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H-TOEPLITZ OPERATORS ON THE BERGMAN SPACE

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Abstract. As an extension to the study of Toeplitz operators on the Bergman space, the notion of H-Toeplitz operators B_ϕ is introduced and studied. Necessary and sufficient conditions under which H-Toeplitz operators become co-isometry and partial isometry are obtained. Some of the invariant subspaces and kernels of H-Toeplitz operators are studied. We have obtained the conditions for the compactness and Fredholmness for H-Toeplitz operators. In particular, it has been shown that a non-zero H-Toeplitz operator can not be a Fredholm operator on the Bergman space. Moreover, we have also discussed the necessary and sufficient conditions for commutativity of H-Toeplitz operators.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex-plane $\mathbb{C}$ and dA be the area measure on $\mathbb{D}$ normalized so that the area of the disc $\mathbb{D}$ is 1. Let $H(\mathbb{D})$ be the space of all analytic functions on $\mathbb{D}$ and $L^2(\mathbb{D}, dA)$ be the Hilbert space of all complex-valued, absolute square integrable Lebesgue measurable functions on $\mathbb{D}$ with the inner product

$$\langle f, g\rangle_{L^2} = \int_{\mathbb{D}} f(z)\overline{g(z)}dA(z) \quad \text{for all } f \text{ and } g \text{ in } L^2(\mathbb{D}, dA).$$

The holomorphic Bergman space or Bergman space denoted by $L^2_a(\mathbb{D})$ is defined as $L^2_a(\mathbb{D}) = \left\{f \in L^2(\mathbb{D}, dA) \mid f \text{ is analytic on } \mathbb{D}\right\}$, equivalently, $L^2_a(\mathbb{D}) = H(\mathbb{D}) \cap L^2(\mathbb{D}, dA)$. The space $L^2_a(\mathbb{D})$ being a closed subspace of the Hilbert space $L^2(\mathbb{D}, dA)$ is a Hilbert space. For non-negative integer n, let $e_n(z) = \sqrt{n+1}z^n$ for all $z \in \mathbb{D}$. Then the collection $\{e_n\}_{n\geq 0}$ forms an orthonormal basis for $L^2_a(\mathbb{D})$. For $z, w \in \mathbb{D}$, the reproducing kernel in the Bergman space, known as Bergman kernel [4] is given by $K_z(w) = \overline{K(z, w)} = \frac{1}{(1-\bar{z}w)^2}$. Let $P_{L^2_a} : L^2(\mathbb{D}, dA) \to L^2_a(\mathbb{D})$ denote the orthogonal projection, known as Bergman projection, from the space $L^2(\mathbb{D}, dA)$ to the space $L^2_a(\mathbb{D})$. Since $P_{L^2_a}(f) \in L^2_a(\mathbb{D})$

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for all $f \in L^2(\mathbb{D}, dA)$ and therefore on using the property of reproducing kernel $K(z, w)$ the explicit formula for $P_{L^2_a}(f)$ is given by

$$P_{L^2_a}(f)(z) = \langle P_{L^2_a} f, K_z \rangle = \langle f, K_z \rangle = \int_{\mathbb{D}} \frac{f(w)}{(1 - \bar{w}z)^2} dA(w).$$

The space $L^\infty(\mathbb{D})$ denote the Banach space of all essentially bounded Lebesgue measurable functions f on $\mathbb{D}$ with the norm given by

$$\|f\|_\infty = \text{ess sup} \left\{ |f(z)| : z \in \mathbb{D} \right\}.$$

Let $L^2_{\text{harm}}(\mathbb{D})$ denote the space of all harmonic functions in $L^2(\mathbb{D}, dA)$. The space $L^2_{\text{harm}}(\mathbb{D})$ is a closed subspace of $L^2(\mathbb{D}, dA)$ and thus is a Hilbert space. Let $P_{\text{harm}} : L^2(\mathbb{D}, dA) \to L^2_{\text{harm}}(\mathbb{D})$ be the orthogonal projection from the space $L^2(\mathbb{D}, dA)$ onto the space $L^2_{\text{harm}}(\mathbb{D})$. Let $C(\overline{\mathbb{D}})$ denote the collection of continuous functions on the closed unit disc $\mathbb{D}$. For $\phi \in L^\infty(\mathbb{D})$, the multiplication operator M_ϕ is defined as the operator $M_\phi : L^2(\mathbb{D}, dA) \to L^2(\mathbb{D}, dA)$ such that $M_\phi(f) = \phi f$ for all $f \in L^2(\mathbb{D}, dA)$. The Toeplitz operator T_ϕ with the symbol $\phi \in L^\infty(\mathbb{D})$ is the operator $T_\phi : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ defined by

$$T_\phi(f) = P_{L^2_a}(\phi f) = \int_{\mathbb{D}} \frac{\phi(z) f(z)}{(1 - \bar{w}z)^2} dA(w)$$

for all $f \in L^2_a(\mathbb{D})$. It is clear that T_ϕ is a bounded operator on $L^2_a(\mathbb{D})$ and $\|T_\phi f\| = \|P_{L^2_a}(\phi f)\| \leq \|\phi\|_\infty \|f\|$ for all $f \in L^2_a(\mathbb{D})$. The study of Toeplitz operators on the Bergman space has began in the seventies by G. Mc Donald and C. Sunderberg in [11]. Since, then Toeplitz operators studied on Bergman space with various symbols. For $\phi \in L^\infty(\mathbb{D})$, the Hankel operator defined as the operator $H_\phi : L^2_a(\mathbb{D}) \to L^2_a(\mathbb{D})$ such that $H_\phi(f) = P_{L^2_a} M_\phi J(f)$ for each $f \in L^2_a(\mathbb{D})$, where the operator $J : L^2_a(\mathbb{D}) \to \overline{L^2_a(\mathbb{D})}$ is given by $J(e_n(z)) = \overline{e_{n+1}(z)}$ for all non-negative integers n. H_ϕ is a bounded linear operator on $L^2_a(\mathbb{D})$. Indeed, $\|H_\phi f\| = \|P_{L^2_a} M_\phi J f\| \leq \|M_\phi\| \|J\| \|f\| = \|\phi\|_\infty \|J\| \|f\|$ for all $f \in L^2_a(\mathbb{D})$. Recently, lots of study about Toeplitz and Hankel operators have been done in the Bergman space (see [5, 7, 9, 12, 14, 16]). Various generalizations of Toeplitz and Hankel operators on spaces of analytic functions have been studied by many mathematicians. In 2007, Arora and Paliwal [3] have introduced and studied the notion of H-Toeplitz operators on the Hardy space, where they have clubbed the notion of Toeplitz and Hankel operators together. The importance of this notion is that it is associated with a class of Toeplitz operators and a class of Hankel operators on the Hardy space where the original operators are neither Toeplitz nor Hankel. Moreover, it can also be observed that an $n \times n$ H-Toeplitz matrix has $2n - 1$ degree of freedom rather than n^2 and therefore for large n, it is comparatively easy to solve the system of linear equations where the coefficient matrix is an H-Toeplitz matrix. In the most recent paper [10], authors introduced the notion of slant H-Toeplitz operators on the Hardy space and studied various properties including compactness and commutativity. Motivated by these developments we have introduced the notion of H-Toeplitz operators B_ϕ on the Bergman space $L^2_a(\mathbb{D})$, where ϕ is either harmonic or in $C(\overline{\mathbb{D}})$.

The organisation of paper is as follows. In Section 2, we have introduced the notion of H-Toeplitz operator on the Bergman space $L^2_a(\mathbb{D})$ and studied various properties of these operators. Some of the invariant subspaces and kernels of H-Toeplitz operators are obtained. In Section 3, we have obtained conditions under which the operator B_ϕ is compact and Fredholm. In Section 4, we have discussed the commutativity of H-Toeplitz operators with harmonic symbols.

2. H-Toeplitz operators on $L^2_a(\mathbb{D})$

We begin this section with the following lemmas which are used in subsequent results:

Lemma 2.1. *In the harmonic Bergman space $L^2_{harm}(\mathbb{D})$, for non-negative integers s and t, the following hold*:

$$P_{harm}(\bar{z}^m z^n) = \begin{cases} \dfrac{m-n+1}{m+1}\bar{z}^{m-n} & \text{if } m > n, \\ \dfrac{n-m+1}{n+1} z^{n-m} & \text{otherwise.} \end{cases} \tag{1}$$

Proof. For the case when $n \geq m$, we compute

$$\begin{aligned} \left\langle P_{\text{harm}}(\bar{z}^m z^n), z^k \right\rangle &= \langle \bar{z}^m z^n, z^k \rangle \\ &= \int_{\mathbb{D}} \bar{w}^m w^n \bar{w}^k dA(w) \\ &= \frac{1}{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^{m+n+k+1} e^{i(n-m-k)\theta} dr d\theta \\ &= \frac{1}{\pi} \frac{1}{(m+n+k+2)} \int_{\theta=0}^{2\pi} e^{i(n-m-k)\theta} d\theta \\ &= \begin{cases} \frac{1}{n+1} & \text{if } k = n-m \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} \frac{k+1}{n+1} \langle z^k, z^k \rangle & \text{if } k = n-m \\ 0 & \text{otherwise} \end{cases} \\ &= \left\langle \frac{n-m+1}{n+1} z^{n-m}, z^k \right\rangle. \end{aligned}$$

On the other side, for $m > n$, we compute

$$\begin{aligned} \left\langle P_{\text{harm}}(\bar{z}^m z^n), \bar{z}^k \right\rangle &= \langle \bar{z}^m z^n, \bar{z}^k \rangle \\ &= \int_{\mathbb{D}} \bar{w}^m w^n w^k dA(w) \\ &= \frac{1}{\pi} \int_{\theta=0}^{2\pi} \int_{r=0}^{1} r^{m+n+k+1} e^{i(n+k-m)\theta} dr d\theta \end{aligned}$$

$$= \frac{1}{\pi} \frac{1}{(m+n+k+2)} \int_{\theta=0}^{2\pi} e^{i(n+k-m)\theta} d\theta$$
$$= \frac{1}{m+1}$$
$$= \frac{k+1}{m+1} \langle \bar{z}^k, \bar{z}^k \rangle$$
$$= \left\langle \frac{m-n+1}{m+1} \bar{z}^{m-n}, \bar{z}^k \right\rangle.$$

Hence, we get the expression (2.1). □

A straightforward calculation gives the following lemma:

Lemma 2.2. *In the Bergman space $L_a^2(\mathbb{D})$, for non-negative integers s and t, the following hold*:

(a) $\langle z^s, z^t \rangle = \begin{cases} \frac{1}{s+1} & \textit{if } s = t \\ 0 & \textit{otherwise.} \end{cases}$

(b) $P_{L_a^2}(\bar{z}^t z^s) = \begin{cases} \frac{s-t+1}{s+1} z^{s-t} & \textit{if } s \geq t \\ 0 & \textit{otherwise.} \end{cases}$

Next, we find the matrices of Toeplitz operator T_ϕ and of Hankel operators H_ϕ on the Bergman space having harmonic symbols.

Let us consider the harmonic symbol $\phi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j \in L^\infty(\mathbb{D})$. Then, the $(m,n)^{th}$ entry of the matrix of T_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ of $L_a^2(\mathbb{D})$ is given by

$$\begin{aligned} \langle T_\phi e_n, e_m \rangle &= \langle P_{L_a^2}(\phi \cdot e_n), e_m \rangle \\ &= \sqrt{n+1}\sqrt{m+1} \left\langle \left(\sum_{i=0}^{\infty} a_i z^{i+n} + \sum_{j=1}^{\infty} b_j \bar{z}^j \right) z^n, z^m \right\rangle \\ &= \sqrt{n+1}\sqrt{m+1} \left(\sum_{i=0}^{\infty} a_i \langle z^{i+n}, z^m \rangle + \sum_{j=1}^{\infty} b_j \langle z^n, z^{m+j} \rangle \right). \end{aligned}$$

So, the following two cases arise:
Case (1): If $m \geq n$, then we have

$$\begin{aligned} \langle T_\phi e_n, e_m \rangle &= \sqrt{n+1}\sqrt{m+1} \sum_{i=0}^{\infty} a_i \langle z^{i+n}, z^m \rangle \\ &= \sqrt{n+1}\sqrt{m+1} \frac{1}{m+1} a_{m-n} = \sqrt{\frac{n+1}{m+1}} a_{m-n}. \end{aligned}$$

Case (2): If $m < n$, then we have

$$\begin{aligned}\langle T_\phi e_n, e_m\rangle &= \sqrt{n+1}\sqrt{m+1}\sum_{j=0}^{\infty} b_j \left\langle z^n, z^{m+j}\right\rangle \\ &= \sqrt{n+1}\sqrt{m+1}\frac{1}{n+1}b_{n-m} = \sqrt{\frac{m+1}{n+1}}b_{n-m}.\end{aligned}$$

Thus, we have

$$\langle T_\phi e_n, e_m\rangle = \begin{cases}\sqrt{\dfrac{n+1}{m+1}}a_{m-n} & \text{for } m \geq n,\\ \sqrt{\dfrac{m+1}{n+1}}b_{n-m} & \text{for } n > m,\end{cases} \tag{2}$$

where m and n are non-negative integers.

Therefore, the matrix of T_ϕ is explicitly given by

$$T_\phi = \begin{bmatrix} a_0 & \frac{1}{\sqrt{2}}b_1 & \frac{1}{\sqrt{3}}b_2 & \frac{1}{2}b_3 & \cdots \\ \frac{1}{\sqrt{2}}a_1 & a_0 & \sqrt{\frac{2}{3}}b_1 & \frac{1}{\sqrt{2}}b_2 & \cdots \\ \frac{1}{\sqrt{3}}a_2 & \sqrt{\frac{2}{3}}a_1 & a_0 & \frac{\sqrt{3}}{2}b_1 & \cdots \\ \frac{1}{2}a_3 & \frac{1}{\sqrt{2}}a_2 & \frac{\sqrt{3}}{2}a_1 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and the adjoint of the matrix of T_ϕ is given by

$$T_\phi^* = \begin{bmatrix} \overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{3}}\overline{a_2} & \frac{1}{2}\overline{a_3} & \cdots \\ \frac{1}{\sqrt{2}}\overline{b_1} & \overline{a_0} & \sqrt{\frac{2}{3}}\overline{a_1} & \frac{1}{\sqrt{2}}\overline{a_2} & \cdots \\ \frac{1}{\sqrt{3}}\overline{b_2} & \sqrt{\frac{2}{3}}\overline{b_1} & \overline{a_0} & \frac{\sqrt{3}}{2}\overline{a_1} & \cdots \\ \frac{1}{2}\overline{b_3} & \frac{1}{\sqrt{2}}\overline{b_2} & \frac{\sqrt{3}}{2}\overline{b_1} & \overline{a_0} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

which is nothing but the matrix of $T_{\overline{\phi}}$ and therefore $T_\phi^* = T_{\overline{\phi}}$.

Next we find the matrix of Hankel operator having harmonic symbols. For $\phi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j \in L^\infty(\mathbb{D})$, the $(m,n)^{th}$ entry of the matrix of H_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ is given by

$$\begin{aligned}\langle H_\phi e_n, e_m\rangle &= \left\langle P_{L_a^2} M_\phi J e_n, e_m\right\rangle \\ &= \sqrt{n+2}\sqrt{m+1}\left\langle \left(\sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j\right)\bar{z}^{n+1}, z^m\right\rangle \\ &= \sqrt{n+2}\sqrt{m+1}\left(\sum_{i=0}^{\infty} a_i \left\langle z^i, z^{m+n+1}\right\rangle + \sum_{j=1}^{\infty} b_j \left\langle \bar{z}^j, z^{m+n+1}\right\rangle\right)\end{aligned}$$

$$= \frac{\sqrt{m+1}\sqrt{n+2}}{m+n+2} a_{m+n+1} \tag{3}$$

for non-negtive integers m and n.

Thus, the matrix of H_ϕ in explicit form is given by

$$H_\phi = \begin{bmatrix} \frac{1}{\sqrt{2}}a_1 & \frac{1}{\sqrt{3}}a_2 & \frac{1}{2}a_3 & \frac{1}{\sqrt{5}}a_4 & \cdots \\ \frac{2}{3}a_2 & \frac{\sqrt{6}}{4}a_3 & \frac{2\sqrt{2}}{5}a_4 & \frac{\sqrt{10}}{6}a_5 & \cdots \\ \frac{\sqrt{6}}{4}a_3 & \frac{3}{5}a_4 & \frac{1}{\sqrt{3}}a_5 & \frac{\sqrt{15}}{7}a_6 & \cdots \\ \frac{2\sqrt{2}}{5}a_4 & \frac{1}{\sqrt{3}}a_5 & \frac{4}{7}a_6 & \frac{\sqrt{5}}{4}a_7 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

One may note that the matrix of Hankel operator H_ϕ is independent of co-analytic term $\sum_{j=1}^{\infty} b_j \bar{z}^j$ of the function ϕ. The adjoint of matrix of operator H_ϕ is given by

$$H_\phi^* = \begin{bmatrix} \frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \cdots \\ \frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{3}{5}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \cdots \\ \frac{1}{2}\overline{a_3} & \frac{2\sqrt{2}}{5}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \frac{4}{7}\overline{a_6} & \cdots \\ \frac{1}{\sqrt{5}}\overline{a_4} & \frac{\sqrt{10}}{6}\overline{a_5} & \frac{\sqrt{15}}{7}\overline{a_6} & \frac{\sqrt{5}}{4}\overline{a_7} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and therefore $H_\phi^* = H_{\hat{\phi}}$ for $\hat{\phi}(z) = \sum_{i=1}^{\infty} \overline{a_i} z^i + \sum_{j=1}^{\infty} \overline{b_j} \bar{z}^j$ and note that each b_j can be zero.

In order to define the notion of H-Toeplitz operator on $L_a^2(\mathbb{D})$, we first consider the operator $K : L_a^2(\mathbb{D}) \to L_{\text{harm}}^2(\mathbb{D})$ defined by

$$K(e_{2n}(z)) = e_n(z) = \sqrt{n+1}z^n \text{ and } K(e_{2n+1}(z)) = \overline{e_{n+1}(z)} = \sqrt{n+2}\bar{z}^{n+1}$$

for all $n \geq 0$ and $z \in \mathbb{D}$. It can be observed that the operator K is bounded linear on $L_a^2(\mathbb{D})$ with $\|K\| = 1$. Moreover, the adjoint K^* of the operator K is given by

$$K^*(e_n(z)) = e_{2n}(z) \text{ and } K^*(\overline{e_{n+1}(z)}) = e_{2n+1}(z) \text{ for all } n \geq 0.$$

Therefore, from the definition of K and K^* it follows that

$$KK^*(e_n) = K(e_{2n}) = e_n \text{ and } KK^*(\overline{e_{n+1}}) = K(e_{2n+1}) = \overline{e_{n+1}}. \tag{4}$$

Also,

$$K^*K(e_{2n}) = K^*(e_n) = e_{2n} \text{ and } K^*K(e_{2n+1}) = K^*(\overline{e_{n+1}}) = e_{2n+1}. \tag{5}$$

Thus, $KK^* = I_{L_{\text{harm}}^2(\mathbb{D})}$ and $K^*K = I_{L_a^2(\mathbb{D})}$. Next, using the definition of operator K, we define H-Toeplitz operator on the Bergman space $L_a^2(\mathbb{D})$.

Definition. For $\phi \in L^\infty(\mathbb{D})$, the H-Toeplitz operator is defined as the operator $B_\phi : L_a^2(\mathbb{D}) \to L_a^2(\mathbb{D})$ such that $B_\phi(f) = P_{L_a^2} M_\phi K(f)$ for all $f \in L_a^2(\mathbb{D})$.

The next proposition directly follows from the definition of the H-Toeplitz operator B_ϕ.

Proposition 2.3. *For $\phi, \psi \in L^\infty(\mathbb{D})$, the operator B_ϕ satisfies the following*:

(i) *B_ϕ is a bounded linear operator on $L_a^2(\mathbb{D})$ with $\|B_\phi\| \leq \|\phi\|_\infty$.*
(ii) *For any scalar α and β, $B_{\alpha\phi+\beta\psi} = \alpha B_\phi + \beta B_\psi$.*
(iii) *The adjoint of H-Toeplitz operator B_ϕ is given by $B_\phi^* = K^* P_{harm} M_{\bar{\phi}}$.*

In general, $B_\phi B_\psi \neq B_{\phi\psi}$ for $\phi, \psi \in L^\infty(\mathbb{D})$ which can be shown with the help of following example:

Example 2.4. Take $\phi(z) = \psi(z) = z$. Then $B_z B_z(1) = P_{L_a^2} M_z K(z) = \frac{1}{\sqrt{2}} P_{L_a^2} M_z(\overline{e_1}(z)) = P(z \cdot \bar{z}) = 1/2$ and $B_{z^2}(1) = P_{L_a^2} M_{z^2}(1) = z^2$. Thus, it follows that $B_z B_z \neq B_{z^2}$.

Next we find the matrix of H-Toeplitz operator B_ϕ for harmonic symbol ϕ on the Bergman space. Take $\phi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{j=1}^\infty b_j \bar{z}^j \in L^\infty(\mathbb{D})$. On using Lemma 2.2, for each non-negative integer n, we have $B_\phi(e_{2n}) = P_{L_a^2} M_\phi K(e_{2n}) = P_{L_a^2} M_\phi(e_n) = T_\phi(e_n)$ and $B_\phi(e_{2n+1}) = P_{L_a^2} M_\phi K(e_{2n+1}) = P_{L_a^2} M_\phi(\overline{e_{n+1}}) = P_{L_a^2} M_\phi J(e_n) = H_\phi(e_n)$. Therefore, from the relations (2) and (3), it follows that

$$\begin{aligned} \langle B_\phi e_{2n}, e_m \rangle &= \langle T_\phi e_n, e_m \rangle \\ &= \begin{cases} \sqrt{\dfrac{n+1}{m+1}} a_{m-n} & \text{for } m \geq n \\ \sqrt{\dfrac{m+1}{n+1}} b_{n-m} & \text{for } n > m \end{cases} \end{aligned}$$

and

$$\langle B_\phi e_{2n+1}, e_m \rangle = \langle H_\phi e_n, e_m \rangle = \frac{\sqrt{m+1}\sqrt{n+2}}{m+n+2} a_{m+n+1},$$

where m and n are non-negative integers. Thus, the matrix representation of B_ϕ with respect to orthonormal basis $\{e_n\}_{n\geq 0}$ of $L_a^2(\mathbb{D})$ in the explicit form is given by

$$B_\phi = \begin{bmatrix} a_0 & \frac{1}{\sqrt{2}} a_1 & \frac{1}{\sqrt{2}} b_1 & \frac{1}{\sqrt{3}} a_2 & \frac{1}{\sqrt{3}} b_2 & \frac{1}{2} a_3 & \frac{1}{2} b_3 & \cdots \\ \frac{1}{\sqrt{2}} a_1 & \frac{2}{3} a_2 & a_0 & \frac{\sqrt{6}}{4} a_3 & \sqrt{\frac{2}{3}} b_1 & \frac{2\sqrt{2}}{5} a_4 & \frac{1}{\sqrt{2}} b_2 & \cdots \\ \frac{1}{\sqrt{3}} a_2 & \frac{\sqrt{6}}{4} a_3 & \sqrt{\frac{2}{3}} a_1 & \frac{3}{5} a_4 & a_0 & \frac{1}{\sqrt{3}} a_5 & \frac{\sqrt{3}}{2} b_1 & \cdots \\ \frac{1}{2} a_3 & \frac{2\sqrt{2}}{5} a_4 & \frac{1}{\sqrt{2}} a_2 & \frac{1}{\sqrt{3}} a_5 & \frac{\sqrt{3}}{2} a_1 & \frac{4}{7} a_6 & a_0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \end{bmatrix}$$

and the matrix of its adjoint is given by

$$B_\phi^* = \begin{bmatrix} \overline{a_0} & \frac{1}{\sqrt{2}}\overline{a_1} & \frac{1}{\sqrt{3}}\overline{a_2} & \frac{1}{2}\overline{a_3} & \cdots \\ \frac{1}{\sqrt{2}}\overline{a_1} & \frac{2}{3}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{2\sqrt{2}}{\sqrt{5}}\overline{a_4} & \cdots \\ \frac{1}{\sqrt{2}}\overline{b_1} & \overline{a_0} & \sqrt{\frac{2}{3}}\overline{a_1} & \frac{1}{\sqrt{2}}\overline{a_2} & \cdots \\ \frac{1}{\sqrt{3}}\overline{a_2} & \frac{\sqrt{6}}{4}\overline{a_3} & \frac{3}{5}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \cdots \\ \frac{1}{\sqrt{3}}\overline{b_2} & \sqrt{\frac{2}{3}}\overline{b_1} & \overline{a_0} & \frac{\sqrt{3}}{2}\overline{a_1} & \cdots \\ \frac{1}{2}\overline{a_3} & \frac{2\sqrt{2}}{5}\overline{a_4} & \frac{1}{\sqrt{3}}\overline{a_5} & \frac{4}{7}\overline{a_6} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{bmatrix}.$$

We can observe that the matrix of T_ϕ can be obtained by deleting every odd column of the matrix of B_ϕ and the matrix of H_ϕ can be obtained by deleting every even column of the matrix of B_ϕ. Also, if the symbol $\phi \in L^\infty(\mathbb{D})$ is co-analytic, then the matrix of B_ϕ is an upper triangular matrix. But, it can not be lower triangular.

Next, we define Bergman H-Toeplitz matrix as follows:

Definition. For $\phi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{j=1}^\infty b_j \bar{z}^j \in L^\infty(\mathbb{D})$, we define an infinite matrix $(c_{m,n})$ as a Bergman H-Toeplitz matrix if its $(m,n)^{th}$ entry satisfies the following relation:

$$c_{m,n} = \begin{cases} \sqrt{\frac{j+1}{m+1}}\,a_{m-j} & \text{for } n = 2j \text{ and } m \geq j, \\ \sqrt{\frac{m+1}{j+1}}\,b_{j-m} & \text{for } n = 2j \text{ and } m < j, \\ \dfrac{\sqrt{m+1}\sqrt{n+2}}{m+n+2}\,a_{m+n+2} & \text{for } n = 2j+1, \end{cases}$$

where m, n and j are non-negative integers. Also, note that Bergman H-Toeplitz matrix $(c_{m,n})$ satisfies the condition $c_{0,0} = c_{j,2j}$ for all $j \geq 0$.

Denote the set of all bounded linear operators on $L_a^2(\mathbb{D})$ by $\mathcal{B}(L_a^2(\mathbb{D}))$. Next, we show that the map $\phi \to B_\phi$ is one-one if the domain is either the space $C(\overline{\mathbb{D}})$ or the space $\{\phi \in L^\infty(\mathbb{D}) : \phi \text{ is harmonic}\}$.

Proposition 2.5. *The function $\gamma : \mathcal{G} \to \mathcal{B}(L_a^2(\mathbb{D}))$ defined by $\gamma(\phi) = B_\phi$ is always one-one, where $\mathcal{G}$ is either $C(\overline{\mathbb{D}})$ or the space $\{\phi \in L^\infty(\mathbb{D}) : \phi$ is harmonic$\}$.*

Proof. Depending upon the space $\mathcal{G}$, the following two cases arise:

Case (1): Let $\mathcal{G}= \{\phi \in L^\infty(\mathbb{D}) : \phi$ is harmonic$\}$. Let $\phi, \psi \in L^\infty(\mathbb{D})$ be harmonic functions on $\mathbb{D}$ defined as $\phi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{j=1}^\infty b_j \bar{z}^j$ and $\psi(z) = \sum_{i=0}^\infty a_i' z^i + \sum_{j=1}^\infty b_j' \bar{z}^j$ such that $B_\phi = B_\psi$. Therefore, $(B_\phi - B_\psi)(e_n) = 0$ for all $n \geq 0$. In particular, we have $B_{\phi-\psi}(e_0) = 0$ which implies that $P_{L_a^2}\left(\sum_{i=0}^\infty (a_i - a_i')z^i + \sum_{j=1}^\infty (b_j - b_j')\bar{z}^j\right) = 0$. This gives $\sum_{i=0}^\infty (a_i - a_i')z^i =$

0 and therefore, $a_i = a_i{}'$ for all $i \geq 0$. Also, $B_{\phi-\psi}(e_2) = 0$, therefore, we get

$$P_{L^2_a}\left(\sum_{i=0}^{\infty}(a_i - a_i{}')z^{i+1} + \sum_{j=1}^{\infty}(b_j - b_j{}')\bar{z}^j z\right) = 0,$$

or equivalently, $(b_1 - b_1{}')P_{L^2_a}(\bar{z} \cdot z) = 0$, i.e., $b_1 = b_1{}'$. Continuing the above process for e_4, e_6, e_8 and so on, it follows that $b_j = b_j^{'}$ for all $j \geq 1$ and hence $\phi = \psi$. Thus, the map γ is one-one.

Case (2): Let $\mathcal{G} = C(\overline{\mathbb{D}})$. Suppose that $\gamma(\phi) = 0$, that is, $B_\phi = 0$ for some $\phi \in C(\overline{\mathbb{D}})$. Then for all non-negative integers m and n it follows that

$$0 = \left\langle B_\phi z^{2m}, z^n \right\rangle = \sqrt{\frac{m+1}{2m+1}}\left\langle T_\phi z^m, z^n \right\rangle = \sqrt{\frac{m+1}{2m+1}}\left\langle \phi z^m, z^n \right\rangle.$$

Therefore, $\int_{\mathbb{D}} \phi z^m \bar{z}^n dA = 0$. Since the linear span of $\{z^m \bar{z}^n : m, n \geq 0\}$ is dense in $C(\overline{\mathbb{D}})$, therefore $\phi = 0$. Thus, the map γ is one-one. □

In the next theorem we obtain the condition under which the H-Toeplitz operator B_ϕ becomes a co-isometry on the Bergman space.

Theorem 2.6. *Let $\phi \in L^\infty(\mathbb{D})$ be a co-analytic function. Then B_ϕ is a co-isometry on $L^2_a(\mathbb{D})$ if and only if $|\phi| = 1$ on $\mathbb{D}$.*

Proof. Let $\phi \in L^\infty(\mathbb{D})$ be a co-analytic function on $\mathbb{D}$. Then, for each non-negative integer n, it follows that

$$\begin{aligned} B_\phi B_\phi^*(z^n) &= (P_{L^2_a} M_\phi K)(K^* M_\phi^*)(z^n) = P_{L^2_a} M_\phi P_{\text{harm}} M_{\bar{\phi}}(z^n) \\ &= P_{L^2_a} M_\phi P_{\text{harm}}(\bar{\phi}(z) z^n) = P_{L^2_a} M_\phi(\bar{\phi}(z) z^n) \\ &= P_{L^2_a}(\bar{\phi}(z)\phi(z) z^n) = P_{L^2_a} M_{|\phi(z)|^2}(z^n) = T_{|\phi(z)|^2}(z^n). \end{aligned} \tag{6}$$

Since the set of polynomials are dense in the Bergman space and therefore from above computation we conclude that $B_\phi B_\phi^* = T_{|\phi|^2}$. Let $|\phi| = 1$. Then $B_\phi B_\phi^* = T_1 = I$ which implies that B_ϕ is a co-isometry on $L^2_a(\mathbb{D})$. Conversely, suppose that the operator B_ϕ is a co-isometry on $L^2_a(\mathbb{D})$. Then $B_\phi B_\phi^* = I$, or equivalently, $T_{1-|\phi|^2} = 0$ which implies that $|\phi| = 1$. Hence, the result holds. □

In the next result, we have obtained the necessary and sufficient condition for H-Toeplitz operator to be partial isometry on the Bergman space.

Theorem 2.7. *Let $\phi \in L^\infty(\mathbb{D})$ be co-analytic function. Then the operator B_ϕ is partial isometry on $L^2_a(\mathbb{D})$ if and only if $|\phi(z)| = 1$ for all $z \in \mathbb{D}$.*

Proof. If $|\phi| = 1$, then by Theorem 2.6, the operator B_ϕ is a co-isometry on $L^2_a(\mathbb{D})$ and hence a partial isometry. Conversely, assume that the operator B_ϕ is a partial isometry on $L^2_a(\mathbb{D})$. Then, $B_\phi B_\phi^* B_\phi = B_\phi$ which on using (6) implies that $T_{|\phi|^2} B_\phi = B_\phi$, equivalently $(1 - T_{|\phi|^2})T_\phi K = 0$. In particular, for all non-negative integers n, $(I - T_{|\phi|^2})T_\phi K e_{2n}(z) = 0$, that is, $(I - T_{|\phi|^2})T_\phi e_n(z) = 0$.

Similarly, $(I-T_{|\phi|^2})T_\phi K e_{2n+1}(z)=0$ which implies that $(I-T_{|\phi|^2})T_\phi \overline{e_{n+1}}(z) = 0$ for all $n \geq 0$. Thus, it follows that $(T_{1-|\phi|^2})T_\phi = 0$, then by [1, Theorem 2] we have $|\phi(z)| = 1$ for all $z \in \mathbb{D}$. □

Recall that on the Bergman space $L^2_a(\mathbb{D})$, an operator T is diagonal if $\langle Tz^i, z^j\rangle = 0$ for all $i \neq j$. Using this we show that a non-zero H-Toeplitz operator having harmonic symbol can never be diagonal operator on the Bergman space.

Theorem 2.8. *Let $\phi \in L^\infty(\mathbb{D})$ be a harmonic symbol. Then the operator B_ϕ is a diagonal operator if and only if $\phi \equiv 0$.*

Proof. Clearly, if $\phi \equiv 0$, then B_ϕ is a diagonal operator on $L^2_a(\mathbb{D})$. Conversely, suppose that B_ϕ is a diagonal operator on $L^2_a(\mathbb{D})$ with the symbol $\phi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{j=1}^\infty b_j \bar{z}^j$. Then for non-negative integers m and n such that $m \neq n$, we have $\langle B_\phi e_m, e_n\rangle = 0$ where $\{e_n\}_{n\geq 0}$ is an orthonormal basis of $L^2_a(\mathbb{D})$. This gives that $\langle P_{L^2_a} M_\phi K e_m, e_n\rangle = 0$ for all $m \neq n$. Then the following two cases arise:

Case (1): If $m = 2k$ for some non-negative integer k, then

$$\begin{aligned}
\langle P_{L^2_a} M_\phi K e_m, e_n\rangle &= \langle M_\phi e_k, e_n\rangle \\
&= \sqrt{k+1}\sqrt{n+1}\Big\langle \Big(\sum_{i=0}^\infty a_i z^{i+k} + \sum_{j=i}^\infty b_j \bar{z}^j\Big) z^k, z^n \Big\rangle \\
&= \sqrt{k+1}\sqrt{n+1}\Big(\sum_{i=0}^\infty a_i \left\langle z^{i+k}, z^n\right\rangle + \sum_{j=i}^\infty b_j \left\langle z^k, z^{n+j}\right\rangle\Big) \\
&= \begin{cases} \dfrac{\sqrt{k+1}\sqrt{n+1}}{n-k+1} a_{n-k} & \text{for } n \geq k \\ \dfrac{\sqrt{k+1}\sqrt{n+1}}{k-n+1} b_{k-n} & \text{otherwise.} \end{cases}
\end{aligned}$$

Case (2): If $m = 2k+1$ for some non-negative integer k, then

$$\begin{aligned}
\langle P_{L^2_a} M_\phi K e_m, e_n\rangle &= \langle M_\phi \overline{e_{k+1}}, e_n\rangle \\
&= \sqrt{k+2}\sqrt{n+1}\left\langle \Big(\sum_{i=0}^\infty a_i z^{i} + \sum_{j=i}^\infty b_j \bar{z}^j\Big) \bar{z}^{k+1}, z^n \right\rangle \\
&= \sqrt{k+2}\sqrt{n+1}\left(\sum_{i=0}^\infty a_i \left\langle z^{i}, z^{k+n+1}\right\rangle + \sum_{j=i}^\infty b_j \left\langle \bar{z}^{j+k+1}, z^{n}\right\rangle\right) \\
&= \frac{\sqrt{k+2}\sqrt{n+1}}{k+n+2} a_{k+n+1}.
\end{aligned}$$

Since B_ϕ is a diagonal operator, therefore, $\langle P_{L^2_a} M_\phi K e_m, e_n\rangle = 0$ for all $m \neq n$ which further implies that $a_i = 0$ and $b_j = 0$ for all i and j. Thus, $\phi \equiv 0$ and hence we get the desired result. □

For a fixed non-negative integer M, define $H_M = \text{Span}\left\{z^l, 0 \le l \le 2M\right\}$. Then, H_M is a closed subspace of the Bergman space $L_a^2(\mathbb{D})$. In fact, following theorem shows that it is invariant under H-Toeplitz operator having co-analytic symbols.

Theorem 2.9. *If $\psi(z) = \sum_{l=M}^{\infty} a_l \bar{z}^l \in L^\infty(\mathbb{D})$, then the subspace H_M of $L_a^2(\mathbb{D})$ is invariant under B_ψ.*

Proof. For any non-negative integers j, the operator K defined on $L_a^2(\mathbb{D})$ satisfies that

$$K(z^j) = \begin{cases} \sqrt{\frac{j+2}{2j+2}} z^{j/2} & \text{if } j \text{ is even,} \\ \sqrt{\frac{j+3}{2j+2}} \bar{z}^{j+1/2} & \text{if } j \text{ is odd.} \end{cases}$$

Consider integers i and j such that $M \le i \le \infty$ and $0 \le j \le 2M$. If j is even, then

$$B_{\bar{z}^i}(z^j) = P_{L_a^2} M_{\bar{z}^i} K(z^j) = \begin{cases} 0 & \text{if } \frac{j}{2} < i, \\ \frac{j-2i+2}{\sqrt{2(j+2)(j+1)}} z^{j/2-i} & \text{otherwise.} \end{cases}$$

If j is odd, then $B_{\bar{z}^i}(z^j) = P_{L_a^2} M_{\bar{z}^i} K(z^j) = \sqrt{\frac{j+3}{2j+2}} p\left(\bar{z}^i \bar{z}^{j+1/2}\right) = 0$. So, it follows that $B_{\bar{z}^i}(z^j) \in H_M$ for all i and j such that $M \le i < \infty$ and $0 \le j \le 2M$. If j is even, then $B_\psi(z^j) = P_{L_a^2} M_\psi K(z^j) = \sqrt{\frac{j+2}{2j+2}} \sum_{l=M}^{\infty} a_l P_{L_a^2}(\bar{z}^l z^{j/2}) = \sum_{l=M}^{\infty} a_l B_{\bar{z}^l}(z^j)$. Again if j is odd, then we have $B_\psi(z^j) = P_{L_a^2} M_\psi K(z^j) = \sqrt{\frac{j+3}{2j+2}} \sum_{l=M}^{\infty} a_l P_{L_a^2}(\bar{z}^l \bar{z}^{j+1/2}) = \sum_{l=M}^{\infty} a_l B_{\bar{z}^l}(z^j)$. Thus, we get that $B_\psi(z^j) = \sum_{l=M}^{\infty} a_l B_{\bar{z}^l}(z^j) \in H_M$ for all $0 \le j \le 2M$ and therefore it follows that $B_\psi(f) \in H_M$ for all $f \in H_M$. Hence, the space H_M is invariant under B_ψ. □

Theorem 2.10. *The subspace H_M of $L_a^2(\mathbb{D})$ is a reducing subspace for the operator $B^*_{z^M}$.*

Proof. We will show that subspaces H_M and ${H_M}^\perp$ are invariant under the H-Toeplitz operator $B^*_{z^M}$. Using Lemma 2.1, we first compute

$$\begin{aligned} B^*_{z^M}(z^j) &= K^* M^*_\phi P_{L_a^2}(z^j) = K^* P_{\text{harm}} M_{\bar{z}^M}(z^j) = K^* P_{\text{harm}}(\bar{z}^M z^j) \\ &= \begin{cases} K^*\left(\frac{j-M+1}{j+1} z^{j-M}\right) & \text{if } j \ge M \\ K^*\left(\frac{M-j+1}{M+1} \bar{z}^{M-j}\right) & \text{if } M > j \end{cases} \\ &= \begin{cases} \dfrac{\sqrt{j-M+1}\sqrt{2j-2M+1}}{j+1} z^{2j-2M} & \text{if } j \ge M \\ \dfrac{\sqrt{M-j+1}\sqrt{2M-2j+1}}{M+1} z^{2M-2j-1} & \text{if } M \ge j \end{cases} \end{aligned}$$

which implies that $B^*_{z^M}(z^j) \in H_M$ for $j = M, M+1, \ldots, 2M$. Since Span $\left\{z^l, 0 \le l \le 2M\right\}$ is H_M only. Thus, $B^*_{z^M}(f) \in H_M$ for all $f \in H_M$ and therefore the subspace H_M is invariant under $B^*_{z^M}$. Next we take $f(z) =$

$\sum_{n=2M+1}^{\infty} a_n z^n + \sum_{m=1}^{\infty} b_m \bar{z}^m \in {H_M}^{\perp}$. Then, $\langle f(z), z^l \rangle = 0$ for all $0 \leq l \leq 2M$. Moreover, using Lemma 2.1 and Lemma 2.2 we have that

$$\begin{aligned} B^*_{z^M}(f)(z) &= K^* P_{\text{harm}} M_{\bar{z}^M}\big(\sum_{n=2M+1}^{\infty} a_n z^n + \sum_{m=1}^{\infty} b_m \bar{z}^m\big) \\ &= K^* P_{\text{harm}}\big(\sum_{n=2M+1}^{\infty} a_n z^n \bar{z}^M + \sum_{m=1}^{\infty} b_m \bar{z}^{m+M}\big) \\ &= K^*\big(\sum_{n=2M+1}^{\infty} a_n \frac{n-M+1}{n+1} z^{n-M} + \sum_{m=1}^{\infty} b_m \bar{z}^{m+M}\big) \\ &= \sum_{n=2M+1}^{\infty} a_n \frac{\sqrt{n-M+1}}{n+1} e_{2n-2M}(z) \\ &\quad + \sum_{m=1}^{\infty} \frac{b_m}{\sqrt{m+M+1}} e_{2m+2M-1}(z) \end{aligned}$$

for all $z \in \mathbb{D}$. Then by using Lemma 2.2, we get that $\langle B^*_{z^M}(f(z)), z^l \rangle = 0$ for all $0 \leq l \leq 2M$. Thus, $B^*_{z^M}(f) \in {H_M}^{\perp}$. But the set of functions of the form f are dense in ${H_M}^{\perp}$ and therefore $B^*_{z^M}\left({H_M}^{\perp}\right) \subseteq {H_M}^{\perp}$. Hence, H_M is a reducing subspace of $B^*_{z^M}$. □

Theorem 2.11. *If $\phi(z) = \sum_{n=N}^{\infty} a_n \bar{z}^n \in L^{\infty}(\mathbb{D})$, then Ker B_ϕ = Span $\{z^k, z^{2l+1} \mid l \geq 0$ and $k = 0, 1, 2, \dots, N-1\}$, where $N \geq 1$ is an arbitrary integer.*

Proof. Let $f \in$ Span $\{z^k, z^{2l+1} \mid l \geq 0$ and $k = 0, 1, 2, \dots, N-1\}$ be arbitrary. Then $f(z) = f_1(z^2) + z f_2(z^2)$, where $f_1(z) = \sum_{t=0}^{N-1} b_t z^t$ and $z f_2(z^2) = \sum_{l=0}^{\infty} c_l z^{2l+1}$. Therefore, on using Lemma 2.2 we compute

$$\begin{aligned} B_\phi(f_1(z^2)) &= P_{L^2_a} M_\phi K\big(\sum_{t=0}^{N-1} b_t z^{2t}\big) \\ &= P_{L^2_a}\left[\big(\sum_{n=N}^{\infty} a_n \bar{z}^n\big) \cdot \big(\sum_{t=0}^{N-1} \sqrt{\frac{t+1}{2t+1}} b_t z^t\big)\right] = 0 \end{aligned}$$

and similarly $B_\phi(z f_2(z^2)) = 0$. Thus, it follows that $B_\phi(f) = 0$ and therefore $f \in \text{Ker} B_\phi$. Conversely, assume that $f \in$ Ker B_ϕ is an arbitrary function and write $f(z) = f_1(z^2) + z f_2(z^2)$ for $f_1, f_2 \in L^2_a(\mathbb{D})$. Then, we obtain $B_\phi(f) = 0$ which on using the definition of the operator K implies that $P_{L^2_a} M_\phi K(f_1(z^2)) + P_{L^2_a} M_\phi K(z f_2(z^2)) = 0$. Further, this implies that $P_{L^2_a} M_\phi K f_1(z^2) = 0$. Therefore, we have $\langle P_{L^2_a} M_\phi K f_1(z^2), z^i \rangle = 0$ for all $i \geq 0$, or equivalently, $\langle f_1(z^2), K^*\left(\sum_{n=N}^{\infty} \overline{a_n} z^{n+i}\right) \rangle = 0$, that is, we have $\sum_{n=N}^{\infty} a_n \sqrt{\frac{2n+2i+1}{n+i+1}} \langle f_1(z^2), z^{2n+2i} \rangle = 0$ for all $i \geq 0$. Since this is true for each i, therefore we get that $f_1(z) \in \text{Span}\ \{z^k \mid\ k = 0, 1, 2, \ldots, N-1\}$. Thus, Ker B_ϕ =Span $\{z^k, z^{2l+1} \mid\ l \geq 0 \text{ and } k = 0, 1, 2, \ldots, N-1\}$. □

Theorem 2.12. *If $\phi \in L^\infty(\mathbb{D})$ is a polynomial harmonic function, then*

$$\dim \mathrm{Ker} B_\phi = \infty.$$

Proof. It can be noted that if ϕ is a co-analytic function in $L^\infty(\mathbb{D})$, then $B_\phi(zf(z^2)) = 0$ for suitable choice of function $f \in L^2_a(\mathbb{D})$. Thus, Ker $B_\phi \neq \{0\}$. Letting $\phi(z) = \sum_{n=1}^N a_n \bar{z}^n + \sum_{m=0}^M b_m z^m$, where $N \geq 1$ and $M > 0$ are arbitrary integers. For $K = \max(M, N)$, we choose $f(z) = \sum_{i=K}^\infty c_i z^{2i+1} \in L^2_a(\mathbb{D})$. Then

$$\begin{aligned} B_\phi(f(z)) &= P_{L^2_a} M_\phi K\big(\sum_{i=K}^\infty c_i z^{2i+1}\big) = P_{L^2_a} M_\phi\big(\sum_{i=K}^\infty \sqrt{\frac{i+2}{2i+2}} c_i \bar{z}^{i+1}\big) \\ &= P_{L^2_a}\left(\big(\sum_{n=1}^N a_n \bar{z}^n + \sum_{m=0}^M b_m z^m\big)\big(\sum_{i=K}^\infty \sqrt{\frac{i+2}{2i+2}} c_i \bar{z}^{i+1}\big)\right) \\ &= P_{L^2_a}\left(\sum_{m=0}^M \sum_{i=k}^\infty \sqrt{\frac{i+2}{2i+2}} b_m c_i z^m \bar{z}^{i+1}\right) = 0, \end{aligned}$$

where the last equality holds in view of the fact that $i + 1 > m$ for each $i \geq k$. Similarly, for all $n \in \mathbb{N}$, we have $B_\phi(z^{2n} f(z)) = 0$. Thus, if a non-zero function $f \in$ Ker B_ϕ, then $\sum_{k=1}^n \lambda_k z^{2k} f \in$ Ker B_ϕ for $n \in \mathbb{N}$ and $\lambda_k \in \mathbb{C}$. In particular, the set $\{z^{2n} f \mid n \in \mathbb{N}\}$ is a linear independent set. Indeed, if $\sum_{k=1}^n \lambda_k z^{2k} f = 0$, then as $f \neq 0$, therefore $\sum_{k=1}^n \lambda_k z^{2k}$ vanish on a positive measure set and consequently we have $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$. Hence, $\{z^2 f, z^4 f, \ldots, z^{2n} f\}$ is linearly independent. Since this is true for all $n \in \mathbb{N}$ and for all such functions in Ker B_ϕ. Therefore, Ker B_ϕ is an infinite dimensional. □

3. Compactness of H-Toeplitz operators

In this section, we study compactness of H-Toeplitz operators B_ϕ on $L^2_a(\mathbb{D})$ using Berezin transform. We also prove that there does not exist any non-zero H-Toeplitz operator on $L^2_a(\mathbb{D})$ which is a Fredholm operator. Let start with the definition of Berezin transform of bounded operator.

Let S be any bounded linear operator defined on a reproducing kernel Hilbert space $\mathbb{H}$ with normalized reproducing kernel k_z. Then the Berezin transform of S is given by $\tilde{S}(z) = \langle Sk_z, k_z \rangle$ for $z \in \mathbb{H}$ (see [13]).

The normalized reproducing kernel of Bergman space is given by

$$k_z(w) = \frac{K_z(w)}{\|K_z\|} = \frac{1-|z|^2}{(1-\bar{z}w)^2} = (1-|z|^2)\sum_{n=0}^\infty (n+1)(\bar{z}w)^n, \tag{7}$$

where $z, w \in \mathbb{D}$. Then, it follows that k_z converges to 0 weakly in $L^2_a(\mathbb{D})$ as $|z| \to 1$. Note that if f is a harmonic function in $L^2_a(\mathbb{D})$, then $\tilde{f}(z) = f(z)$ for all $z \in \mathbb{D}$.

Proposition 3.1. *Let ϕ be a harmonic function in $L^\infty(\mathbb{D})$. Then $\|\tilde{B_\phi}\| \leq \|\phi\|_\infty$.*

Proof. For $z \in \mathbb{D}$, we have $\langle B_\phi K_z, K_z\rangle \leq \|B_\phi\|\|K_z\|^2$. Therefore, it follows that $|\tilde{B_\phi}(z)| = \left|\left\langle B_\phi \frac{K_z}{\|K_z\|}, \frac{K_z}{\|K_z\|}\right\rangle\right| \leq \|B_\phi\| \leq \|T_\phi\|\|K\| \leq \|\phi\|_\infty$ for all $z \in \mathbb{D}$. □

Lemma 3.2. *The operator K defined on $L^2_a(\mathbb{D})$ satisfies following*

$$K(k_z(w)) = (1-|z|^2)\left(\sum_{n=0}^{\infty}\sqrt{(2n+1)(n+1)}(\bar{z}w)^n + \sum_{n=0}^{\infty}\sqrt{2(n+1)(n+2)}(z\bar{w})^{n+1}\right) \text{ for } z, w \in \mathbb{D}.$$

Proof. For a fixed $z \in \mathbb{D}$, using the equation (7), we compute

$$\begin{aligned}
K(k_z(w)) &= (1-|z|^2)\sum_{n=0}^{\infty}(n+1)K(\bar{z}w)^n \\
&= (1-|z|^2)\sum_{n=0}^{\infty}\sqrt{n+1}\bar{z}^n K(e_n(w)) \\
&= (1-|z|^2)\left(\sum_{n=0}^{\infty}\sqrt{2n+1}\bar{z}^{2n}e_n(w) + \sum_{n=0}^{\infty}\sqrt{2n+2}\bar{z}^{2n+1}\overline{e_{n+1}}(w)\right) \\
&= (1-|z|^2)\left(\sum_{n=0}^{\infty}\sqrt{(2n+1)(n+1)}\bar{z}^{2n}w^n \right.\\
&\qquad \left. + \sum_{n=0}^{\infty}\sqrt{2(n+1)(n+2)}\bar{z}^{2n+1}\bar{w}^{n+1}\right) \qquad (8)
\end{aligned}$$

which hold for all $w \in \mathbb{D}$. □

Let us denote the expression (8) of $K(k_z(w))$ by $h(z, w)$ for all $z, w \in \mathbb{D}$. Then, note that $h(z, w) \to 0$ as $z \to \partial\mathbb{D}$ for $z, w \in \mathbb{D}$. Consider the set $\mathcal{A} = C(\overline{\mathbb{D}}) \cup \{\phi \in L^\infty(\mathbb{D}) : \phi \text{ is harmonic}\}$.

Proposition 3.3. *If $\phi \in \mathcal{A}$, then B_ϕ is not bounded below.*

Proof. For a fixed $z \in \mathbb{D}$, let k_z be the normalized reproducing kernel. Then, $\|B_\phi k_z\|^2 = \|T_\phi h\|^2 \leq \|\phi \cdot h\|^2 = \int_{\mathbb{D}} |\phi(w)|^2|h(z, w)|^2 dA(w) \to 0$ as $z \to \partial\mathbb{D}$. Since $h(z, w) \to 0$ as $z \to \partial\mathbb{D}$ and the convergence of the integral comes from the dominated convergence theorem. Therefore, we conclude that $\|B_\phi k_z\| \to 0$ as $z \to \partial\mathbb{D}$. Hence, B_ϕ is not bounded below. □

Recall that for a bounded linear operator T defined on a Hilbert space $\mathbb{H}$, the approximated point spectrum of operator T is defined as the set $\sigma_{ap}(T) = \{\lambda \in \mathbb{C} : T - \lambda I \text{ is not bounded below}\}$. Since, for $\phi \in \mathcal{A}$, the operator B_ϕ

is not bounded below. Hence, $0 \in \sigma_{ap}(B_\phi)$, the approximate point spectrum. Next, we obtain the Berezin transform of the operator B_ϕ and discuss the compactness of B_ϕ.

For each $z \in \mathbb{D}$, define an automorphism φ_z as $\varphi_z(w) = \frac{z-w}{1-\bar{z}w}$ for all $w \in \mathbb{D}$. Then, $\varphi_z^{'}(w) = \frac{|z|^2-1}{(1-\bar{z}w)^2}$ and if we substitute $w = \varphi_z(\lambda)$, the Jacobian change in measure is $dA(w) = \left|\varphi_z^{'}(\lambda)\right|^2 dA(\lambda)$. Thus, for a Lebesgue integrable function f on $\mathbb{D}$, we have the following change of variables formula:

$$\int_{\mathbb{D}} f(w)dA(w) = (1-|z|^2)^2 \int_{\mathbb{D}} (fo\varphi_z)(\lambda)\frac{1}{|1-\bar{z}\lambda|^4}dA(\lambda). \tag{9}$$

Now in the next proposition we obtain the Berezin transform for B_ϕ.

Proposition 3.4. *If $\phi \in L^\infty(\mathbb{D})$, then the Berezin transform of B_ϕ is given by $\tilde{B_\phi}(z) = (1-|z|^2)\int_{\mathbb{D}} \frac{\phi(w)h(z,w)}{(1-z\bar{w})^2}dA(w)$.*

Proof. The Berezin transform of the operator B_ϕ is given by

$$\begin{aligned}
\tilde{B_\phi}(z) &= \langle B_\phi k_z, k_z\rangle = \left\langle P_{L_a^2} M_\phi K k_z, k_z\right\rangle \\
&= \langle \phi(w)\cdot h(z,w), k_z(w)\rangle \\
&= \int_{\mathbb{D}} \phi(w)h(z,w)\overline{k_z(w)}dA(w) \\
&= (1-|z|^2)\int_{\mathbb{D}} \frac{\phi(w)h(z,w)}{(1-z\bar{w})^2}dA(w) \text{ for all } w \in \mathbb{D},
\end{aligned}$$

where the equality in the second line comes from (8). □

It is clear from the above theorem that $\tilde{B_\phi}(z) \to 0$ as $|z| \to 1$.

Zeng [15] proved the following lemma in order to study about the compactness of Toeplitz operator T_ϕ.

Lemma 3.5. *Let M be a compact subset of $\mathbb{D}$. Let $\phi \in L^\infty(\mathbb{D})$ be such that $\phi \equiv 0$ on $\mathbb{D} \setminus M$. Then, T_ϕ is a compact operator on $L_a^2(\mathbb{D})$.*

Theorem 3.6. *Let ϕ be a harmonic function in $\mathbb{D}$ and continuous in $\overline{D}$. Then B_ϕ is a compact operator on $L_a^2(\mathbb{D})$ if and only if $\phi_{|\partial\mathbb{D}} = 0$.*

Proof. Suppose that $\phi_{|\partial\mathbb{D}} = 0$. Then ϕ can be uniformly approximated by functions with compact support in $\mathbb{D}$. Then by Lemma 3.5, T_ϕ is a compact operator in $L_a^2(\mathbb{D})$ and hence the operator B_ϕ is compact. Now conversely suppose that the operator B_ϕ is compact on $L_a^2(\mathbb{D})$. Then, both T_ϕ and H_ϕ are compact operators. Let $z_0 \in \partial\mathbb{D}$. Then k_z converges to 0 weakly in $L_a^2(\mathbb{D})$ as $z \to z_0$. Therefore, $\langle T_\phi k_z, k_z\rangle \to 0$ as $z \to z_0$. On the other hand by replacing φ by $fo\varphi_z$ for $z \in \mathbb{D}$ and using equation (9), we have

$$|\langle T_\phi k_z, k_z\rangle| = \left|\int_{\mathbb{D}} \phi\frac{|K_z|^2}{\|K_z\|^2}dA\right| \cong \left|\int_{\mathbb{D}} \phi\cdot|K_z|^2(1-|z|^2)^2 dA\right|$$

$$= \left| \int_{\mathbb{D}} ((\phi) o \varphi_z) \, dA \right| \to |\phi(z_0)| \text{ as } z \to z_0.$$

The convergence comes from the dominated theorem since $\varphi_z(w) \to z_0$ for all $z \in \mathbb{D}$. Thus, $|\phi(z_0)| = 0$ for $z_0 \in \partial\mathbb{D}$. This implies that $\phi(z_0) = 0$ and therefore $\phi_{|\partial\mathbb{D}} = 0$. □

Since every Hilbert-Schmidt operator is a compact operator and therefore from the above result we see that a non-zero H-Toeplitz operator can not be Hilbert-Schmidt on $L_a^2(\mathbb{D})$. Alternatively, we prove this result using the definition of Hilbert-Schmidt operator.

Theorem 3.7. *Let $\phi \in L^\infty(\mathbb{D})$ be a harmonic function. Then B_ϕ is a Hilbert-Schmidt operator on $L_a^2(\mathbb{D})$ if and only if $\phi \equiv 0$.*

Proof. For $\phi(z) = \sum_{i=0}^\infty a_i z^i + \sum_{j=1}^\infty b_j \bar{z}^j \in L^\infty(\mathbb{D})$, assume that B_ϕ is a Hilbert-Schmidt operator. This means that $\sum_{n=0}^\infty \langle B_\phi e_n, B_\phi e_n \rangle < \infty$. Using Lemma 2.2, we compute

$$\begin{aligned}
&\|B_\phi e_n\|^2 \\
&= \sum_{n=0}^\infty \langle B_\phi e_n(z), B_\phi e_n(z) \rangle \\
&= \sum_{n=0}^\infty \langle B_\phi e_{2n}(z), B_\phi e_{2n}(z) \rangle + \sum_{n=0}^\infty \langle B_\phi e_{2n+1}(z), B_\phi e_{2n+1}(z) \rangle \\
&= \sum_{n=0}^\infty \left\langle P_{L_a^2} M_\phi e_n(z), P_{L_a^2} M_\phi e_n(z) \right\rangle + \sum_{n=0}^\infty \left\langle P_{L_a^2} M_\phi \overline{e_{n+1}}(z), P_{L_a^2} M_\phi \overline{e_{n+1}}(z) \right\rangle \\
&= \sum_{n=0}^\infty (n+1) \left\langle P_{L_a^2} \left(\sum_{i=0}^\infty a_i z^{i+n} + \sum_{j=1}^\infty b_j \bar{z}^j z^n \right), P_{L_a^2} \left(\sum_{i=0}^\infty a_i z^{i+n} + \sum_{j=1}^\infty b_j \bar{z}^j z^n \right) \right\rangle \\
&\quad + \sum_{n=0}^\infty (n+2) \left\langle P_{L_a^2} \left(\sum_{i=0}^\infty a_i z^i \bar{z}^{n+1} + \sum_{j=1}^\infty b_j \bar{z}^j \bar{z}^{n+1} \right), P_{L_a^2} \left(\sum_{i=0}^\infty a_i z^i \bar{z}^{n+1} \right.\right. \\
&\quad \left.\left. + \sum_{j=1}^\infty b_j \bar{z}^j \bar{z}^{n+1} \right) \right\rangle \\
&= \sum_{n=0}^\infty (n+1) \left\langle \sum_{i=0}^\infty a_i z^{i+n} + \sum_{j=1}^n \frac{n-j+1}{n+1} b_j z^{n-j}, \sum_{i=0}^\infty a_i z^{i+n} \right. \\
&\quad \left. + \sum_{j=1}^n \frac{n-j+1}{n+1} b_j z^{n-j} \right\rangle + \sum_{n=0}^\infty (n+2) \left\langle \sum_{i=n+1}^\infty \frac{i-n}{i+1} a_i z^{i-n-1}, \right. \\
&\quad \left. \sum_{i=n+1}^\infty \frac{i-n}{i+1} a_i z^{i-n-1} \right\rangle
\end{aligned}$$

$$= \sum_{n=0}^{\infty}(n+1)\left(\sum_{i=0}^{\infty}\frac{|a_i|^2}{i+n+1}+\sum_{j=1}^{n}(\frac{n-j+1}{(n+1)^2})|b_j|^2\right)$$
$$+\sum_{n=0}^{\infty}(n+2)\left(\sum_{i=n+1}^{\infty}\frac{i-n}{(i+1)^2}|a_i|^2\right).$$

Therefore, we have

$$\sum_{n=0}^{\infty}\left[(n+1)\left(\sum_{i=0}^{\infty}\frac{|a_i|^2}{i+n+1}+\sum_{j=0}^{n}\frac{n-j+1}{(n+1)^2}|b_j|^2\right)\right.$$
$$\left.+(n+2)\left(\sum_{i=n+1}^{\infty}\frac{i-n}{(i+1)^2}|a_i|^2\right)\right]<\infty.$$

This implies that $(n+1)\left(\sum_{i=0}^{\infty}\frac{|a_i|^2}{i+n+1}+\sum_{j=0}^{n}\frac{n-j+1}{(n+1)^2}|b_j|^2\right)\to 0$ as $n\to\infty$ and this further implies that $a_i = 0\ \forall\ i\geq 0$ and $b_j = 0\ \forall\ j\geq 1$. Thus, the symbol $\phi = 0$. Hence, B_ϕ is a Hilbert-Schmidt operator if and only if $\phi\equiv 0$. □

An operator T defined on a Hilbert space is said to be Fredholm if its range is closed, dimensions of Ker T and Ker T^* are finite. Then, the Fredholm spectrum [2] of the operator $\mathbb{T}$ is defined as the set $\sigma_e(T) = \{\lambda\in\mathbb{C} : T-\lambda I$ is not Fredholm$\}$. We conclude this section by finding the condition under which H-Toeplitz operator is Fredholm on $L_a^2(\mathbb{D})$.

Lemma 3.8 ([8])**.** *Let A be a bounded linear operator on an Hilbert space $\mathbb{H}$. Then A is a Fredholm operator if and only if there is no sequence $\{h_n\}$ of unit vectors in $\mathbb{H}$ such that $h_n\to 0$ weakly on $\mathbb{H}$ and $\lim\|Ah_n\| = 0$.*

Using Lemma 3.8, we have the next proposition.

Proposition 3.9. *Let A be a Fredholm operator on $L_a^2(\mathbb{D})$. Then, there is no net $(h_z)_{z\in\mathbb{D}}$ of unit norm in $L_a^2(\mathbb{D})$ such that $h_z\to 0$ weakly as $z\to\partial\mathbb{D}$ and $\lim\|Ah_z\| = 0$.*

Proof. Let us suppose that $\{h_z\}_{z\in\mathbb{D}}$ be a net of functions in $L_a^2(\mathbb{D})$ of unit norm which converges weakly to 0 in $L_a^2(\mathbb{D})$ and $\lim\|Ah_z\| = 0$. Since the operator A is Fredholm on $L_a^2(\mathbb{D})$ therefore there exists an operator B on $L_a^2(\mathbb{D})$ such that $BA = I + J$ for some compact operator J on $L_a^2(\mathbb{D})$. Now consider $|1-\|BAh_z\|| = |\|h_z\|-\|BAh_z\||\leq\|Jh_z\|$. But as J is compact therefore $\|Jh_z\|\to 0$. Thus, $\|BAh_z\|\to 1$ as $z\to\partial\mathbb{D}$. So, it is impossible for $\|Ah_z\|\to 0$ as $z\to\partial\mathbb{D}$. Thus, such a net does not exist in $L_a^2(\mathbb{D})$. □

Theorem 3.10. *There does not exist any non-zero H-Toeplitz operator on $L_a^2(\mathbb{D})$ which is Fredholm.*

Proof. Let us assume that the H-Toeplitz operator B_ϕ is Fredholm on $L_a^2(\mathbb{D})$ for some $\phi\in L^\infty(\mathbb{D})$. Consider the net $(k_z)_{z\in\mathbb{D}}$ consisting of all normalised reproducing kernels on $L_a^2(\mathbb{D})$. Since $k_z \to 0$ weakly in $L_a^2(\mathbb{D})$ as $z \to \partial\mathbb{D}$ and also $\|B_\phi k_z\| \to 0$ as $z \to \partial\mathbb{D}$. Therefore, in view of Proposition 3.9, this contradict the fact that B_ϕ is a Fredholm operator. Hence, B_ϕ is a Fredholm operator on $L_a^2(\mathbb{D})$ if and only if $\phi \equiv 0$. □

In particular, from the above theorem it follows that $0 \in \sigma_e(B_\phi)$, the Fredholm spectrum of the operator B_ϕ and hence the Fredholm spectrum of the operator B_ϕ is non-empty.

4. Commutativity of H-Toeplitz operator

In this section we study the commutativity of H-Toeplitz operators for analytic and harmonic symbols. In general two H-Toeplitz operators need not commute which can be seen with the help of the following example:

Example 4.1. Let $\phi(z) = z$ and $\psi(z) = \bar{z}$. Then, $B_z(e_2(z)) = P_{L_a^2} M_z K(e_2(z))$ $= \sqrt{\frac{2}{3}} e_2(z)$ and $B_{\bar{z}}(e_2(z)) = P_{L_a^2} M_{\bar{z}} e_1(z) = \frac{1}{\sqrt{2}}$. Therefore,

$$B_z B_{\bar{z}}(e_2(z)) = B_z(\frac{1}{\sqrt{2}}) = P_{L_a^2} M_z(\frac{1}{\sqrt{2}}) = \frac{1}{\sqrt{2}} z \text{ and}$$

$$B_{\bar{z}} B_z(e_2(z)) = \sqrt{\frac{2}{3}} B_{\bar{z}}(e_2(z)) = \sqrt{\frac{2}{3}} P_{L_a^2} M_{\bar{z}} e_1(z) = \frac{2}{\sqrt{3}} P_{L_a^2}(\bar{z}z) = \frac{1}{\sqrt{3}}.$$

Thus, $B_z B_{\bar{z}} \neq B_{\bar{z}} B_z$.

In the next result, we obtain conditions under which two H-Toeplitz operators having analytic symbols commute on $L_a^2(\mathbb{D})$.

Theorem 4.2. *Let $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ and $\psi(z) = \sum_{m=0}^{\infty} b_m z^m$ be bounded analytic functions, where $\phi(0) = 0 = \psi(0)$ and a_i, b_j are non-zero scalars for each i and j. If $\frac{b_{n+k}}{a_{n+k}} \geq \frac{b_{2n+1}}{a_{2n+1}}$ for all non-negative integers n and k, then following are equivalent*:

(i) $B_\phi B_\psi = B_\psi B_\phi$.
(ii) *ϕ and ψ are linearly dependent.*

Proof. Clearly if ϕ and ψ are linearly dependent, then B_ϕ and B_ψ commute. Conversely assume that $B_\phi B_\psi = B_\psi B_\phi$. In particular $B_\phi B_\psi(1) = B_\psi B_\phi(1)$, which means $P_{L_a^2} M_\phi K P_{L_a^2} M_\psi(1) = P_{L_a^2} M_\psi K P_{L_a^2} M_\phi(1)$, or equivalently, it follows that $P_{L_a^2} M_\phi K\left(\sum_{m=0}^{\infty} b_m z^m\right) = P_{L_a^2} M_\psi K\left(\sum_{n=0}^{\infty} a_n z^n\right)$. Therefore, we have $P_{L_a^2} M_\phi K\left(\sum_{m=0}^{\infty} \frac{b_m}{\sqrt{m+1}} K e_m\right) = P_{L_a^2} M_\psi K\left(\sum_{n=0}^{\infty} \frac{a_n}{\sqrt{n+1}} K e_n\right)$, that is,

$$P_{L_a^2} M_\phi \left(\sum_{m=0}^{\infty} \frac{b_{2m}}{\sqrt{2m+1}} e_m(z) + \frac{b_{2m+1}}{\sqrt{2m+2}} \overline{e_{m+1}}(z) \right)$$

$$= P_{L_a^2} M_\psi \left(\sum_{n=0}^{\infty} \frac{a_{2n}}{\sqrt{2n+1}} e_n(z) + \frac{a_{2n+1}}{\sqrt{2n+2}} \overline{e_{n+1}}(z) \right)$$

or, equivalently,

$$P_{L_a^2}\left[\left(\sum_{n=0}^{\infty} a_n z^n\right)\left(\sum_{m=0}^{\infty}\sqrt{\frac{m+1}{2m+1}}b_{2m}z^m+\sqrt{\frac{m+2}{2m+2}}b_{2m+1}\bar{z}^{m+1}\right)\right]$$
$$= P_{L_a^2}\left[\left(\sum_{m=0}^{\infty} b_m z^m\right)\left(\sum_{n=0}^{\infty}\sqrt{\frac{n+1}{2n+1}}a_{2n}z^n+\sqrt{\frac{n+2}{2n+2}}a_{2n+1}\bar{z}^{n+1}\right)\right].$$

This follows that

(10)
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sqrt{\frac{m+1}{2m+1}}a_n b_{2m} z^{m+n}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sqrt{\frac{m+2}{2m+2}}a_n b_{2m+1} P_{L_a^2}\left(z^n\bar{z}^{m+1}\right)$$
$$=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sqrt{\frac{n+1}{2n+1}}a_{2n} b_m z^{m+n}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sqrt{\frac{n+2}{2n+2}}a_{2n+1} b_m P_{L_a^2}\left(z^m\bar{z}^{n+1}\right).$$

Then, on comparing the coefficient of z^0, by the equation (10) we get that

$$a_0b_0+\sum_{m=0}^{\infty}\sqrt{\frac{m+2}{2m+2}}\frac{1}{m+2}a_{m+1}b_{2m+1}=a_0b_0+\sum_{n=0}^{\infty}\sqrt{\frac{n+2}{2n+2}}\frac{1}{n+2}a_{2n+1}b_{n+1}$$

or, equivalently,

$$\sum_{n=0}^{\infty}\frac{1}{\sqrt{(n+2)(2n+2)}}\left(a_{n+1}b_{2n+1}-a_{2n+1}b_{n+1}\right)=0$$

which implies that $\frac{b_{n+1}}{a_{n+1}}=\frac{b_{2n+1}}{a_{2n+1}}$ for each non-negative integer n. On comparing the coefficient of z, by the equation (10) we get that

$$\sqrt{\frac{2}{3}}a_0b_2+a_1b_0+\sum_{m=0}^{\infty}\frac{2}{m+3}\sqrt{\frac{m+2}{2m+2}}a_{m+2}b_{2m+1}$$
$$=\sqrt{\frac{2}{3}}a_2b_0+a_0b_1+\sum_{n=0}^{\infty}\frac{2}{n+3}\sqrt{\frac{n+2}{2n+2}}a_{2n+1}b_{n+2},$$

which gives that $\sum_{n=0}^{\infty}\frac{1}{n+3}\sqrt{\frac{n+2}{2n+2}}\left(a_{n+2}b_{2n+1}-b_{n+2}a_{2n+1}\right)=0$. This further implies that $\frac{b_{n+2}}{a_{n+2}}=\frac{b_{2n+1}}{a_{2n+1}}$ for each non-negative integer n. Similarly, on comparing the coefficient of z^2, by the equation (10) we have $\frac{b_{n+3}}{a_{n+3}}=\frac{b_{2n+1}}{a_{2n+1}}$ for each non-negative integer n. On continuing the above manner we obtain that $\frac{b_{n+k}}{a_{n+k}}=\frac{b_1}{a_1}$ for each non-negative integer n and k. Therefore, $b_i=\lambda a_i$ for all $i\geq 0$, where $\lambda=\frac{b_1}{a_1}$ is a constant. Hence, it follows that $\phi=\lambda\psi$. □

Using the same techniques as in Theorem 4.2, we have obtained the necessary and sufficient conditions for the commutativity of H-Toeplitz operators having harmonic symbols given as follows.

Theorem 4.3. *Let* $\phi(z) = \sum_{i=0}^{\infty} a_i z^i + \sum_{j=1}^{\infty} b_j \bar{z}^j$ *and* $\psi(z) = \sum_{m=0}^{\infty} c_m z^m + \sum_{n=1}^{\infty} d_n \bar{z}^n$ *be two harmonic functions in* $L^\infty(\mathbb{D})$ *such that* $\phi(0) = 0 = \psi(0)$ *and* a_i, b_j, c_m, d_n *are non-zero scalars for each* $i, m \geq 0$ *and* $j, n \geq 1$. *Also, assume that* $a_{m+k} c_{2m+1} \geq c_{m+k} a_{2m+1}$ *and* $b_n c_{2(n+m)} \geq d_n a_{2(n+m)}$ *for all* $n, k \geq 1$ *and* $m \geq 0$. *Then following are equivalent*:

(i) $B_\phi B_\psi = B_\psi B_\phi$.
(ii) ϕ *and* ψ *are linearly dependent.*

Axler [6] proved the following theorem for the commutativity of Toeplitz operators with bounded harmonic symbols.

Theorem 4.4. *Suppose* ϕ *and* ψ *are bounded harmonic functions on* $\mathbb{D}$. *Then* $T_\phi T_\psi = T_\psi T_\phi$ *if and only if*

(i) ϕ *and* ψ *are analytic on* $\mathbb{D}$, *or*
(ii) $\overline{\phi}$ *and* $\overline{\psi}$ *are analytic on* $\mathbb{D}$, *or*
(iii) *there exist constants* $a, b \in \mathbb{C}$ *not both zero such that* $a\phi + b\psi$ *is constant on* $\mathbb{D}$.

Thus, in view of Theorems 4.2 and 4.4, it follows that an H-Toeplitz operator need not be a Toeplitz operator. Since if we take $\phi(z) = \sum_{n=1}^{\infty} a_n z^n$ and $\psi(z) = \sum_{m=1}^{\infty} b_m z^m$ with $a_i, b_i \neq 0 \ \forall i \geq 1$ such that ϕ and ψ are not linearly dependent. Then, $B_\phi B_\psi \neq B_\psi B_\phi$ and hence B_ϕ, B_ψ are not Toeplitz operators on $L_a^2(\mathbb{D})$.

Conclusion and future work

This paper initiate the study of H-Toeplitz operators, an invariant to Toeplitz operators, on the Bergman space. Various properties namely co-isometry, partial isometry, invariant subspaces, compactness and Fredholmness of H-Toeplitz operators B_ϕ have been explored on the Bergman space. The properties of normality, hyponormality and self-adjointness of H-Toeplitz operators for various symbols on the Bergman space are yet to be known. Also, the notion can be further extended to slant H-Toeplitz operators and generalized slant H-Toeplitz operators on the Bergman space.

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