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RELATIVE SELF-CLOSENESS NUMBERS

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Abstract. We define the relative self-closeness number $N\mathcal{E}(g)$ of a map $g : X \to Y$, which is a generalization of the self-closeness number $N\mathcal{E}(X)$ of a connected CW complex X defined by Choi and Lee [1]. Then we compare $N\mathcal{E}(p)$ with $N\mathcal{E}(X)$ for a fibration $X \to E \overset{p}{\to} Y$. Furthermore we obtain its rationalized result.

1. Introduction

Let $\mathcal{E}(X)$ be the group of the self-homotopy equivalence classes of a connected CW complex X. In 2015, H. W. Choi and K. Y. Lee [1] introduced the following concept:

Definition 1. For a connected CW complex X, the subset $\mathcal{A}^k_\sharp(X)$ of $[X,X]$ is defined by

$$\mathcal{A}^k_\sharp(X) = \{\, f \in [X,X] \mid f_\sharp : \pi_i(X) \xrightarrow{\cong} \pi_i(X) \text{ is an isomorphism for any } i \le k\},$$

and the *self-closeness number* $N\mathcal{E}(X)$ *of* X by

$$N\mathcal{E}(X) = \min\{\, k \mid \mathcal{A}^k_\sharp(X) = \mathcal{E}(X)\}.$$

In this paper, we define the relative version:

Definition 2. For a map $g : X \to Y$ between connected CW complexes, let $\mathcal{E}(g) := \{[f] \in \mathcal{E}(X) \mid g \circ f \simeq g\}$ (the group of relative self-homotopy equivalence classes) and

$$\mathcal{A}^k_\sharp(g) := \{f \in [X,X] \mid f_\sharp : \pi_i(X) \xrightarrow{\cong} \pi_i(X) \text{ is an isomorphism for any } i \le k \text{ and } g \circ f \simeq g\}.$$

Then the relative self-closeness number of a map g is defined as

$$N\mathcal{E}(g) := \min\{k \mid \mathcal{A}^k_\sharp(g) = \mathcal{E}(g)\}.$$

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In [6], N. Oda and the author gived evaluations of self-closeness numbers in fibrations. We compare $N\mathcal{E}(p)$ with $N\mathcal{E}(X)$ for a fibration $X \to E \xrightarrow{p} Y$ in §2. Furthermore we obtain its rationalized result by using Sullivan model [7] in §3. In this paper, we often confuse a map and its homotopy class.

2. An upper bound in a fibration

Lemma 3. (1) *It is a homotopy invariant, i.e.,* $N\mathcal{E}(g_1) = N\mathcal{E}(g_2)$ *if* $g_1 \simeq g_2 : X \to Y$.

(2) *For any map* $g : X \to Y$, $N\mathcal{E}(g) \leq N\mathcal{E}(X)$. *In particular,* $N\mathcal{E}(id_X) = 0$ *for* $id_X : X \overset{=}{\to} X$ *and* $N\mathcal{E}(c) = N\mathcal{E}(X)$ *for the constant map* $c : X \to *$.

(3) *For maps* $g_i : X \to Y_i$, $N\mathcal{E}(g_1) \leq N\mathcal{E}(g_2)$ *if* $h \circ g_1 \simeq g_2$ *for a map* $h : Y_1 \to Y_2$.

Proof. (1) It is obvious from [1, Theorem 1] and the definition.

(2) It is obvious since $\mathcal{A}^k_\sharp(g) \subset \mathcal{A}^k_\sharp(X)$.

(3) Let $N\mathcal{E}(g_2) = k$. Suppose $\pi_{\leq k}(f)$ is an isomorphism for a map $f : X \to X$. If $g_1 \circ f \simeq g_1$, then $g_2 \circ f \simeq g_2$. Then $f \in \mathcal{E}(X)$ from the assumption. Thus we have $N\mathcal{E}(g_1) \leq k$. □

Example 4. (1) For the projection $g : S^m \times S^n \to S^n$, $N\mathcal{E}(g) = m$.

(2) For the Hopf map $\eta : S^3 \to S^2$, $N\mathcal{E}(\eta) = 0$.

Theorem 5. *Let* $X \xrightarrow{j} E \xrightarrow{p} Y$ *be a fibration. Then* $N\mathcal{E}(X) + 1 \geq N\mathcal{E}(p)$.

Proof. Let $k := N\mathcal{E}(X)$. Suppose that $f \in [E, E]$ with $p \circ f \simeq p$ and $\pi_{\leq k+1}(f)$ isomorphic. Then there is the restriction map f' of f in the homotopy commutative diagram:

$$\begin{array}{ccccc} X & \xrightarrow{j} & E & \xrightarrow{p} & Y \\ \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle =} \\ X & \xrightarrow{j} & E & \xrightarrow{p} & Y \end{array}$$

in which $\pi_{\leq k}(f')$ is isomorphic from the five lemma about the commutative diagram between homotopy exact sequences:

$$\begin{array}{ccccccccc} \pi_{i+1}(E) & \xrightarrow{\pi_{i+1}(p)} & \pi_{i+1}(Y) & \xrightarrow{\partial} & \pi_i(X) & \xrightarrow{\pi_i(j)} & \pi_i(E) & \xrightarrow{\pi_i(p)} & \pi_i(Y) \\ {\scriptstyle \pi_{i+1}(f)}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle =} & & {\scriptstyle \pi_i(f')}\downarrow & & {\scriptstyle \pi_i(f)}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle =} \\ \pi_{i+1}(E) & \xrightarrow{\pi_{i+1}(p)} & \pi_{i+1}(Y) & \xrightarrow{\partial} & \pi_i(X) & \xrightarrow{\pi_i(j)} & \pi_i(E) & \xrightarrow{\pi_i(p)} & \pi_i(Y) \end{array}$$

for $i \le k$. From the definition of k, $f' \in \mathcal{E}(X)$. Then $f_\sharp : \pi_*(E) \to \pi_*(E)$ is isomorphic from the five lemma about the commutative diagram:

$$\begin{array}{ccccccccc}
\pi_{i+1}(Y) & \xrightarrow{\pi_{i+1}(p)} & \pi_i(X) & \xrightarrow{\pi_i(j)} & \pi_i(E) & \xrightarrow{\pi_i(p)} & \pi_i(Y) & \xrightarrow{\partial} & \pi_{i-1}(X) \\
\downarrow{=} & & \pi_i(f')\downarrow{\cong} & & \pi_i(f)\downarrow & & \downarrow{=} & & \pi_{i-1}(f')\downarrow{\cong} \\
\pi_{i+1}(Y) & \xrightarrow{\pi_{i+1}(p)} & \pi_i(X) & \xrightarrow{\pi_i(j)} & \pi_i(E) & \xrightarrow{\pi_i(p)} & \pi_i(Y) & \xrightarrow{\partial} & \pi_{i-1}(X)
\end{array}$$

for all i. Thus we have $f \in \mathcal{E}(E)$ from Whitehead theorem. That means $k+1 \ge N\mathcal{E}(p)$. □

3. The rationalized version

In this section, we assume that a space is a simply connected CW complex of finite type. Let X_0 be the rationalization of a space X [5]. Then $\pi_*(X_0) = \pi_*(X) \otimes \mathbb{Q}$ and $H_*(X_0;\mathbb{Z}) = H_*(X;\mathbb{Q})$. We assume familiarity with rational homotopy theory as in the text [2].

Let $M(X) = (\Lambda V, d)$ be the Sullivan minimal model of a space X [7]. It is a free commutative differential graded algebra over $\mathbb{Q}$ (DGA) with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i>1} V^i$ where $\dim V^i < \infty$ and a decomposable differential, namely $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of ΛV generated by elements of positive degree. The degree of a homogeneous element x of a graded algebra is denoted by $|x|$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. Note that $M(X)$ determines the rational homotopy type of X. In particular, $V^n \cong \mathrm{Hom}(\pi_n(X), \mathbb{Q})$ for all n and $H^*(\Lambda V, d) \cong H^*(X;\mathbb{Q})$ as graded $\mathbb{Q}$-algebras.

Now we recall "DGA-homotopy" in [3, Chapter X]: In general, two maps $f : M(Y) \to M(X)$ and $g : M(Y) \to M(X)$ are DGA-homotopic (denote as $f \simeq g$) if there is a DGA-map $H : M(Y) \to M(X) \otimes \Lambda(t, dt)$ such that $H\mid_{t=0,dt=0} = f$ and $H\mid_{t=1,dt=0} = g$. Here $|t| = 0$ and $|dt| = 1$ with $d(t) = dt$, $d(dt) = 0$. Then we have $[X_0, Y_0] \cong [M(Y), M(X)]$ as homotopy sets. Let $\mathrm{Aut}M$ be the group of DGA-automorphisms of a DGA M. For a nilpotent space X and the model $M(X)$, there is a group isomorphism $\mathcal{E}(X_0) \cong \mathcal{E}(M(X)) := \mathrm{Aut}M(X)/\sim$, which is the group of self-DGA-homotopy equivalence classes of $M(X)$. Thus we have the *rational self-closeness number* of X as $N\mathcal{E}(X_0) = N\mathcal{E}(M(X))$.

A fibration $p : E \to Y$ with fibre X has a minimal model which is a DGA-map $M(p) : M(Y) \to M(E)$. It is induced by a relative or Koszul-Sullivan (KS-)model

$$i \ : \ M(Y) = (\Lambda W, d_Y) \to (\Lambda W \otimes \Lambda V, D),$$

where $D|_W = d_Y$ and $(\Lambda V, \overline{D}) = (\Lambda V, d_X) = M(X)$ and there is a quasi-isomorphism $\rho_E : M(E) = (\Lambda U, d_E) \xrightarrow{\sim} (\Lambda W \otimes \Lambda V, D)$ such that $\rho_E \circ M(p) \simeq i$. Let D_1 be the indecomposable part of D.

Theorem 6. *Let $\xi : X \stackrel{j}{\to} E \stackrel{p}{\to} Y$ be a fibration of simply connected complexes. Then $N\mathcal{E}(X_0) \geq N\mathcal{E}(p_0)$. In particular, $N\mathcal{E}(X_0) = N\mathcal{E}(p_0)$ if ξ is rationally fibre-trivial.*

Proof. Let $k := N\mathcal{E}(X_0) = N\mathcal{E}(\Lambda V, d_X)$. Suppose that $f \in [E_0, E_0]$ with $p_0 \circ f = p_0$ and $\pi_{\leq k}(f)$ isomorphic. Let $F : (\Lambda W \otimes \Lambda V, D) \to (\Lambda W \otimes \Lambda V, D)$ be the corresponding DGA-map for f and let $\rho_E : (\Lambda U, d_E) \to (\Lambda W \otimes \Lambda V, D)$ a minimal model. Then

$$\begin{array}{ccccc} (\Lambda W, d_Y) & \xrightarrow{i} & (\Lambda W \otimes \Lambda V^{\leq k}, D) & \xleftarrow{\rho_E} & (\Lambda U^{\leq k}, d_E) \\ \downarrow = & & \downarrow F & & \cong \downarrow M(f) \\ (\Lambda W, d_Y) & \xrightarrow{i} & (\Lambda W \otimes \Lambda V^{\leq k}, D) & \xleftarrow{\rho_E} & (\Lambda U^{\leq k}, d_E) \end{array}$$

induces that $V^{\leq k} \stackrel{F}{\to} \Lambda W \otimes \Lambda V^{\leq k} \stackrel{\text{proj.}}{\to} V^{\leq k}$ is isomorphic. Indeed, for $V_2 := \ker(D_1|_V)$ and a decomposition $V = V_1 \oplus V_2$ with $D_1(V_1) \subset W$, we obtain $\rho_E : U \cong W_2 \oplus V_2$ with a decomposition $W = D_1(V_1) \oplus W_2$. Then $\text{proj.} \circ F|_{V_1^{\leq k}} : V_1^{\leq k} \to V_1^{\leq k}$ is isomorphic from the above left commutative diagram and $\text{proj.} \circ F|_{V_2^{\leq k}} : V_2^{\leq k} \to V_2^{\leq k}$ is isomorphic from the right homotopy commutative diagram.

Let $\overline{F} : (\Lambda V, d_X) \to (\Lambda V, d_X)$ be the induced map of F. From the assumption, $\overline{F}$ is isomorphic since $\text{proj.} \circ \overline{F} : V^{\leq k} \to V^{\leq k}$ is isomorphic. Then the commutative diagram between the KS-models of ξ

$$\begin{array}{ccccc} (\Lambda W, d_Y) & \xrightarrow{i} & (\Lambda W \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, d_X) \\ \downarrow = & & \downarrow F & & \cong \downarrow \overline{F} \\ (\Lambda W, d_Y) & \xrightarrow{i} & (\Lambda W \otimes \Lambda V, D) & \longrightarrow & (\Lambda V, d_X) \end{array}$$

induces that E_2-terms of the Serre spectral sequences are isomorphic, i.e., $id^*_{\Lambda W} \otimes \overline{F}^* : H^*(Y;\mathbb{Q}) \otimes H^*(X;\mathbb{Q}) \cong H^*(Y;\mathbb{Q}) \otimes H^*(X;\mathbb{Q})$, we have $f^*(= F^*) : H^*(E;\mathbb{Q}) \cong H^*(E;\mathbb{Q})$. Thus $f : E_0 \to E_0$ is a homotopy equivalence. That means $k \geq N\mathcal{E}(p_0)$.

Furthermore, if ξ is rationally fibre-trivial, i.e., $D = d_Y + d_X$, we have $F \equiv id_{\Lambda W} \otimes \overline{F} \bmod \Lambda^+\mathrm{W} \otimes \Lambda\mathrm{V}$, where $\Lambda^+ W$ is the positive degree elements' subspace of ΛW. Then $F \in \mathcal{E}(\Lambda W \otimes \Lambda V, D)$ if and only if $\overline{F} \in \mathcal{E}(\Lambda V, d_X)$. Thus $N\mathcal{E}(p_0) = N\mathcal{E}(X_0) = k$. □

Example 7. Let $X = S^3 \times S^5 \times S^9$. Of course $N\mathcal{E}(X_0) = 9$. Let $M(X) = (\Lambda(v_1, v_2, v_3), 0)$ with $|v_1| = 3$, $|v_2| = 5$, $|v_3| = 9$. Note that $[X_0, X_0] = [(\Lambda(v_1, v_2, v_3), 0), (\Lambda(v_1, v_2, v_3), 0)] \cong \mathbb{Q}^{\times 3}$ and $\mathcal{E}(X_0) \cong (\mathbb{Q}^*)^{\times 3}$ with $\mathbb{Q}^* = \mathbb{Q} - 0$ by $f(v_i) = a_i v_i$ ($a_i \in \mathbb{Q}$) for $i = 1, 2, 3$. In the following, we see that there are 3-types' rationally free circle actions on X from [4]. When a KS-model

$$(\mathbb{Q}[t], 0) \to (\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D) \to (\Lambda(v_1, v_2, v_3), 0)$$

with $|t| = 2$ induces $\dim H^*(\mathbb{Q}[t] \otimes \Lambda(v_1, v_2, v_3), D) < \infty$, there is a rationally free S^1-action on X where the rational Borel fibration is given by the model [4]. Note that the DGA-map $f : (\Lambda(t, v_1, v_2, v_3), D) \to (\Lambda(t, v_1, v_2, v_3), D)$ preserving t ($f(t) = t$) is given by

$$f(v_1) = a_1 v_1,\ f(v_2) = a_2 v_2 + b_1 v_1 t,\ f(v_3) = a_3 v_3 + b_2 v_2 t^2 + b_3 v_1 t^3$$

with $a_i, b_i \in \mathbb{Q}$. Then from $D \circ f = f \circ D$ we obtain

(1) When $Dv_1 = Dv_2 = 0$ and $Dv_3 = v_1 v_2 t + t^5$, then $N\mathcal{E}(p_0) = 0$.
(2) When $Dv_1 = Dv_2 = 0$ and $Dv_3 = t^5$, then $N\mathcal{E}(p_0) = 5$.
(3) When $Dv_1 = t^2$ and $Dv_2 = Dv_3 = 0$, then $N\mathcal{E}(p_0) = 9$.
(4) When $Dv_1 = 0$, $Dv_2 = t^3$ and $Dv_3 = 0$, then $N\mathcal{E}(p_0) = 9$.

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