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ON NONLINEAR ELLIPTIC EQUATIONS WITH SINGULAR LOWER ORDER TERM

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Abstract. We prove existence and regularity results of solutions for a class of nonlinear singular elliptic problems like

$$\begin{cases} -\operatorname{div}\Big((a(x)+|u|^q)\nabla u\Big) = \dfrac{f}{|u|^\gamma} \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases}$$

where Ω is a bounded open subset of $\mathbb{R}^{\mathbb{N}}(N \geq 2)$, $a(x)$ is a measurable nonnegative function, $q, \gamma > 0$ and the source f is a nonnegative (not identicaly zero) function belonging to $L^m(\Omega)$ for some $m \geq 1$. Our results will depend on the summability of f and on the values of $q, \gamma > 0$.

1. Introduction

Let us consider the following boundary value problem

$$\begin{cases} -\operatorname{div}\Big((a(x)+|u|^q)\nabla u\Big) = \dfrac{f}{|u|^\gamma} \ \text{ in } \Omega, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{1}$$

where Ω is any bounded open subset of $\mathbb{R}^{\mathbb{N}}$ $(N \geq 2)$, $q, \gamma > 0$, f is a nonnegative function belonging to some Lebesgue space $L^m(\Omega)$, $m \geq 1$, and let $a(x)$ be a measurable function satisfying

$$0 < \alpha \leq a(x) \leq \beta \ \text{ a.e. in } \Omega, \tag{2}$$

where α, β are fixed real numbers.

In the linear case, problems of the form

$$\begin{cases} -\operatorname{div}\Big(M(x)\nabla u\Big) = \dfrac{f}{u^\gamma} \ \text{ in } \Omega, \\ u > 0, \\ u = 0 \ \text{ on } \partial\Omega, \end{cases} \tag{3}$$

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where M is a bounded elliptic measurable matrix and f is smooth, have been largely studied in the past by many authors. We refer to the pioneer work of Stuart in [17], Crandall, Rabinowitz and Tartar in [6] and to the one of Lazer and McKenna in [12].

The linear problem (3) has been deeply studied by Boccardo and Orsina in [4] when the datum f belongs $L^m(\Omega)$, $m \geq 1$. They have proved the existence and regularity of solutions depending on the values of γ (by distinguishing between the cases $\gamma > 1$, $\gamma = 1$ and $\gamma < 1$), and on the summability m of the datum f. We emphasize that the main idea used by the authors in [4] in order to deal with the singular term $\frac{f}{u^\gamma}$ is strongly based on the standard maximum principle for elliptic equations which insures the strict positivity of the solutions u. A non existence result was also given in [4] if f is a bounded Radon measure concentrated on a Borel set E of zero capacity for every $\gamma > 0$. After that a large number of papers was devoted to the study the existence of solutions of problems like (1) in both linear and nonlinear cases and in different contexts, for a review of such results we refer to [5, 7–10, 14, 15, 18] and the references therein.

The motivations in studying problem (1) are mainly arise by the papers [1] and [4]. In [1] (see also [2, 13]), the existence of solutions of the following quasilinear elliptic problem of the type

$$(4) \qquad \begin{cases} -\operatorname{div}\Big((a(x) + |u|^q)\nabla u\Big) + b(x)|u|^{p-1}u|\nabla u|^2 = f & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

was investigated when f is nonnegative, f belongs to $L^1(\Omega)$, $a(x)$ satisfying (2), $0 < \mu \leq b(x) \leq \nu$, a.e. in Ω and $p \geq 2q$ (see also the improvements in [13], when the existence of solutions has been proved without any restriction on p, q and on the sign of f).

The aim of this paper is to prove the existence and regularity of solutions of problem (1) depending on the summability of the datum f and the parameters γ and q. As we will see, our growth assumption on the function $a(x) + |u|^q$ has a regularization effect on the solution u and its gradient ∇u, allowing in some cases to have finite energy solutions (i.e., solutions in $H_0^1(\Omega)$) even if f belongs to $L^1(\Omega)$.

Notations. Hereafter, we will make use of two truncation functions T_k and G_k: for every $k \geq 0$ and $r \in \mathbb{R}$, let

$$T_k(r) = \min(k, \max(r, -k)), \quad G_k(r) = r - T_k(r).$$

For the sake of simplicity we will use when referring to the integrals the following notation

$$\int_\Omega f = \int_\Omega f(x)\, dx.$$

Finally, throughout this paper, C will indicate any positive constant which depends only on data and whose value may change from line to line.

Our aim is to prove the existence of weak solutions to problem (1). Here is the definition of solutions we will consider.

Definition 1. A solution of (1) is a function $u \in W_0^{1,1}(\Omega)$ such that

$$\forall \omega \subset\subset \Omega, \quad \exists c_\omega > 0 : u \geq c_\omega \text{ in } \omega, \tag{5}$$

$$(a(x) + u^q)|\nabla u| \in L^1_{loc}(\Omega), \tag{6}$$

and that

$$\int_\Omega (a(x) + u^q)\nabla u \nabla \varphi = \int_\Omega \frac{f\varphi}{u^\gamma}, \quad \forall \varphi \in C_c^1(\Omega). \tag{7}$$

2. Approximation of problem (1)

Let f be a nonnegative measurable function which belongs to some Lebesgue space, let $n \in \mathbb{N}$, $f_n = \frac{f}{1+\frac{1}{n}f}$ and let us consider the following approximate problem

$$\begin{cases} -\operatorname{div}\Big((a(x) + u_n^q)\nabla u_n\Big) = \dfrac{f_n}{(u_n + \frac{1}{n})^\gamma} \quad \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{8}$$

Lemma 2.1. *Problem* (8) *has a nonnegative weak solution* $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$.

Proof. Let $k, n \in \mathbb{N}$ be fixed, $v \in L^2(\Omega)$ and define $w = F(v)$ to be the unique solution of

$$\begin{cases} -\operatorname{div}\Big((a(x) + |T_k(v)|^q)\nabla w\Big) = \dfrac{f_n}{(|v| + \frac{1}{n})^\gamma} \quad \text{in } \Omega, \\ w = 0 \quad \text{on } \partial\Omega, \end{cases}$$

using w as test function, we have using (2)

$$\alpha \int_\Omega |\nabla w|^2 \leq n^{\gamma+1} \int_\Omega |w|,$$

by Hölder inequality together with Poincaré inequality, it follows that

$$\int_\Omega |w|^2 \leq C n^{\gamma+1} \Big(\int_\Omega |w|^2\Big)^{\frac{1}{2}},$$

and so,

$$\int_\Omega |w|^2 \leq C n^{\frac{\gamma+1}{2}}.$$

Hence, the ball of radius $Cn^{\frac{\gamma+1}{2}}$ is invariant for F. Now, let us choose a sequence $v_r \to v$ in $L^2(\Omega)$, then by Lebesgue convergence theorem:

$$\frac{f_n}{(|v_r| + \frac{1}{n})^\gamma} \to \frac{f_n}{(|v| + \frac{1}{n})^\gamma} \quad \text{in } \ L^2(\Omega),$$

and the uniqueness of solution for linear problem yields that $w_r = F(v_r) \to w = F(v)$ in $L^2(\Omega)$. Therefore, we proved that F is continuous.

As we proved before, we have that

$$\int_\Omega |\nabla F(v)|^2 \leq C(\gamma, n) \quad \text{for any } v \in L^2(\Omega),$$

then, $F(v)$ is relatively compact in $L^2(\Omega)$, and by Schauder's fixed point theorem, there exists $u_{n,k} \in H_0^1(\Omega)$ such that $F(u_{n,k}) = u_{n,k}$ for each n, k fixed. Moreover, $u_{n,k}$ belongs to $L^\infty(\Omega)$ for all $k, n \in \mathbb{N}$. Indeed, for $t \geq 1$ fixed, using $G_t(u_{n,k})$ as test function, we obtain, since $u_{n,k} + \frac{1}{n} \geq t \geq 1$ on $\{u_{n,k} \geq t\}$.

$$\int_\Omega |\nabla G_t(u_{n,k})|^2 \leq \int_\Omega f_n G_t(u_{n,k}),$$

and so, the result of [16] implies that $u_{n,k} \in L^\infty(\Omega)$. Furthermore, the estimate of $u_{n,k}$ in $L^\infty(\Omega)$ is independent from $k \in \mathbb{N}$, then for k large enough and for n fixed, $u_n \in H_0^1(\Omega) \cap L^\infty(\Omega)$ is the solution of the following approximate problem

$$\begin{cases} -\operatorname{div}\Big((a(x) + |u_n|^q)\nabla u_n\Big) = \dfrac{f_n}{(|u_n| + \frac{1}{n})^\gamma} & \text{in } \Omega, \\ u_n = 0 \quad \text{on } \partial\Omega. \end{cases}$$

Since $\frac{f_n}{(|u_n| + \frac{1}{n})^\gamma} \geq 0$, the maximum principle implies that $u_n \geq 0$ and this conclude the proof. □

Lemma 2.2. *The sequence u_n is such that for every $\omega \subset\subset \Omega$ there exists c_ω not depending on n such that*

$$u_n \geq c_\omega > 0 \ \text{in } \omega, \quad \forall n \in \mathbb{N}.$$

Proof. We emphasize that since we have an unbouded divergence operator, the method developed in the proof of Lemma 2.2 in [4] does not apply directly here, so, we use the idea in the proof of Lemma 2.3 of [2]. In order to do that, let us first define for $s \geq 0$ the function

$$\Psi_\delta(s) = \begin{cases} 1 \ \text{ if } \ 0 \leq s < 1, \\ \dfrac{1}{\delta}(1 + \delta - s) \ \text{ if } \ 1 \leq s < \delta + 1, \\ 0 \ \text{ if } \ s \geq \delta + 1. \end{cases}$$

We choose $\Psi_\delta(u_n)\varphi$ as test function in (8) with $\varphi \in H_0^1(\Omega) \cap L^\infty(\Omega)$, $\varphi \geq 0$, then we have

$$\int_\Omega (a(x) + u_n^q)\nabla u_n \nabla\varphi \ \Psi_\delta(u_n) = \frac{1}{\delta}\int_{\{1 \leq u_n \leq \delta+1\}} (a(x) + u_n^q)|\nabla u_n|^2 \ \varphi + \int_\Omega \frac{f_n}{(u_n + \frac{1}{n})^\gamma} \ \Psi_\delta(u_n)\varphi,$$

thus, dropping the nonnegative term and letting δ goes to zero, we obtain

$$\int_\Omega (a(x)+u_n^q)\nabla u_n \nabla\varphi \ \chi_{\{0\le u_n<1\}} \ge \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\ \varphi\chi_{\{0\le u_n<1\}}.$$

Therefore

$$\int_\Omega (a(x)+T_1(u_n)^q)\nabla T_1(u_n)\nabla\varphi \ \ge \int_\Omega \frac{f}{2^\gamma(1+f)}\chi_{\{0\le u_n<1\}}\varphi$$

for every $\varphi \in H_0^1(\Omega)\cap L^\infty(\Omega)$ $\varphi\ge 0$.

Since $\frac{f}{2^\gamma(1+f)}\chi_{\{0\le u_n<1\}}$ is not identically zero and $\alpha \le a(x)+T_1(u_n)^q \le \beta+1$, the strong maximum principle (see [11]) implies that there exists $c_\omega>0$ such that $T_1(u_n)\ge c_\omega$ in every $\omega \subset\subset \Omega$, and so $u_n \ge c_\omega$ (since $T_1(u_n)\le u_n$). Therefore, Lemma 2.2 is completely proved. □

In order to prove the existence of solution for problem (1), we need a priori estimates on the approximate solutions u_n, depending on f, q and γ, so that we distinguish between different cases.

3. The case $\gamma < 1$

3.1. The case $\gamma < 1$ and $q > 1-\gamma$

Lemma 3.1. *Let u_n be the solution of problem (8), with $\gamma<1$ and $q>1-\gamma$. Suppose that f belongs to $L^1(\Omega)$. Then u_n is bounded in $H_0^1(\Omega)$.*

Proof. For n fixed, we choose $\varepsilon < \frac{1}{n}$ and use $\Big((u_n+\varepsilon)^\gamma-\varepsilon^\gamma\Big)\Big(1-(1+u_n)^{1-(q+\gamma)}\Big)$ as test function, then we have

$$\begin{aligned}\gamma\int_\Omega &(u_n+\varepsilon)^{\gamma-1}\Big(1-(1+u_n)^{1-(q+\gamma)}\Big)(a(x)+u_n^q)|\nabla u_n|^2 \\ &+(q+\gamma-1)\int_\Omega \Big((u_n+\varepsilon)^\gamma-\varepsilon^\gamma\Big)(a(x)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \\ =&\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\Big((u_n+\varepsilon)^\gamma-\varepsilon^\gamma\Big)\Big(1-(1+u_n)^{1-(q+\gamma)}\Big).\end{aligned} \tag{9}$$

Dropping the first nonnegative term in the left hand side of (9), using (2) and since $\varepsilon<\frac{1}{n}$, we thus obtain

$$\begin{aligned}(q+\gamma-1)&\int_\Omega \Big((u_n+\varepsilon)^\gamma-\varepsilon^\gamma\Big)(\alpha+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \\ \le &\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\Big((u_n+\varepsilon)^\gamma-\varepsilon^\gamma\Big)\Big(1-(1+u_n)^{1-(q+\gamma)}\Big) \le \int_\Omega f,\end{aligned}$$

and passing to the limit on ε

$$\int_\Omega (\alpha u_n^\gamma + u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}} \le C\int_\Omega f. \tag{10}$$

Since we have

$$\int_{\{u_n\geq 1\}}(\alpha+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\leq\int_\Omega(\alpha u_n^\gamma+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}},$$

then it follows from (10) that

$$\frac{\min(\alpha,1)}{2^{q+\gamma-1}}\int_{\{u_n\geq 1\}}|\nabla u_n|^2\leq\min(\alpha,1)\int_{\{u_n\geq 1\}}\frac{1+u_n^{q+\gamma}}{(1+u_n)^{q+\gamma}}|\nabla u_n|^2\leq C\int_\Omega f.$$

Hence

$$\int_{\{u_n\geq 1\}}|\nabla u_n|^2\leq C. \tag{11}$$

Now, we choose $(T_k(u_n)+\varepsilon)^\gamma-\varepsilon^\gamma$ as test function with $\varepsilon<\frac{1}{n}$ in (8), using (2) and dropping the nonnegative term, we get

$$\alpha\int_\Omega\frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\varepsilon)^{1-\gamma}}\leq\int_\Omega\frac{f_n}{(u_n+\frac{1}{n})^\gamma}((T_k(u_n)+\varepsilon)^\gamma-\varepsilon^\gamma)\leq\int_\Omega f.$$

Therefore

$$\int_\Omega|\nabla T_k(u_n)|^2=\int_\Omega\frac{|\nabla T_k(u_n)|^2}{(T_k(u_n)+\varepsilon)^{1-\gamma}}(T_k(u_n)+\varepsilon)^{1-\gamma}\leq C(k+\varepsilon)^{1-\gamma}.$$

Letting ε goes to zero

$$\int_\Omega|\nabla T_k(u_n)|^2\leq Ck^{1-\gamma}. \tag{12}$$

Combining (11) and (12) we obtain

$$\int_\Omega|\nabla u_n|^2=\int_{\{u_n>1\}}|\nabla u_n|^2+\int_{\{u_n\leq 1\}}|\nabla u_n|^2\leq C.$$

Hence, u_n is bounded in $H_0^1(\Omega)$ as desired. □

3.2. The case $\gamma<1$ and $q\leq 1-\gamma$

In this case, we can not have an estimate of u_n in $H_0^1(\Omega)$, but in a larger Sobolev space.

Lemma 3.2. *Let u_n be the solution of problem* (8), *with $\gamma<1$ and $q\leq 1-\gamma$. Suppose that f belongs to $L^1(\Omega)$. Then u_n is bounded in $W_0^{1,r}(\Omega)$, $r=\frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$.*

Proof. For fixed n, we choose $\varepsilon<\frac{1}{n}$ and use $(u_n+\varepsilon)^\gamma-\varepsilon^\gamma$ as test function, we obtain, using (2)

$$\begin{aligned}\gamma\frac{\min(\alpha,1)}{2^{q-1}}\int_\Omega(u_n+\varepsilon)^{q+\gamma-1}|\nabla u_n|^2&\leq\gamma\int_\Omega(\alpha+u_n^q)(u_n+\varepsilon)^{\gamma-1}|\nabla u_n|^2\\&\leq\int_\Omega\frac{f_n}{(u_n+\frac{1}{n})^\gamma}((u_n+\varepsilon)^\gamma-\varepsilon^\gamma)\leq\int_\Omega f,\end{aligned}$$

and by the Sobolev inequality

$$\Big(\int_\Omega \Big((u_n+\varepsilon)^{\frac{q+\gamma+1}{2}} - \varepsilon^{\frac{q+\gamma+1}{2}}\Big)^{2^*}\Big)^{\frac{2}{2^*}} \le C\int_\Omega f. \tag{13}$$

Letting ε goes to zero, then (13) becomes

$$\int_\Omega u_n^{\frac{2^*(q+\gamma+1)}{2}} \le C. \tag{14}$$

Therefore, u_n is bounded in $L^{\frac{N(q+\gamma+1)}{N-2}}(\Omega)$. Now if $r<2$ as in the statement of Lemma 3.2, we have by Hölder inequality

$$\begin{aligned}\int_\Omega |\nabla u_n|^r &= \int_\Omega \frac{|\nabla u_n|^2}{(u_n+\varepsilon)^{(1-(q+\gamma))\frac{r}{2}}}\ (u_n+\varepsilon)^{(1-(q+\gamma))\frac{r}{2}}\\ &\le C\Big(\int_\Omega (u_n+\varepsilon)^{(1-(q+\gamma))\frac{r}{2-r}}\Big)^{1-\frac{r}{2}}.\end{aligned}$$

Thanks to (14), the value of r is such that $\frac{(1-(q+\gamma))r}{2-r} = \frac{N(q+\gamma+1)}{N-2}$, so that the right hand side of the above inequality is bounded, and then u_n is bounded in $W_0^{1,r}(\Omega)$, $r=\frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$ as desired. □

Remark 3.3. As consequence of both Lemmas 3.1 and 3.2, there exist a subsequence (not relabeled) and a function u such that u_n converges weakly to u in $W_0^{1,r}(\Omega)$ (with $r=\frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$) and almost everywhere in Ω.

In the next Lemma we give an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(\Omega)$ for any $\rho<\frac{N}{N-1}$.

Lemma 3.4. *Let u_n be the solution of problem* (8)*, with $\gamma<1$. Suppose that f belongs to $L^1(\Omega)$. Then $u_n^q|\nabla u_n|$ is bounded in $L^\rho(\Omega)$ for every $\rho<\frac{N}{N-1}$.*

Proof. For n fixed, we choose $\varepsilon<\frac{1}{n}$ and we take as test function $\Big((T_1(u_n)+\varepsilon)^\gamma-\varepsilon^\gamma\Big)\Big(1-(1+u_n)^{1-\lambda}\Big)$, with $\lambda>1$, to obtain,

$$\begin{aligned}&\gamma\int_\Omega (T_1(u_n)+\varepsilon)^{\gamma-1}\Big(1-(1+u_n)^{1-\lambda}\Big)(a(x)+u_n^q)|\nabla T_1(u_n)|^2\\ &\quad+(\lambda-1)\int_\Omega\Big((T_1(u_n)+\varepsilon)^\gamma-\varepsilon^\gamma\Big)(a(x)+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^\lambda}\\ &=\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\Big((T_1(u_n)+\varepsilon)^\gamma-\varepsilon^\gamma\Big)\Big(1-(1+u_n)^{1-\lambda}\Big).\end{aligned} \tag{15}$$

Dropping the first nonnegative term in the left hand side of (15), using (2) and the fact that $\alpha+u_n^q \ge \frac{\min(\alpha,1)}{2^{q-1}}(1+u_n)^q$ yield

$$\begin{aligned}&(\lambda-1)\frac{\min(\alpha,1)}{2^{q-1}}\int_\Omega\Big((T_1(u_n)+\varepsilon)^\gamma-\varepsilon^\gamma\Big)(1+u_n)^{q-\lambda}|\nabla u_n|^2\\ &\le\int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\Big((T_1(u_n)+\varepsilon)^\gamma-\varepsilon^\gamma\Big)\Big(1-(1+u_n)^{1-\lambda}\Big).\end{aligned}$$

Since $\varepsilon < \frac{1}{n}$ and $\lambda > 1$, we obtain

$$\int_\Omega \Big((T_1(u_n)+\varepsilon)^\gamma - \varepsilon^\gamma\Big)(1+u_n)^{q-\lambda}|\nabla u_n|^2 \le C\int_\Omega f. \tag{16}$$

Letting ε goes to zero, then (16) becomes

$$\int_{\{u_n>1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2 \le \int_\Omega T_1(u_n)^\gamma(1+u_n)^{q-\lambda}|\nabla u_n|^2 \le C\int_\Omega f. \tag{17}$$

Combining (12) and (17) lead to

$$\begin{aligned}\int_\Omega (1+u_n)^{q-\lambda}|\nabla u_n|^2 &= \int_{\{u_n>1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2+\int_{\{u_n\le 1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2\\ &\le C.\end{aligned}$$

Now let us set $\rho = \frac{N(2+q-\lambda}{N(q+1)-(\lambda+q)}$, using the previous result together with Hölder inequality, we thus have

$$\int_\Omega u_n^{q\rho}|\nabla u_n|^\rho \le \int_\Omega (1+u_n)^{\frac{\rho(q+\lambda)}{2}}\frac{|\nabla u_n|^\rho}{(1+u_n)^{\frac{\rho(\lambda-q)}{2}}} \le C\Big(\int_\Omega (1+u_n)^{\frac{\rho(q+\lambda)}{2-\rho}}\Big)^{\frac{2-\rho}{2}},$$

and the Sobolev inequality yields that

$$\Big(\int_\Omega u_n^{\rho^*(q+1)}\Big)^{\frac{\rho}{\rho^*}} \le C\Big(\int_\Omega u_n^{\frac{\rho(q+\lambda)}{2-\rho}}\Big)^{\frac{2-\rho}{2}},$$

the previous choice of ρ implies that $\rho^*(q+1) = \frac{\rho(q+\lambda)}{2-\rho}$, and since $\lambda > 1$, we obtain an estimate of $u_n^q|\nabla u_n|$ in $L^\rho(\Omega)$ for every $\rho < \frac{N}{N-1}$, as desired. □

In order to pass to the limit in the approximate equations, the almost everywhere convergence of the ∇u_n to ∇u is required, this result will be proved following the same techniques as in [2] (see also [3, 13]).

Lemma 3.5. *The sequence $\{\nabla u_n\}$ converges to ∇u a.e..*

Proof. Let $\varphi \in C_c^1(\Omega)$, $\varphi \ge 0$, $\varphi \equiv 1$ on $\omega \subset\subset \Omega$ and use $T_h(u_n - T_k(u))\varphi$ as test function in (8), we thus have thanks to Lemmas 2.2 and 3.4

$$\begin{aligned}&\int_\Omega (a(x)+u_n^q)|\nabla T_h(u_n-T_k(u))|^2\,\varphi\\ &\le Ch\|\nabla\varphi\|_{L^\infty(\Omega)} + h\|\varphi\|_{L^\infty(\Omega)}\,\frac{1}{c_\omega^\gamma}\int_\Omega f\\ &\quad - \int_\Omega (a(x)+u_n^q)\nabla T_k(u)\nabla T_h(u_n-T_k(u))\,\varphi.\end{aligned}$$

Since $\nabla T_h(u_n - T_k(u)) \ne 0$ (which implies $u_n \le h+k$), we can easily pass to the limit as n tends to ∞, thanks to Remark 3.3, in the right hand side of the above inequality, so that

$$\alpha \limsup_{n\to\infty}\int_\Omega |\nabla T_h(u_n-T_k(u))|^2\,\varphi \le Ch. \tag{18}$$

Let now s be such that $s < r < 2$, where r is as in the statement of Lemma 3.2, we can write

$$\begin{aligned}
(19) \qquad \int_\omega |\nabla u_n - \nabla u|^s &\le \int_\Omega |\nabla u_n - \nabla u|^s \varphi \\
&= \int_{\{|u_n-u|\le h, u\le k\}} |\nabla u_n - \nabla u|^s \varphi \\
&\quad + \int_{\{|u_n-u|\le h, u> k\}} |\nabla u_n - \nabla u|^s \varphi \\
&\quad + \int_{\{|u_n-u|> h\}} |\nabla u_n - \nabla u|^s \varphi .
\end{aligned}$$

If we denote by R the bound of u_n in $W_0^{1,r}(\Omega)$, we have

$$\begin{aligned}
\int_\Omega |\nabla u_n - \nabla u|^s \varphi \le & \int_\Omega |\nabla T_h(u_n - T_k(u))|^s \varphi \\
& + \|\varphi\|_{L^\infty(\Omega)} \Big(2^s R^s (meas\{u > k\})^{1-\frac{s}{r}} \\
& + 2^s R^s (meas\{|u_n - u| > h\})^{1-\frac{s}{r}} \Big).
\end{aligned}$$

Thus, combining (18) and (19), we obtain for every $h > 0$ and every $k > 0$

$$\begin{aligned}
\limsup_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^s \varphi \le & \Big(\frac{2h}{\alpha} \int_\Omega f \Big)^{\frac{s}{2}} \|\varphi\|_{L^\infty(\Omega)} meas(\Omega)^{1-\frac{s}{2}} \\
& + \|\varphi\|_{L^\infty(\Omega)} 2^s R^s (meas\{u > k\})^{1-\frac{s}{r}} .
\end{aligned}$$

Letting h tends to zero and then k tends to infinity, we finally have

$$\limsup_{n\to\infty} \int_\Omega |\nabla u_n - \nabla u|^s \varphi = 0, \quad \forall s < 2.$$

Therefore, up to subsequence, $\{\nabla u_n\}$ converges to ∇u a.e., and Lemma 3.5 is completely proved. □

Now we are in position to prove our existence result given by the following Theorem.

Theorem 3.6. *Let $\gamma < 1$ and f be a nonnegative function in $L^1(\Omega)$. Then there exists a nonnegative solution u of problem* (1) *in the sense of Definition 1. Moreover, the solution u belongs to $H_0^1(\Omega)$ if $q > 1 - \gamma$ and it belongs to $W_0^{1,r}(\Omega)$* (*with r as in the statement of Lemma 3.2*) *if $q \le \gamma - 1$.*

Proof. As we have already said (see Remark 3.3), there exists a function $u \in W_0^{1,r}(\Omega)$, such that u_n converges weakly to u in $W_0^{1,r}(\Omega)$. On the other hand, Lemma 3.4, Lemma 3.5 and Remark 3.3 imply that the sequence $u_n^q|\nabla u_n|$ converges weakly to $u^q|\nabla u|$ in $L^\rho(\Omega)$ for every $\rho < \frac{N}{N-1}$. Hence for every φ in $C_c^1(\Omega)$

$$\lim_{n\to\infty} \int_\Omega (a(x) + u_n^q) \nabla u_n \nabla \varphi = \int_\Omega (a(x) + u^q) \nabla u \nabla \varphi .$$

For the limit of the right hand of (8). Let $\omega = \{\varphi \neq 0\}$, then by Lemma 2.2, one has, for φ in $C_c^1(\Omega)$

$$\Big| \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} \Big| \leq \frac{\|\varphi\|_{L^\infty}}{c_\omega^\gamma} f.$$

Therefore, by Lebesgue convergence theorem, we obtain

$$\lim_{n\to\infty} \int_\Omega \frac{f_n \varphi}{(u_n + \frac{1}{n})^\gamma} = \int_\Omega \frac{f\varphi}{u^\gamma}.$$

Hence, we conclude that the solution u satisfies the conditions (5)-(7) of Definition 1, so that the proof of Theorem 3.6 is now completed. □

4. The case $\gamma = 1$

Lemma 4.1. *Let u_n be the solution of problem* (8)*, with $\gamma = 1$ and suppose that f belongs to $L^1(\Omega)$. Then u_n is bounded in $H_0^1(\Omega) \cap L^{\frac{N(q+2)}{N-2}}(\Omega)$.*

Remark 4.2. In contrast with the case $\gamma < 1$, we have no restriction over q in order to have finite energy solutions. Furthermore, the solution u have an additional summability in $L^s(\Omega)$ with $s = \frac{N(q+2)}{N-2}$.

Proof. We take u_n as a test function in (8), using (2), we have since $\frac{f_n u_n}{u_n + \frac{1}{n}} \leq f$,

$$\alpha \int_\Omega |\nabla u_n|^2 + \int_\Omega u_n^q |\nabla u_n|^2 \leq \int_\Omega f, \tag{20}$$

which implies the boundedness of u_n in $H_0^1(\Omega)$. On the other hand, from (20), by Sobolev embedding, it follows that

$$\Big(\int_\Omega u_n^{\frac{2^*(q+2)}{2}} \Big)^{\frac{2}{2^*}} \leq \int_\Omega f,$$

Hence u_n is bounded in $L^{\frac{N(q+2)}{N-2}}(\Omega)$. □

Lemma 4.3. *Let u_n be the solution of problem* (8)*, with $\gamma = 1$. Suppose that f belongs to $L^1(\Omega)$. Then $u_n^q|\nabla u_n|$ is bounded in $L^\rho(\Omega)$ for every $\rho < \frac{N}{N-1}$.*

Proof. We choose $T_1(u_n)\Big(1 - (1+u_n)^{1-\lambda}\Big)$, with $\lambda > 1$, as test function to obtain,

$$\begin{aligned} &\gamma \int_\Omega T_1(u_n)\Big(1 - (1+u_n)^{1-\lambda}\Big)(a(x) + u_n^q)|\nabla T_1(u_n)| \\ &\quad + (\lambda - 1) \int_\Omega T_1(u_n)(a(x) + u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^\lambda} \\ &= \int_\Omega \frac{f_n}{u_n + \frac{1}{n}} T_1(u_n)\Big(1 - (1+u_n)^{1-\lambda}\Big). \end{aligned}$$

Dropping the nonnegative term, using (2), we have, since $\alpha + u_n^q \geq \frac{\min(\alpha,1)}{2^{q-1}}(1+u_n)^q$,

$$\int_\Omega T_1(u_n)(1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C\int_\Omega f.$$

Then, we obtain

$$\int_{\{u_n>1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq \int_\Omega T_1(u_n)^\gamma(1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C\int_\Omega f. \tag{21}$$

Thanks to Lemma 4.1 and (21), we thus have

$$\int_\Omega (1+u_n)^{q-\lambda}|\nabla u_n|^2 = \int_{\{u_n>1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2 + \int_{\{u_n\leq 1\}}(1+u_n)^{q-\lambda}|\nabla u_n|^2 \leq C.$$

Following exactly the same proof as in Lemma 3.4, we conclude the proof of Lemma 4.1. □

Theorem 4.4. *Let $\gamma = 1$ and f be a function in $L^1(\Omega)$. Then there exists a solution u in $H_0^1(\Omega) \cap L^{\frac{N(q+2)}{N-2}}(\Omega)$ of problem (1) in the sense of Definition 1 .*

Proof. Thanks to Lemmas 2.2, 3.5, 4.1 and 4.3, the proof of Theorem 4.4 is identical to the one of Theorem 3.6. □

5. The strongly singular case $\gamma > 1$

In this case we can not have an estimate on u_n in $H_0^1(\Omega)$. However, we can prove that u_n is bounded in $H_{loc}^1(\Omega)$ such that the boundary condition can be satisfied through the fact that $u_n^{\frac{q+\gamma+1}{2}}$ in $H_0^1(\Omega)$.

Lemma 5.1. *Let u_n be the solution of the problem (8), with $\gamma > 1$. Suppose that f belongs to $L^1(\Omega)$. Then $u_n^{\frac{q+\gamma+1}{2}}$ is bounded in $H_0^1(\Omega)$, and u_n is bounded in $H_{loc}^1(\Omega)$. Moreover if $q \leq \gamma - 1$, then $u_n^q|\nabla u_n|$ is bounded in $L^2(\omega)$ for every $\omega \subset\subset \Omega$.*

Proof. We choose u_n^γ as test function in (8), dropping the nonnegative term, we obtain since $\frac{u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq 1$

$$\int_\Omega u_n^{q+\gamma-1}|\nabla u_n|^2 \leq \int_\Omega \frac{f_n u_n^\gamma}{(u_n+\frac{1}{n})^\gamma} \leq \int_\Omega f, \tag{22}$$

by observing that

$$\int_\Omega u_n^{q+\gamma-1}|\nabla u_n|^2 = \frac{4}{(q+\gamma+1)^2}\int_\Omega |\nabla u_n^{\frac{q+\gamma+1}{2}}|^2,$$

we easily deduce the first result of the Lemma. Next, we take $u_n^\gamma\Big(1-(1+u_n)^{1-(q+\gamma)}\Big)$ as test function, dropping the nonnegative term, using (2), we have

$$(q+\gamma-1)\int_\Omega u_n^\gamma(\alpha+u_n^q)\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\leq\int_\Omega\frac{f_n u_n^\gamma}{(u_n+\frac{1}{n})^\gamma}\leq\int_\Omega f,$$

and so,

$$\int_{\{u_n\geq 1\}}(\alpha+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\leq\int_\Omega(\alpha u_n^\gamma+u_n^{q+\gamma})\frac{|\nabla u_n|^2}{(1+u_n)^{q+\gamma}}\leq C\int_\Omega f,$$

which yields that

$$\frac{\min(\alpha,1)}{2^{q+\gamma-1)}}\int_{\{u_n\geq 1\}}|\nabla u_n|^2\leq\min(\alpha,1)\int_{\{u_n\geq 1\}}\frac{1+u_n^{q+\gamma}}{(1+u_n)^{q+\gamma}}|\nabla u_n|^2\leq C\int_\Omega f,$$

and then

$$\int_{\{u_n\geq 1\}}|\nabla u_n|^2\leq C. \tag{23}$$

Now we take $T_k^\gamma(u_n)$ as test function in (8), using (2), Lemma 2.2 and recalling that $\frac{T_k^\gamma(u_n)}{(u_n+\frac{1}{n})^\gamma}\leq\frac{u_n}{(u_n+\frac{1}{n})^\gamma}\leq 1$ we then obtain

$$\alpha c_\omega^{\gamma-1}\int_\omega|\nabla T_k(u_n)|^2\leq\alpha\int_\Omega T_k^{\gamma-1}(u_n)|\nabla T_k(u_n)|^2\leq\int_\Omega f\quad\forall\omega\subset\subset\Omega,$$

and we arrive at

$$\int_\omega|\nabla T_k(u_n)|^2\leq C\ \ \forall\omega\subset\subset\Omega. \tag{24}$$

Finally, using (23) together with (24) yield that

$$\int_\omega|\nabla u_n|^2=\int_{\omega\cap\{u_n\geq 1\}}|\nabla u_n|^2+\int_\omega|\nabla T_1(u_n)|^2\leq C\ \ \forall\omega\subset\subset\Omega,$$

so that u_n is bounded in $H^1_{loc}(\Omega)$, as desired.

Now starting from (22), we have

$$\int_{\{u_n\geq 1\}}u_n^{q+\gamma-1}|\nabla u_n|^2\leq\int_\Omega\frac{f_n u_n^\gamma}{(u_n+\frac{1}{n})^\gamma}\leq\int_\Omega f.$$

Then we obtain since $2q\leq q+\gamma-1$

$$\int_\omega u_n^{2q}|\nabla u_n|^2=\int_{\omega\cap\{u_n\geq 1\}}u_n^{q+\gamma-1}|\nabla u_n|^2+\int_\omega|\nabla T_1(u_n)|^2\leq C,\ \ \forall\omega\subset\subset\Omega,$$

and then we deduce that $u_n^q|\nabla u_n|$ is bounded in $L^2(\omega)$ for every $\omega\subset\subset\Omega$. □

Remark 5.2. We note that by virtue of Lemma 5.1 we easily deduce the almost everywhere convergence of ∇u_n to ∇u following exactly the same proof as the one of Lemma 3.5.

Now we are in position to prove our existence result given by the following Theorem.

Theorem 5.3. *Let $\gamma > 1$, $q \leq \gamma - 1$ and f be a nonnegative function in $L^1(\Omega)$. Then there exists a nonnegative solution $u \in H^1_{loc}(\Omega)$ of problem* (1) *in the sense of Definition 1.*

Proof. Thanks to Lemmas 2.2, 3.5, 5.1, the proof of Theorem 5.3 is identical to the one of Theorem 3.6. □

6. Regularity results

In this section we study the regularity of solutions of the problem (1) depending on $q, \gamma > 0$ and the summability of f.

6.1. The case $\gamma < 1$

Theorem 6.1. *Let $\gamma < 1$, f be a nonnegative function in $L^m(\Omega)$, $1 < m < \frac{N}{2}$ and we set $m_1 = \frac{2N}{N(q+1)+\gamma(N-2)+2(1-q)} \leq m < \frac{N}{2}$. Then there exists a nonnegative solution u of problem* (1) *given by Theorem 3.6 such that*

(i) *if $m_1 \leq m < \frac{N}{2}$, $q \leq 1-\gamma$, then u belongs to $H^1_0(\Omega) \cap L^s(\Omega)$ with $s = \frac{Nm(q+\gamma+1)}{N-2m}$.*

(ii) *if $1 < m < \frac{N}{2}$, $q > 1-\gamma$, then u belongs to $H^1_0(\Omega) \cap L^s(\Omega)$ with $s = \frac{Nm(q+\gamma+1)}{N-2m}$.*

Proof. We choose u_n^{1-q} as test function to obtain by Hölder inequality

$$(1-q)\int_\Omega |\nabla u_n|^2 \leq \|f_n\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(1-q-\gamma)m'}\Big)^{\frac{1}{m'}},$$

and by Sobolev embedding it follows,

$$\Big(\int_\Omega u_n^{2^*}\Big)^{\frac{2}{2^*}} \leq C\Big(\int_\Omega u_n^{(1-q-\gamma)m'}\Big)^{\frac{1}{m'}}. \tag{25}$$

Now if $m = m_1$, we have $(1-q-\gamma)m' = 2^*$, and since $m < \frac{N}{2}$, we have also that $\frac{2}{2^*} > \frac{1}{m'}$, so from (25) we deduce that u_n is bounded in $H^1_0(\Omega)$ as desired and so u belongs to $H^1_0(\Omega)$.

Next, we choose u_n^r as test function, with $r \geq 1-q$, dropping the nonnegative term and by Hölder inequality we have

$$\int_\Omega u_n^{r+q-1}|\nabla u_n|^2 \leq \|f_n\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(r-\gamma)m'}\Big)^{\frac{1}{m'}},$$

using again Sobolev embedding,

$$\Big(\int_\Omega u_n^{\frac{2^*(r+q+1)}{2}}\Big)^{\frac{2}{2^*}} \leq C\Big(\int_\Omega u_n^{(r-\gamma)m'}\Big)^{\frac{1}{m'}}.$$

Choosing r such that $(r-\gamma)m' = \frac{2^*(r+q+1)}{2}$ which is equivalent to $s = \frac{Nm(q+\gamma+1)}{N-2m}$ and $r \geq 1-q$ implies that $m \geq \frac{2N}{N(q+1)+\gamma(N-2)+2(1-q)}$, then we deduce that u_n is bounded in $L^s(\Omega)$ so that u belongs to $L^s(\Omega)$.

Now it remains to prove (ii), we take u_n^r as test function, with $r > \gamma$, dropping the nonnegative term, and using Hölder inequality together with Sobolev embedding yield that

$$\Big(\int_\Omega u_n^{\frac{2^*(r+q+1)}{2}}\Big)^{\frac{2}{2^*}} \leq \|f_n\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(r-\gamma)m'}\Big)^{\frac{1}{m'}}.$$

Choosing r such that $(r-\gamma)m' = \frac{2^*(r+q+1)}{2}$ which is equivalent to $s = \frac{Nm(q+\gamma+1)}{N-2m}$ and that $r > \gamma$ implies that $m > 1$, then we deduce that u_n is bounded in $L^s(\Omega)$ and so u belongs to $L^s(\Omega)$. □

Remark 6.2. The result of Theorem 6.1 improves that of [4] (see Lemma 5.5). Indeed, we need only f to belong in $L^{m_1}(\Omega)$ ($m_1 < \frac{2N}{N+\gamma(N-2)+2}$) in order to get a finite energy solution. Moreover, the summability in $L^s(\Omega)$ with $s = \frac{Nm(q+\gamma+1)}{N-2m}$ is better than the summability $\frac{Nm(\gamma+1)}{N-2m}$ obtained in [4].

As proved in Lemma 3.2, if $1 \leq m < \frac{2N}{N(q+1)+\gamma(N-2)+2(1-q)}$, then we do not have a finite energy solution.

Theorem 6.3. *Let $\gamma < 1$, $q \leq 1-\gamma$ and f be a function in $L^m(\Omega)$, $1 \leq m < \frac{2N}{N(q+1)+\gamma(N-2)+2(1-q)}$. Then the solution u of problem* (1) *belongs to $W_0^{1,r}(\Omega)$, $r = \frac{N(q+\gamma+1)}{N-(1-(q+\gamma))}$.*

6.2. The case $\gamma = 1$

Theorem 6.4. *Let $\gamma = 1$, f be a nonnegative function in $L^m(\Omega)$, $1 \leq m < \frac{N}{2}$. Then there exists a nonnegative solution u of problem* (1) *given by Theorem 4.4 such that u belongs to $H_0^1(\Omega) \cap L^s(\Omega)$ with $s = \frac{Nm(q+2)}{N-2m}$.*

Proof. We choose u_n^r as test function, with $r \geq 1$, dropping the nonnegative term and by Hölder inequality we have

$$\int_\Omega u_n^{r+q-1}|\nabla u_n|^2 \leq \|f_n\|_{L^m(\Omega)}\Big(\int_\Omega u_n^{(r-1)m'}\Big)^{\frac{1}{m'}},$$

by Sobolev embedding,

$$\Big(\int_\Omega u_n^{\frac{2^*(r+q+1)}{2}}\Big)^{\frac{2}{2^*}} \leq C\Big(\int_\Omega u_n^{(r-1)m'}\Big)^{\frac{1}{m'}}.$$

Choosing r such that $(r-1)m' = \frac{2^*(r+2)}{2}$ which is equivalent to $s = \frac{Nm(q+2)}{N-2m}$ and $r \geq 1$ implies that $m \geq 1$, then we deduce that u_n is bounded in $L^s(\Omega)$ and so u belongs to $L^s(\Omega)$. □

6.3. The case $\gamma > 1$

Theorem 6.5. *Let $\gamma > 1$, $q > \gamma - 1$ and f be a function in $L^m(\Omega)$, $m > 1$. Then there exists a solution u of problem (1) such that if*

$$\max\{1, \frac{N(2q-\gamma+1)}{4q-2\gamma+2+N(q+\gamma+1)}\} < m < \frac{N}{2},$$

then u belongs to $L^s(\Omega)$, $s = \frac{Nm(q+\gamma+1)}{N-2m}$.

Proof. Following the proof of Theorem 6.1, we deduce that u_n is bounded in $L^s(\Omega)$ and so u belongs to $L^s(\Omega)$. Next, we choose $u_n^\gamma\, T_1(u_n - T_k(u_n))$ as test function, we have

$$\begin{aligned}\gamma \int_\Omega u_n^{\gamma-1}(a(x)+u_n^q)|\nabla u_n|^2 T_1(u_n - T_k(u_n)) \\ + \int_{\{k\le u_n\le k+1\}} u_n^\gamma (a(x)+u_n^q)|\nabla u_n|^2 \\ = \int_\Omega \frac{f_n}{(u_n+\frac{1}{n})^\gamma}\, u_n^\gamma\, T_1(u_n - T_k(u_n)),\end{aligned} \tag{26}$$

dropping the second nonnegative term in the left hand side of (26) and using assumption (2), we obtain

$$\int_{\{u_n\ge k+1\}} u_n^{\gamma-1}|\nabla u_n|^2 \le \frac{1}{\gamma\alpha}\int_{\{u_n\ge k+1\}} f. \tag{27}$$

Thus, thanks to the estimate (27), we have

$$\begin{aligned}\int_{\{u_n>k\}} u_n^q|\nabla u_n| &\le \Big(\int_{\{u_n>k\}} u_n^{2q-\gamma+1}\Big)^{\frac12}\Big(\int_{\{u_n>k\}} u_n^{\gamma-1}|\nabla u_n|^2\Big)^{\frac12} \\ &\le \Big(\int_{\{u_n>k\}} u_n^{2q-\gamma+1}\Big)^{\frac12}\Big(\frac{1}{\gamma\alpha}\int_{\{u_n>k\}} f\Big)^{\frac12}.\end{aligned}$$

Since u_n is bounded in $L^s(\Omega)$, then $2q-\gamma+1 \le s$ is equivalent to $m \ge \frac{N(2q-\gamma+1)}{4q-2\gamma+2+N(q+\gamma+1)}$, and we thus have

$$\int_{\{u_n>k\}} u_n^q|\nabla u_n| \le C\Big(\int_{\{u_n>k\}} f\Big)^{\frac12}. \tag{28}$$

Now let $\varphi \in C_c^1(\Omega)$, $\varphi \ge 0$, $\varphi \equiv 1$ on $\omega \subset\subset \Omega$ and E be a measurable subset of Ω, using (28) and Lemma 5.1, we obtain

$$\begin{aligned}\int_{E\cap\omega} u_n^q|\nabla u_n| \le \int_E u_n^q|\nabla u_n|\varphi \le \int_{\{u_n>k\}} u_n^q|\nabla u_n|\varphi + k^q\int_E |\nabla u_n|\varphi \\ \le C\|\varphi\|_{L^\infty}\Big(\int_{\{u_n>k\}} f\Big)^{\frac12} + \|\varphi\|_{L^\infty} k^q meas(E)^{\frac12}\Big(\int_\omega |\nabla u_n|^2\Big)^{\frac12}.\end{aligned}$$

Taking the limit as $meas(E)$ tends to zero, k tends to infinity and since $u_n^q|\nabla u_n|$ converges to $u^q|\nabla u|$ almost everywhere, we easily verify thanks to Vitali's theorem that

$$u_n^q|\nabla u_n| \text{ converge strongly to } u^q|\nabla u| \text{ in } L^1_{loc}(\Omega). \tag{29}$$

Therefore, putting together (29), Lemma 2.2 and Lemma 5.1, we conclude the proof of Theorem 6.5. □

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