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SOME INVERSE RESULTS OF SUMSETS

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Abstract. Let $h \geq 2$ and $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of integers. It is well-known that $|hA| = hk - h + 1$ if and only if A is a k-term arithmetic progression. In this paper, we give some nontrivial inverse results of the sets A with some extremal the cardinalities of hA.

1. Introduction

Let $[a, b]$ denote the interval of integers n such that $a \leq n \leq b$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a finite set of integers such that $a_0 < a_1 < \cdots < a_{k-1}$, we define

$$d(A) = \gcd(a_1 - a_0, a_2 - a_0, \ldots, a_{k-1} - a_0).$$

Let $a_i' = (a_i - a_0)/d(A),\ i = 0, 1, \ldots, k-1$. We call

$$A^{(N)} = \{a_0', a_1', \ldots, a_{k-1}'\}$$

the normal form of the set A. For any integer c, we define the set

$$c + A = \{c + a : a \in A\}.$$

For any finite set of integers A and any positive integer $h \geq 2$, let

$$hA = \{a_1 + \cdots + a_h : a_1, \ldots, a_h \in A\}.$$

It is easy to see that $|hA| = \left|hA^{(N)}\right|$. For given set A, a direct problem is to determine the structure and properties of the h-fold sumset hA when the set A is known. An inverse problem is to deduce properties of the set A from properties of the sumset hA.

The following two results gave the simple lower bound of the cardinality of hA and showed that the lower bound is attained if and only if the set is an arithmetic progression.

Theorem A ([11], Theorem 1.3). *Let $h \geq 2$. Let A be a finite set of integers with $|A| = k$. Then*

$$|hA| \geq hk - h + 1.$$

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Theorem B ([11], Theorem 1.6)**.** *Let $h \geq 2$. Let A be a finite set of integers with $|A| = k$. Then $|hA| = hk - h + 1$ if and only if A is a k-term arithmetic progression.*

In 1959, Freiman [2] proved the following result:

Theorem C. *Let $k \geq 3$. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $\gcd(A) = 1$. Then*

$$|2A| \geq \min\{a_{k-1}, 2k-3\} + k = \begin{cases} a_{k-1} + k, & \text{if } a_{k-1} \leq 2k-3, \\ 3k-3, & \text{if } a_{k-1} \geq 2k-2. \end{cases}$$

In [1], [3], [8], [12], the authors generalized the above theorem to the case of summation of two distinct sets. In 1959, Freiman [2] (see also [11]) investigated the structure of set A if the cardinality of $2A$ is between $2k-1$ and $3k-4$.

Theorem D ([11], Theorem 1.16)**.** *Let A be a set of integers such that $|A| = k \geq 3$. If $|2A| = 2k - 1 + b \leq 3k - 4$, then A is a subset of an arithmetic progression of length $k + b \leq 2k - 3$.*

In 1996, Lev [7] gave the following result:

Theorem E ([7], Theorem 1)**.** *Let h, $k \geq 2$ be integers. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $\gcd(A) = 1$. Then*

$$|hA| \geq |(h-1)A| + \min\{a_{k-1}, h(k-2) + 1\}.$$

For other related problems, see [4–6], [9–11], [13].

In this paper, we consider the following inverse problem: assume that A is a finite integer set and the cardinalities of hA are extremal cases, how to determine the structure of the set A? We obtain the following results:

Theorem 1.1. *Let $h \geq 2$ and $k \geq 5$ be integers. Let A be an integer set with $|A| = k$. If $hk - h + 1 < |hA| \leq hk + h - 2$, then*

$$A^{(N)} = [0, k] \setminus \{i\}, \quad 1 \leq i \leq k - 1.$$

Moreover, $|hA| = hk$ for $i = 1$ or $k - 1$, and $|hA| = hk + 1$ for $2 \leq i \leq k - 2$.

Theorem 1.2. *Let $h \geq 2$ and $k \geq 5$ be integers. Let A be an integer set with $|A| = k$. If $hk + h - 2 < |hA| \leq hk + 2h - 3$, then*

$$A^{(N)} = [0, k+1] \setminus \{i, j\}, \quad 1 \leq i < j \leq k + 1.$$

Moreover, we have

(a) *$|hA| = hk + h - 1$ for $h = 2$, $2 \leq i \leq k - 2$, $j = k + 1$ and $\{i, j\} = \{1, 2\}, \{k-1, k\}, \{1, k\}, \{1, 3\}, \{k-2, k\}$;*

(b) *$|hA| = hk + h$ for $i = 1$ and $4 \leq j \leq k - 1$ when $h \geq 2$; or $2 \leq i \leq k - 3$ and $j = k$ when $h \geq 2$, or $\{i, j\} = \{2, 3\}, \{k-2, k-1\}$ when $h = 2$;*

(c) *$|hA| = hk + h + 1$ for $2 \leq i < j \leq k - 1$, except for $\{i, j\} = \{2, 3\}, \{k-2, k-1\}$ when $h = 2$.*

Remark 1.3. By Theorem 1.1 and Theorem 1.2 we know that there is no set A such that $|3A| = 3k - 1$.

2. Lemmas

Lemma 2.1. *Let $h \geq 2$ and $k \geq 5$ be integers. Let $A = \{a_0, a_1, \ldots, a_{k-1}\}$ be a set of integers such that $0 = a_0 < a_1 < \cdots < a_{k-1}$ and $\gcd(A) = 1$. If $|hA| \leq hk + 2h - 3$, then $a_{k-1} \leq k + 1$. Moreover, if $|hA| \leq hk + h - 2$, then $a_{k-1} \leq k$.*

Proof. By Theorem E, we have

$$\begin{aligned} |hA| \geq\ & |(h-1)A| + \min\{a_{k-1}, h(k-2)+1\} \\ \geq\ & |(h-2)A| + \min\{a_{k-1}, h(k-2)+1\} + \min\{a_{k-1}, (h-1)(k-2)+1\} \\ \geq\ & \cdots\cdots \\ & \vdots \\ \geq\ & |A| + \min\{a_{k-1}, h(k-2)+1\} + \cdots + \min\{a_{k-1}, 2(k-2)+1\}. \end{aligned} \tag{2.1}$$

If $|hA| \leq hk + 2h - 3$, then $a_{k-1} \leq 2(k-2) + 1$. Otherwise, if $a_{k-1} \geq 2k - 2$, then by (2.1) and $k \geq 5$, we have

$$|hA| \geq k + (h-2)(2k-2) + 2k - 3 > hk + 2h - 3,$$

which is impossible. Thus, again by (2.1) we have

$$hk + 2h - 3 \geq |hA| \geq k + (h-1)a_{k-1},$$

hence $a_{k-1} \leq k + 1$.

If $|hA| \leq hk + h - 2$, then by the above discussion, we have $a_{k-1} \leq 2(k-2) + 1$. Thus, by (2.1) we have $hk + h - 2 \geq |hA| \geq k + (h-1)a_{k-1}$, hence $a_{k-1} \leq k$.

This completes the proof of Lemma 2.1. □

Lemma 2.2. *Let i, j be positive integers such that $i \geq 2$ and $j \geq i + 2$. Put $A = [0, i-1] \cup [i+1, j]$. Then $hA = [0, hj]$ for all $h \geq 2$.*

Proof. We have

$$[0, hi - h] \cup [hi + h, hj] \subset hA. \tag{2.2}$$

Write

$$A_1 = \{i-2, i-1\}, \quad A_2 = \{i+1, i+2\}.$$

Since $i \geq 2$ and $j \geq i + 2$, we have $A_1 \cup A_2 \subset A$.

For $h \geq 2$, we have $hi - h + 3l + 1 \geq hi - 2h + 3(l+1)$ for all $0 \leq l \leq h$. Thus

$$\begin{aligned} h(A_1 \cup A_2) &= \bigcup_{l=0}^{h} ((h-l)A_1 + lA_2) \\ &= \bigcup_{l=0}^{h} \big([(i-2)(h-l), (i-1)(h-l)] + [l(i+1), l(i+2)]\big) \end{aligned}$$

$$= \bigcup_{l=0}^{h}[hi-2h+3l, hi-h+3l] \tag{2.3}$$
$$= [hi-2h, hi+2h].$$

By (2.2) and (2.3), we have $hA = [0, hj]$.

This completes the proof of Lemma 2.2. □

Lemma 2.3. *Let i, j be positive integers such that $i \geq 2$ and $j \geq i+3$. Put $A = [0, i-1] \cup [i+2, j]$. Then $hA = [0, hj]$ for all $h \geq 3$.*

Proof. We have

$$[0, hi-h] \cup [hi+2h, hj] \subset hA. \tag{2.4}$$

Write

$$A_1 = \{i-2, i-1\}, \quad A_2 = \{i+2, i+3\}.$$

Since $i \geq 2$ and $j \geq i+3$, we have $A_1 \cup A_2 \subset A$.

For $h \geq 3$, we have $hi - h + 4l + 1 \geq hi - 2h + 4(l+1)$ for all $0 \leq l \leq h$. Thus

$$h(A_1 \cup A_2) = \bigcup_{l=0}^{h}((h-l)A_1 + lA_2)$$
$$= \bigcup_{l=0}^{h}\big([(i-2)(h-l), (i-1)(h-l)] + [l(i+2), l(i+3)]\big)$$
$$= \bigcup_{l=0}^{h}[hi-2h+4l, hi-h+4l] \tag{2.5}$$
$$= [hi-2h, hi+3h].$$

By (2.4) and (2.5), we have $hA = [0, hj]$.

This completes the proof of Lemma 2.3. □

3. Propositions

Proposition 3.1. *Let $h \geq 2$, $k \geq 4$ be positive integers and $A^{(N)} = [0, k] \setminus \{i\}$ for some $i \in [1, k]$. Then*

(1) *If $i = k$, then $\left|hA^{(N)}\right| = hk - h + 1$;*

(2) *If $i = 1$ or $k-1$, then $\left|hA^{(N)}\right| = hk$;*

(3) *If $2 \leq i \leq k-2$, then $\left|hA^{(N)}\right| = hk + 1$.*

Proof. (1) If $A^{(N)} = [0, k-1]$, then $hA^{(N)} = [0, hk-h]$, we have $\left|hA^{(N)}\right| = hk - h + 1$.

(2) If $i = 1$, then $A^{(N)} = \{0\} \cup [2, k]$. We have

$$1 \notin hA^{(N)}, \ \{0\} \cup [2h, hk] \subset hA^{(N)}.$$

For $2 \le m \le 2h-1$, let r_m be the least nonnegative residue of m modulo 2, we have $2 + r_m \in A^{(N)}$ and

$$m = \underbrace{2 + \cdots + 2}_{\frac{m-r_m}{2} - 1 \text{ copies}} + \underbrace{0 + \cdots + 0}_{h - \frac{m-r_m}{2} \text{ copies}} + (2 + r_m).$$

Hence, we have $|hA^{(N)}| = hk$.

If $i = k-1$, then by

$$A^{(N)} = [0, k-2] \cup \{k\} = k - (\{0\} \cup [2, k]),$$

we have $\left|hA^{(N)}\right| = hk$.

(3) If $2 \le i \le k-2$, then $A^{(N)} = [0, i-1] \cup [i+1, k]$. By Lemma 2.2, we have $hA^{(N)} = [0, hk]$. Thus $\left|hA^{(N)}\right| = hk + 1$.

This completes the proof of Proposition 3.1. □

Proposition 3.2. *Let $h \ge 2$, $k \ge 5$ be positive integers and $A^{(N)} = [0, k+1] \backslash \{i, i+1\}$ for some $i \in [1, k]$. Then*

(1) *If $i = k$, then $\left|hA^{(N)}\right| = hk - h + 1$;*

(2) *If $i = 1$ or $k-1$, then $\left|hA^{(N)}\right| = hk + h - 1$;*

(3) *If $2 \le i \le k-2$, then $\left|hA^{(N)}\right| = hk + h + 1$ for $h \ge 3$. For $h = 2$ and $i = 2$ or $k-2$, we have $\left|2A^{(N)}\right| = 2k + 2$; For $h = 2$ and $3 \le i \le k-3$, we have $\left|2A^{(N)}\right| = 2k + 3$.*

Proof. (1) If $A^{(N)} = [0, k-1]$, then by Theorem B, we have $hA^{(N)} = [0, hk-h]$.

(2) If $A^{(N)} = \{0\} \cup [3, k+1]$, then

$$1, 2 \notin hA^{(N)},\ \{0\} \cup [3h, hk+h] \subset hA^{(N)}.$$

For $3 \le m \le 3h-1$, let r_m be the least nonnegative residue of m modulo 3. Noting that $r_m + 3 \in A^{(N)}$ and $1 \le \left\lfloor \frac{m-r_m}{3} \right\rfloor \le h-1$, we have

$$m = \underbrace{3 + \cdots + 3}_{\frac{m-r_m}{3} - 1 \text{ copies}} + (3 + r_m) + \underbrace{0 + \cdots + 0}_{h - \frac{m-r_m}{3} \text{ copies}}.$$

Hence, $|hA^{(N)}| = hk + h - 1$.

If $A^{(N)} = [0, k-2] \cup \{k+1\}$, then by

$$A^{(N)} = (k+1) - (\{0\} \cup [3, k+1]),$$

we have $|hA^{(N)}| = hk + h - 1$.

(3) If $2 \le i \le k-2$, then

$$A^{(N)} = [0, i-1] \cup [i+2, k+1].$$

If $h \ge 3$, then by Lemma 2.3 we have $hA^{(N)} = [0, hk+h]$, thus $|hA^{(N)}| = hk + h + 1$.

If $i = 2$ and $h = 2$, then

$$A^{(N)} = \{0, 1\} \cup [4, k+1],$$

thus $2A^{(N)} = \{0,1,2\} \cup [4, 2k+2]$, we have $|2A^{(N)}| = 2k+2$. If $3 \le i \le k-2$ and $h = 2$, then

$$2A^{(N)} = [0, 2i-2] \cup [i+2, k+i] \cup [2i+4, 2k+2].$$

If $i \le k-3$, then $2A^{(N)} = [0, 2k+2]$; if $i = k-2$, then $2A^{(N)} = [0, 2k-2] \cup [2k, 2k+2]$. Hence, $|2A^{(N)}| = 2k+2$ or $2k+3$.

This completes the proof of Proposition 3.2. □

Proposition 3.3. *Let $h \ge 2$, $k \ge 5$ be positive integers and $A^{(N)} = [0, k+1] \backslash \{i, i+2\}$ for some $i \in [1, k-1]$. Then*

(1) *If $i = k-1$, then $\left|hA^{(N)}\right| = hk$;*

(2) *If $i = 1$ or $k-2$, then $\left|hA^{(N)}\right| = hk + h - 1$;*

(3) *If $2 \le i \le k-3$, then $\left|hA^{(N)}\right| = hk + h + 1$.*

Proof. (1) If $i = k-1$, then $A^{(N)} = [0, k-2] \cup \{k\}$. By Proposition 3.1(2), we have $|hA^{(N)}| = hk$.

(2) If $i = 1$, then $A^{(N)} = \{0\} \cup \{2\} \cup [4, k+1]$. We have

$$1, 3 \notin hA^{(N)}, \ \{0, 2\} \cup [4h, hk+h] \subset hA^{(N)}.$$

For $4 \le m \le 4h-1$, let r_m be the least nonnegative residue of m modulo 4. Then $1 \le \left\lfloor \frac{m-r_m}{4} \right\rfloor \le h-1$. If $r_m = 0$ or 1, then

$$m = \underbrace{4 + \cdots + 4}_{\frac{m-r_m}{4} - 1 \text{ copies}} + \underbrace{0 + \cdots + 0}_{h - \frac{m-r_m}{4} \text{ copies}} + (4 + r_m).$$

If $r_m = 2$ or 3, then

$$m = \underbrace{4 + \cdots + 4}_{\frac{m-r_m}{4} - 1 \text{ copies}} + \underbrace{0 + \cdots + 0}_{h - \frac{m-r_m}{4} - 1 \text{ copies}} + 2 + (2 + r_m).$$

Hence, $|hA^{(N)}| = hk + h - 1$.

If $i = k-2$, then

$$A^{(N)} = (k+1) - (\{0\} \cup \{2\} \cup [4, k+1]).$$

Thus $|hA^{(N)}| = hk + h - 1$.

(3) If $2 \le i \le k-3$, then $A^{(N)} = [0, i-1] \cup \{i+1\} \cup [i+3, k+1]$. Thus

$$[0, hi-h] \cup \{hi+h\} \cup \{hi+3h, hk+h\} \subset hA^{(N)}.$$

Now we shall show that $h(i-1)+m, h(i+1)+m \in hA^{(N)}$ for $1 \le m \le 2h-1$. For $m = 1$ we have

$$h(i-1)+1 = \underbrace{(i-1) + \cdots + (i-1)}_{h-2 \text{ copies}} + (i-2) + (i+1),$$

$$h(i+1)+1 = \underbrace{(i+1) + \cdots + (i+1)}_{h-2 \text{ copies}} + (i-1) + (i+4).$$

For $2 \le m \le 2h-1$, let r_m be the least nonnegative residue of m modulo 2. Then $1 \le \left\lfloor \frac{m-r_m}{2} \right\rfloor \le h-1$, we have

$$\begin{aligned} h(i-1)+m = &\underbrace{(i-1)+\cdots+(i-1)}_{h-1-\frac{m-r_m}{2}\text{ copies}}+(i-1-r_m)\\ &+\underbrace{(i+1)+\cdots+(i+1)}_{\frac{m-r_m}{2}-1\text{ copies}}+(i+1+2r_m),\\ h(i+1)+m = &\underbrace{(i+1)+\cdots+(i+1)}_{h-\frac{m-r_m}{2}\text{ copies}}+\underbrace{(i+3)+\cdots+(i+3)}_{\frac{m-r_m}{2}-1\text{ copies}}\\ &+(i+3+r_m), \end{aligned}$$

Hence, $|hA^{(N)}| = hk+h+1$.

This completes the proof of Proposition 3.3. □

Proposition 3.4. *Let $h \ge 2$, $k \ge 5$ be positive integers and $A^{(N)} = [0, k+1] \backslash \{i, j\}$ for some $i \in [1, k-2]$, $j \ge i+3$.*

(1) *If $i = 1$ and $j = k+1$, then $\left|hA^{(N)}\right| = hk$;*

(2) *If $i = 1$ and $j = k$, then $\left|hA^{(N)}\right| = hk+h-1$;*

(3) *If $i = 1$, $4 \le j \le k-1$; or $2 \le i \le k-3$, $j = k$, then $\left|hA^{(N)}\right| = hk+h$;*

(4) *If $2 \le i \le k-2$, $j = k+1$, then $\left|hA^{(N)}\right| = hk+1$;*

(5) *If $2 \le i \le k-3$ and $j \le k-1$, then $\left|hA^{(N)}\right| = hk+h+1$.*

Proof. (1) If $i = 1$ and $j = k+1$, then $A^{(N)} = \{0\} \cup [2, k]$. By Proposition 3.1(2), we have $\left|hA^{(N)}\right| = hk$.

(2) If $i = 1$ and $j = k$, then $A^{(N)} = \{0\} \cup [2, k-1] \cup \{k+1\}$. By the proof of Proposition 3.1(2), we have $\{0\} \cup [2, hk-h] \subset hA^{(N)}$.

For $1 \le m \le 2h-2$, let r_m be the least nonnegative residue of m modulo 2. Then

$$h(k-1)+m = \underbrace{(k-1)+\cdots+(k-1)}_{h-1-\frac{m+r_m}{2}\text{ copies}}+\underbrace{(k+1)+\cdots+(k+1)}_{\frac{m+r_m}{2}\text{ copies}}+(k-1-r_m).$$

Noting that $hk+h-1 \notin hA^{(N)}$, we have $|hA^{(N)}| = hk+h-1$.

(3) If $i = 1$ and $4 \le j \le k-1$, then

$$A^{(N)} = \{0\} \cup [2, j-1] \cup [j+1, k+1].$$

By the proof of Proposition 3.1(2) we have

$$\{0\} \cup [2, hj-h] \subset hA^{(N)}.$$

Noting that

$$[j-2, j-1] \cup [j+1, j+2] \subset A^{(N)},$$

by the proof of Lemma 2.2 we have $[hj-2h, hj+2h] \subset hA^{(N)}$.

Hence, $|hA^{(N)}| = hk+h$.

If $2 \le i \le k-3$ and $j = k$, then

$$A^{(N)} = [0, i-1] \cup [i+1, k-1] \cup \{k+1\} =: A_1 \cup \{k+1\}.$$

By Lemma 2.2, we have $hA_1 = [0, h(k-1)]$. By the proof of Proposition 3.4(2), we have

$$[hk-h+1, hk+h-2] \subset hA^{(N)} \text{ and } hk+h-1 \notin hA^{(N)}.$$

Hence, $|hA^{(N)}| = hk+h$.

(4) If $i = k-2$ and $k = k+1$, then $A^{(N)} = [0, k-3] \cup \{k-1, k\}$. By Lemma 2.2 we have $|hA^{(N)}| = hk+1$.

If $2 \le i \le k-3$ and $j = k+1$, then

$$A^{(N)} = [0, i-1] \cup [i+1, k].$$

By Lemma 2.2, we have $hA^{(N)} = [0, hk]$, thus $|hA^{(N)}| = hk+1$.

(5) If $2 \le i \le k-3$ and $j \le k-1$, then

$$A^{(N)} = [0, i-1] \cup [i+1, j-1] \cup [j+1, k+1].$$

By Lemma 2.2 we have $[0, h(j-1)] \subset hA^{(N)}$. Noting that

$$[j-2, j-1] \cup [j+1, j+2] \subset A^{(N)},$$

by the proof of Lemma 2.2 we have $[hj-2h, hj+2h] \subset hA^{(N)}$.

Hence, $|hA^{(N)}| = hk+h+1$.

This completes the proof of Proposition 3.4. □

4. Proof of Theorem 1.1

If $hk-h+1 < |hA| \le hk+h-2$, then by Lemma 2.1, we have $A^{(N)} = [0, k] \backslash \{i\}$ for some $i \in [1, k]$. By Proposition 3.1, we have $|hA| = hk$ or $|hA| = hk+1$.

Again by Proposition 3.1, we have $|hA| = hk$ if and only if $A^{(N)} = [0, k] \setminus \{i\}$ with $i = 1$ or $k-1$; $|hA| = hk+1$ if and only if $A^{(N)} = [0, k] \setminus \{i\}$ with some $2 \le i \le k-2$.

This completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2

If $hk+h-2 < |hA| \le hk+2h-3$, then by Lemma 2.1, we have $A^{(N)} = [0, k+1] \backslash \{i, j\}$ for some $1 \le i < j \le k+1$. By Propositions 3.2-3.4, we have $|hA| = hk+h-1$, $hk+h$ or $|hA| = hk+h+1$.

Again by Proposition 3.1, we have (a), (b) and (c).

This completes the proof of Theorem 1.2.

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