,\delta r) \subset \mathbb{B}$.

(4) *For each $\delta > 1$ and $q > p$, there is a positive constant C such that*

$$\Big(\fint_{\mathbb{B}(x,r)} |\nabla u(y)|^q dv(y)\Big)^{\frac{1}{q}} \le C\Big(\fint_{\mathbb{B}(x,\delta r)} |\nabla u(y)|^p dv(y)\Big)^{\frac{1}{p}},$$

whenever $\mathbb{B}(x,\delta r) \subset \mathbb{B}$.

3. Proofs of Theorems 1.1 and 1.2

In order to prove Theorem 1.1, we need a Hardy-Littlewood type integral estimate for p-harmonic functions in $\mathbb{B}$ which was proved in [14].

Proposition 3.1. *Let $\gamma > -1$, $0 < k < \infty$. Then*

$$\int_{\mathbb{B}} |u(x)|^k dv_\gamma(x) \approx |u(0)|^k + \int_{\mathbb{B}} (1-|x|)^k |\nabla u(x)|^k dv_\gamma(x)$$

for all $u \in \mathcal{H}(\mathbb{B})$.

Proof of Theorem 1.1. We first prove (b)$\Rightarrow$(a). Assume that (b) holds. Then for each fixed x and all y sufficiently close to x

$$\Big|\frac{u(x)-u(y)}{x-y}\Big| \le \frac{\sigma(x,y)}{|x-y|}\big(g(x)+g(y)\big),\ x \ne y.$$

Since u is differentiable at x, and hence letting y approach x in the direction of a real coordinate axis, we have

$$(1-|x|^2)|D_i u(x)| \le 2g(x),\ 1 \le i \le n,$$

so that $(1-|x|)|\nabla u(x)| \le 2\sqrt{n}g(x)$ for all $x \in \mathbb{B}$. It follows from the assumption $g \in L^k(\mathbb{B}, dv_\gamma)$ that

$$\int_{\mathbb{B}} (1-|x|)^k |\nabla u(x)|^k dv_\gamma(x) < \infty.$$

Thus we obtain that $u \in \mathcal{A}^k_\gamma$ by Proposition 3.1.

For the converse, we assume $u \in \mathcal{A}^k_\gamma$. Fix $r \in (0, \frac{1}{18})$ and consider any two points $x, y \in \mathbb{B}$ with $\sigma(x,y) < r$. Since $E(x,r)$ is a Euclidean ball, by Lemma 2.1, it is given that

$$\begin{aligned}
|u(x) - u(y)| &= \left| \int_0^1 \frac{du}{ds}(sy + (1-s)x) ds \right| \\
&\le \sqrt{n}|x-y| \int_0^1 |\nabla u(sy + (1-s)x)| ds \\
&\lesssim \sigma(x,y) \sup\{(1-|x|)|\nabla u(\xi)| : \xi \in E(x,r)\} \\
&\lesssim \sigma(x,y) h(x),
\end{aligned}$$

where

$$h(x) = C(r) \sup\{(1-|\xi|)|\nabla u(\xi)| : \xi \in E(x,r)\}.$$

If $\sigma(x,y) \ge r$, the triangle inequality implies

$$\begin{aligned}
|u(x) - u(y)| &\le |u(x)| + |u(y)| \\
&\le \sigma(x,y)\Big(\frac{|u(x)|}{r} + \frac{|u(y)|}{r}\Big).
\end{aligned}$$

Now, letting $g(x) = h(x) + \frac{|u(x)|}{r}$, then

$$|u(x) - u(y)| \le \sigma(x,y)\big(g(x) + g(y)\big)$$

for all $x, y \in \mathbb{B}$. Note that $g(x) = h(x) + \frac{|u(x)|}{r}$ is the desired function provided that $h \in L^k(\mathbb{B}, dv_\gamma)$. From properties of a pseudo-hyperbolic ball in $\mathbb{B}$, we know that $E(\xi, r) \subset \mathbb{B}(x, \frac{1-|x|}{4})$ for every $\xi \in E(x,r)$. It follows from [14] and Lemma 2.3 that

$$\begin{aligned}
\sup_{\xi \in E(x,r)} |\nabla u(\xi)| &\le C\Big(⨍_{\mathbb{B}(x, \frac{1-|x|}{4})} |\nabla u(x)|^p dv(y)\Big)^{\frac{1}{p}} \\
&\lesssim \Big((1-|x|)^{-p} ⨍_{\mathbb{B}(x, \frac{1-|x|}{3})} |u(x)|^p v(y)\Big)^{\frac{1}{p}} \\
&\lesssim (1-|x|)^{-1}\Big(⨍_{\mathbb{B}(x, \frac{1-|x|}{2})} |u(x)|^k v(y)\Big)^{\frac{1}{k}}.
\end{aligned}$$

Hence by Fubini's theorem and Lemma 2.1

$$\begin{aligned}
\|h\|^k_{L^k_\gamma} &\lesssim \int_{\mathbb{B}} (1-|x|)^\gamma ⨍_{\mathbb{B}(x, \frac{1-|x|}{2})} |u(y)|^k dv(y) dv(x) \\
&\lesssim \int_{\mathbb{B}} |u(y)|^k \int_{\mathbb{B}(x, \frac{1-|x|}{2})} (1-|x|)^{-n} dv(x) dv_\gamma(y) \\
&\lesssim \|u\|^k_{\mathcal{A}^k_\gamma}.
\end{aligned}$$

This proves (a)$\Leftrightarrow$(b).

(a)$\Leftrightarrow$(c). Since $\sigma \le \rho$, it follows a discussion similar to the above, the result follows.

(a)$\Leftrightarrow$(d). Assume that (d) holds. Then it can be deduced that

$$(1-|x|)|\nabla u(x)| \leq 2(1-|x|)\tau(x)$$

for all $x \in \mathbb{B}$. The assumption $\tau \in L^k(\mathbb{B}, dv_{\gamma+k})$ implies $(1-|x|)|\nabla u(x)| \in L^k(\mathbb{B}, dv_\gamma)$ and thus, according to Proposition 3.1, means that $u \in \mathcal{A}^k_\gamma$.

Conversely, suppose that $u \in \mathcal{A}^k_\gamma$. Then (b) implies that there exists a positive continuous function $g \in L^k(\mathbb{B}, dv_\gamma)$ such that

$$|u(x)-u(y)| \leq \sigma(x,y)\big(g(x)+g(y)\big)$$

for all $x, y \in \mathbb{B}$. Since for $x, y \in \mathbb{B}$,

$$[x,y] \geq 1-|x|,\ [x,y] \geq 1-|y|,$$

we see that

$$\begin{aligned}|u(x)-u(y)| &\leq |x-y|\Big(\frac{g(x)}{[x,y]}+\frac{g(y)}{[x,y]}\Big)\\ &\lesssim |x-y|\big(\tau(x)+\tau(y)\big),\ x, y \in \mathbb{B},\end{aligned}$$

where

$$\tau(x) = \frac{g(x)}{1-|x|}.$$

Hence $\tau \in L^k(\mathbb{B}, dv_{\gamma+k})$ from the assumption $g \in L^k(\mathbb{B}, dv_\gamma)$. The proof of Theorem 1.1 is finished. □

In the following we discuss some applications of Theorem 1.1.

Consider a symmetric lifting operator L which is defined by

$$Lu(x,y) = \frac{u(x)-u(y)}{x-y},\ x \neq y$$

for $u \in \mathcal{H}(\mathbb{B})$.

Theorem 3.1. *Let $\gamma > -1$, $0 < k < n+\gamma$. Then $L : \mathcal{A}^k_\gamma \to L^k(\mathbb{B}\times\mathbb{B}, dv_\gamma \times dv_\gamma) \cap \mathcal{H}(\mathbb{B}\times\mathbb{B})$ is bounded.*

Proof. Let $u \in \mathcal{A}^k_\gamma$. Then there exists a positive continuous function $g \in L^k(\mathbb{B}, dv_\gamma)$ such that

$$|Lu(x,y)|^k = \Big|\frac{u(x)-u(y)}{x-y}\Big|^k \lesssim \frac{|g(x)|^k+|g(y)|^k}{[x,y]^k},\ x \neq y,$$

by Theorem 1.1. It follows from Lemma 2.2 that

$$\begin{aligned}\int_{\mathbb{B}}\int_{\mathbb{B}} |Lu(x,y)|^k dv_\gamma(x) dv_\gamma(y) &\lesssim \int_{\mathbb{B}} |g(x)|^k \int_{\mathbb{B}} \frac{dv_\gamma(y)}{[x,y]^k} dv_\gamma(x)\\ &\lesssim \int_{\mathbb{B}} |g(x)|^k dv_\gamma(x).\end{aligned}$$

One can see that

$$\int_{\mathbb{B}} |g(x)|^k dv_\gamma(x) \lesssim \int_{\mathbb{B}} |u(x)|^k dv_\gamma(x)$$

from the proof of Theorem 1.1. Consequently, $L : \mathcal{A}^k_\gamma \to L^k(\mathbb{B}\times\mathbb{B}, dv_\gamma \times dv_\gamma) \cap \mathcal{H}(\mathbb{B}\times\mathbb{B})$ is bounded. $\square$

For the case of $n+\gamma < k < \infty$, we can also prove the following result by using an argument similar to the one in the proof of Theorem 3.1.

Theorem 3.2. *Let $\gamma > -1$, $n+\gamma < k < \infty$ and $t = (k+\gamma-n)/2$. Then $L : \mathcal{A}^k_\gamma \to L^k(\mathbb{B}\times\mathbb{B}, dv_t \times dv_t) \cap \mathcal{H}(\mathbb{B}\times\mathbb{B})$ is bounded.*

Remark 3.1. For $\gamma > -1, k \geq n+\gamma$, L never maps $\mathcal{A}^k_\gamma$ into $L^k(\mathbb{B}\times\mathbb{B}, dv_\gamma \times dv_\gamma) \cap \mathcal{H}(\mathbb{B}\times\mathbb{B})$, see Example 3.9 in [7].

Proof of Theorem 1.2. Assume that (1) holds. Fixing $x \in \mathbb{B}$, it follows from the proof of Theorem 1.1 that

$$(1-|x|)|\nabla u(x)| \leq C\Big(\fint_{\mathbb{B}(x,\frac{1-|x|}{4})} |u(y)|^k dv(y)\Big)^{\frac{1}{k}}.$$

Replacing u by $u - u(x)$ leads to

$$\begin{aligned}(1-|x|)^k|\nabla u(x)|^k &\lesssim \fint_{\mathbb{B}(x,\frac{1-|x|}{4})} |u(x)-u(y)|^k dv(y)\\ &\lesssim \frac{(1-|x|)^{k-t}}{(1-|x|)^n}\int_{\mathbb{B}(x,\frac{1-|x|}{4})} \frac{|u(x)-u(y)|^k}{[x,y]^k} dv_t(y)\\ &\lesssim (1-|x|)^{k-t-n}\int_{\mathbb{B}} \frac{|u(x)-u(y)|^k}{[x,y]^k} dv_t(y).\end{aligned}$$

From the assumption $k-t-n = s-\gamma$, we concludes that

$$\int_{\mathbb{B}} (1-|x|)^k|\nabla u(x)|^k dv_\gamma(x) \lesssim \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(x)-u(y)|^k}{[x,y]^k} dv_s(x)dv_t(y).$$

Hence $u \in \mathcal{A}^k_\gamma$ by Proposition 3.1.

Conversely, suppose that $u \in \mathcal{A}^k_\gamma$. An elementary triangle inequality gives that

$$I \leq C\int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(x)|^k + |u(y)|^k}{[x,y]^k} dv_s(x)dv_t(y).$$

So the integral I will be finite if both of the following integrals are finite:

$$I_1 = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(x)|^k}{[x,y]^k} dv_s(x)dv_t(y),\ \ I_2 = \int_{\mathbb{B}}\int_{\mathbb{B}} \frac{|u(y)|^k}{[x,y]^k} dv_s(x)dv_t(y).$$

Since $s-\gamma = k-t-n$, it follows from Fubini's theorem and Lemma 2.2 that

$$\begin{aligned}I_1 &= \int_{\mathbb{B}} |u(x)|^k dv_s(x)\int_{\mathbb{B}} \frac{dv_t(y)}{[x,y]^k}\\ &\lesssim \int_{\mathbb{B}} \frac{|u(x)|^k dv_s(x)}{(1-|x|^2)^{s-\gamma}}\\ &\lesssim \int_{\mathbb{B}} |u(x)|^k dv_\gamma(x).\end{aligned}$$

A similar argument shows that I_2 is also finite. This proves Theorem 1.2. $\square$

4. Characterizations of Bloch space

Proof of Theorem 1.3. Assume that (2) holds. Fixing $x \in \mathbb{B}$, it follows from the proof of Theorem 1.2 that

$$|\nabla u(x)| \leq \frac{C}{1-|x|} \fint_{\mathbb{B}(x, \frac{1-|x|}{4})} |u(x) - u(y)| dv(y).$$

By Lemma 2.1,

$$\begin{aligned}
&\omega\big((1-|x|)^\alpha\big)|\nabla u(x)| \\
&\leq C \fint_{\mathbb{B}(x, \frac{1-|x|}{4})} \omega\big((1-|x|)^\alpha\big) \frac{|u(x)-u(y)|}{[x,y]} dv(y) \\
&\lesssim \fint_{\mathbb{B}(x, \frac{1-|x|}{4})} \omega\big((1-|x|)^\beta (1-|y|)^{\alpha-\beta}\big) \frac{|u(x)-u(y)|}{[x,y]^\theta |x-y|^{1-\theta}} dv(y) \\
&< \infty.
\end{aligned}$$

Hence $f \in \mathcal{B}^\alpha_\omega$.

Conversely, we assume that $u \in \mathcal{B}^\alpha_\omega$. For any $x, y \in \mathbb{B}$,

$$\begin{aligned}
|u(x)-u(y)| &= \left| \int_0^1 \frac{du}{dt}((1-t)x + ty) dt \right| \\
&\leq \sqrt{n} |x-y| \int_0^1 |\nabla u((1-t)x+ty)| dt \\
&\leq C|x-y| \int_0^1 \frac{dt}{\omega\big((1-|(1-t)x+ty|)^\alpha\big)}.
\end{aligned}$$

Since for $x, y \in \mathbb{B}$ and $t \in [0,1]$,

$$\begin{aligned}
(1-|(1-t)x+ty|)^\alpha &\geq (1-(1-t)|x| - t|y|)^\alpha \\
&= ((1-t)(1-|x|) + t(1-|y|))^\alpha \\
&\geq t^{\alpha-\beta}(1-|y|)^{\alpha-\beta}(1-t)^\beta(1-|x|)^\beta,
\end{aligned}$$

we get

$$\begin{aligned}
\frac{|u(x)-u(y)|}{[x,y]^\theta |x-y|^{1-\theta}} &\leq C \int_0^1 \frac{dt}{\omega\big((1-|(1-t)x+ty|)^\alpha\big)} \\
&\leq C \int_0^1 \frac{dt}{\omega\Big(t^{\alpha-\beta}(1-|y|)^{\alpha-\beta}(1-t)^\beta(1-|x|)^\beta\Big)} \\
&\leq \frac{C}{\omega\Big((1-|x|)^\beta(1-|y|)^{\alpha-\beta}\Big)} \int_0^1 \frac{dt}{t^{\alpha-\beta}(1-t)^\beta}
\end{aligned}$$

$$\lesssim \frac{1}{\omega\Big((1-|x|)^{\beta}(1-|y|)^{\alpha-\beta}\Big)},$$

where the last integral converges since $\alpha < 1+\beta$. Thus

$$|u(x)-u(y)| \le \frac{C[x,y]^{\theta}|x-y|^{1-\theta}}{\omega\big((1-|x|)^{s}(1-|y|)^{\alpha-s}\big)}.$$

This completes the proof of Theorem 1.3. □

Proof of Theorem 1.4. We first assume that (3) holds. For $x \in \mathbb{B}$, by Lemma 2.3, we have

$$\begin{aligned}
|\nabla u(x)| &\le C\Big(⨍_{\mathbb{B}(x,\frac{1-|x|}{4})}|\nabla u(y)|^{p}dv(y)\Big)^{\frac{1}{p}}\\
&\lesssim (1-|x|)^{-1}\Big(⨍_{\mathbb{B}(x,\frac{1-|x|}{3})}|u(x)-u(y)|^{p}dv(y)\Big)^{\frac{1}{p}}\\
&\lesssim (1-|x|)^{-1}\Big(⨍_{\mathbb{B}(x,\frac{1-|x|}{2})}|u(x)-u(y)|^{q}v(y)\Big)^{\frac{1}{q}}\\
&\lesssim \frac{1}{\omega\big((\frac{1-|x|}{2})^{\alpha}\big)}\\
&\lesssim \frac{1}{\omega\big((1-|x|)^{\alpha}\big)},
\end{aligned}$$

which implies $u \in \mathcal{B}^{\alpha}_{\omega}$.

Conversely, we assume that $u \in \mathcal{B}^{\alpha}_{\omega}$. For $x \in \mathbb{B}$ and $y \in \mathbb{B}(x,r)$, it follows from the proof of Theorem 1.1 that

$$\begin{aligned}
|u(x)-u(y)| &\le \sqrt{n}|x-y|\int_0^1 |\nabla u((1-s)x+sy)|ds\\
&\le C|x-y|\int_0^1 \frac{ds}{\omega\big(((1-s)x+sy|)^{\alpha}\big)}\\
&\le C|x-y|\int_0^1 \frac{ds}{\omega\big((1-|(1-s)x+sy|)^{\alpha}\big)}\\
&\le C|x-y|\int_0^1 \frac{ds}{\omega\big((1-|x|-s|x-y|)^{\alpha}\big)}\\
&\le C\int_0^{|x-y|} \frac{ds}{\omega\big((1-|x|-s)^{\alpha}\big)}.
\end{aligned}$$

Let $x-y=\zeta \in \mathbb{B}$. By a similar argument as in the proof of [5, Theorem 2], we have

$$⨍_{\mathbb{B}(x,r)}|u(x)-u(y)|^{q}dv(y) \le ⨍_{\mathbb{B}(x,r)}\Big(\int_0^{|x-y|}\frac{ds}{\omega\big((1-|x|-s)^{\alpha}\big)}\Big)^{q}dv(y)$$

$$
\begin{aligned}
&= \fint_{r\mathbb{B}} \Big(\int_0^{|\zeta|} \frac{ds}{\omega\big((1-|x|-s)^\alpha\big)}\Big)^q dv(\zeta)\\
&\le \fint_{r\mathbb{B}} \Big(\int_0^{|\zeta|} \frac{|\zeta|^{q-1}ds}{\omega^q\big((1-|x|-s)^\alpha\big)}\Big) dv(\zeta)\\
&\le \frac{C}{r^n}\int_0^r \rho^{n+q-2}\Big(\int_0^\rho \frac{ds}{\omega^q\big((1-|x|-s)^\alpha\big)}\Big)d\rho\\
&\le \frac{C}{r^n}\int_0^r \Big(\int_s^r \rho^{n+q-2}d\rho\Big)\frac{ds}{\omega^q\big((r-s)^\alpha\big)}\\
&\le Cr^{q-2}\int_0^r \frac{(r-s)}{\omega^q\big((r-s)^\alpha\big)}ds\\
&\le Cr^{q-2}\int_0^r \Big[\frac{(r-s)^\alpha}{\omega\big((r-s)^\alpha\big)}\Big]^q \frac{ds}{(r-s)^{\alpha q-1}}\\
&\le \frac{Cr^{\alpha q+q-2}}{\omega^q(r^\alpha)}\int_0^r (r-s)^{1-\alpha q}ds\\
&\le \frac{Cr^q}{\omega^q(r^\alpha)},
\end{aligned}
$$

as desired. The proof of Theorem 1.4 is finished. □

5. Integral means of p-harmonic functions

In this section, we shall prove Theorem 1.5. Before the proof, we recall a lemma which comes from [3].

Lemma 5.1. *Suppose that $\alpha > 0$, $\beta \le \alpha$ and ω is a majorant. Then, for $\rho \in (0,1)$, $\psi(\rho)$ and $\psi(\rho)/\omega(\psi(\rho))$ are decreasing in $(0,1)$, where ψ is the same as in Theorem 1.5.*

Proof of Theorem 1.5. We first prove (I). Let $x = r\xi \in \mathbb{B}$, where $r = |x|$, $\xi \in \mathbb{S}$. Then

$$
\begin{aligned}
|u(x)| &= \Big|u(0) + \int_0^1 \langle \nabla u(tx), x\rangle dt\Big|\\
&\le |u(0)| + \int_0^r |\nabla u(\rho\xi)|d\rho.
\end{aligned}
$$

By Minkowski's inequality, we see that

$$
\begin{aligned}
M_s(r,u) &= \Big(\int_{\mathbb{S}} |u(r\xi)|^s d\sigma(\xi)\Big)^{\frac{1}{s}}\\
&\le |u(0)| + \Big(\int_{\mathbb{S}}\Big(\int_0^r |\nabla u(\rho\xi)|d\rho\Big)^s d\sigma(\xi)\Big)^{\frac{1}{s}}\\
&\le |u(0)| + \int_0^r \Big(\int_{\mathbb{S}} |\nabla u(\rho\xi)|^s d\sigma(\xi)\Big)^{\frac{1}{s}} d\rho
\end{aligned}
$$

$$= |u(0)| + \int_0^r M_s(\rho, \nabla u) d\rho.$$

Since $M_s(\rho, \nabla u) \le C/\omega\big(\psi(\rho)\big)$, it follows from Lemma 5.1 that

$$\begin{aligned}
M_s(r, u) &\le |u(0)| + C \int_0^r \frac{1}{\omega\big(\psi(\rho)\big)} d\rho \\
&\le |u(0)| + C \int_0^r \frac{\psi(\rho)}{\omega\big(\psi(\rho)\big)} \frac{1}{\psi(\rho)} d\rho \\
&\le |u(0)| + \frac{C}{\omega(1)} \int_0^1 \frac{r}{\psi(rt)} dt \\
&\le |u(0)| + C \int_0^1 \frac{r}{\psi(rt)} dt.
\end{aligned}$$

Now, we prove (II). For $x \in \mathbb{B}$, it follows from Lemma 2.3 and the proof of Theorem 1.2 that

$$\begin{aligned}
(1-|x|)^s |\nabla u(x)|^s &\le \frac{C}{(1-|x|)^n} \int_{\mathbb{B}(x, \frac{1-|x|}{4})} |u(y)|^s dv(y) \\
&\le \frac{C}{(1-|x|)^n} \int_{\mathbb{B}(0, \frac{1+2|x|}{2+|x|})} |u(y)|^s v(y) \\
&\le \frac{C}{(1-|x|)^n} \int_0^{\frac{1+2|x|}{2+|x|}} n\rho^{n-1} \int_{\mathbb{S}} |u(\rho\xi)|^s d\sigma(\xi) d\rho \\
&\le \frac{C}{(1-|x|)^n} \int_0^{\frac{1+2|x|}{2+|x|}} \frac{\rho^{n-1}}{\omega^s\big((1-\rho)^\alpha\big)} d\rho \\
&\le \frac{C}{(1-|x|)^n} \int_0^{\frac{1+2|x|}{2+|x|}} \frac{\rho^{n-1}}{(1-\rho)^{\alpha s}} d\rho \\
&\le \frac{C}{(1-|x|)^{\alpha s + n - 1}}.
\end{aligned}$$

Thus

$$|\nabla u(x)| \le \frac{C}{(1-|x|)^{\alpha + 1 + \frac{n-1}{s}}}.$$

Consequently, for all $q \in (0, \infty)$

$$\begin{aligned}
M_q(r, \nabla u) &= \Big(\int_{\mathbb{S}} |\nabla u(r\xi)|^q d\sigma(\xi) \Big)^{\frac{1}{q}} \\
&\le \frac{C}{(1-r)^{\alpha + 1 + \frac{n-1}{s}}}.
\end{aligned}$$

This completes the proof of Theorem 1.5. $\square$

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